

University Advanced Quantum Mechanics · Midterm Examination

The first eight units of Advanced Quantum Mechanics on one paper: time reversal and Kramers degeneracy, identical particles scattering with spin, the Klein–Gordon equation and its exact Coulomb levels, Dirac's free spinors and their boosts, the exact Dirac–Coulomb spectrum, and the Klein step, Zitterbewegung and hole theory. The extended derivation takes the Dirac equation to its non-relativistic limit and derives $g = 2$, the spin-orbit term with its Thomas half and the Darwin term, which Quantum Mechanics II quoted.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	74	14	8

SECTION **UAQM · Midterm · University Advanced Quantum Mechanics — Midterm examination**

IN THE FORMAT OF A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation

DEMAND **Recall 3 · Routine 6 · Demanding 3 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS UAQM-01 · UAQM-02 · UAQM-03 · UAQM-04 · UAQM-05 · UAQM-06 · UAQM-07 · UAQM-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Advanced Quantum Mechanics · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

1 MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 [1]

CORE COURSE

For a system of total angular momentum J , the time-reversal operator satisfies $\Theta^2 = (-1)^{(2J)}$, whatever phase is chosen for it. Options A to C place an ion at a crystal site with no point symmetry at all, so the crystal field removes every degeneracy that symmetry does not protect; option D is a free atom. In which case does time-reversal invariance guarantee that every level of the ground multiplet stays at least twofold degenerate?

- A Ho^{3+} , whose ground multiplet has $J = 8$, in the crystal field alone
- B Er^{3+} , whose ground multiplet has $J = 15/2$, in the crystal field and a uniform static magnetic field of 0.10 T
- C Er^{3+} , whose ground multiplet has $J = 15/2$, in the crystal field and a uniform static electric field of 1.0×10^7 V/m
- D A free hydrogen atom in its 1s level, the proton's spin included, in a uniform static electric field of 1.0×10^7 V/m

Answer

2 MULTIPLE CHOICE · ROUTINE · UAQM-02.06 [1]

CORE COURSE

Two electrons scatter in their centre-of-mass frame through a spin-independent interaction whose direct amplitude is $f(\theta)$. Beam and target are unpolarised, and a detector at $\theta = 90^\circ$ counts an electron whichever of the two it is. In units of $|f(90^\circ)|^2$, what differential cross-section does the detector measure?

- A 0
- B 1
- C 2
- D 4

Answer

3CORE
COURSE

MULTIPLE CHOICE · RECALL · UAQM-03.07 · UAQM-03.08

[1]

For a spinless particle bound to a point nucleus of charge Ze , the exact Klein-Gordon energies are $E = mc^2[1 + (Z\alpha)^2/(n - l - \frac{1}{2} + s)^2]^{-1/2}$, with $s = \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}$ and $\alpha = 1/137.04$. Above about what Z does the formula first fail, giving a complex energy for the $1s$ state?

- A 69
- B 137
- C 172
- D 206

Answer

4CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-04.07 · UAQM-04.08

[1]

A free Dirac particle has $H = c\alpha \cdot p + \beta mc^2$, where α_i and β are the 4×4 Dirac matrices, $\Sigma = \text{diag}(\sigma, \sigma)$ is the spin matrix and $L = r \times p$. Which statement is correct?

- A $[H, L] = 0$, because a free particle experiences no torque
- B $[H, \Sigma_z] = 0$, because H contains no magnetic field to turn the spin
- C $[H, \beta] = 0$, so the upper and lower components of ψ evolve independently
- D $[H, \Sigma \cdot p] = 0$, so helicity is conserved, although Σ_z on its own is not

Answer

5CORE
COURSE

MULTIPLE CHOICE · RECALL · UAQM-05.05 · UAQM-05.06

[1]

Under parity a Dirac field transforms as $\psi(r, t) \rightarrow \gamma^0 \psi(-r, t)$, with $\bar{\psi} = \psi^\dagger \gamma^0$ and $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. Which bilinear is unchanged by every proper Lorentz transformation but changes sign under parity?

- A $\bar{\psi}\psi$
- B $\bar{\psi}\gamma^0\psi$
- C $\bar{\psi}\gamma^5\psi$
- D $\bar{\psi}\gamma^0\gamma^5\psi$

Answer

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-06.03 · UAQM-06.04

[1]

An electron is held in a Penning trap whose magnetic field is 4.70 T. Its cyclotron frequency is $f_c = eB/(2\pi m)$, with $e/m = 1.75882 \times 10^{11} \text{ C kg}^{-1}$. In the exact Dirac Landau levels, where $g = 2$, the level with n quanta of cyclotron motion and spin up along B coincides with the level with $n + 1$ quanta and spin down. Taking $g = 2(1 + a)$, with $a = 1.15965 \times 10^{-3}$, and neglecting relativistic shifts and the trap's electric field, at what frequency must a drive be applied to connect these two levels?

- A 152.6 MHz
- B 305.1 MHz
- C 131.72 GHz
- D 76.3 MHz

Answer

7

CORE
COURSE

MULTIPLE CHOICE · RECALL · UAQM-07.06 · UAQM-07.07

[1]

In the exact Dirac-Coulomb spectrum of a hydrogen-like ion with a point nucleus and $Z < 137$, which statement is correct?

- A $2S_{1/2}$ and $2P_{1/2}$ coincide only to order $(Z\alpha)^4$; the exact energies separate them at order $(Z\alpha)^6$, and that separation is the Lamb shift.
- B $3P_{3/2}$ and $3D_{3/2}$ coincide exactly, because the energies depend only on n and on $|\kappa| = j + 1/2$.
- C $3D_{3/2}$ and $3D_{5/2}$ coincide exactly, because the energies depend only on n and l .
- D $2S_{1/2}$, with $\kappa = -1$, and $2P_{1/2}$, with $\kappa = +1$, are separated, because the energies depend on the sign of κ .

Answer

8

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-08.02 · UAQM-08.06

[1]

An electron of total energy $E = 2mc^2$, so kinetic energy mc^2 , meets an electrostatic step on which its potential energy rises by V_0 . Beyond the step its momentum $\hbar q$ obeys $c^2 \hbar^2 q^2 = (E - V_0)^2 - m^2 c^4$. For which step heights is the transmitted wave oscillatory rather than evanescent?

- A $V_0 < mc^2$ only: any higher step is penetrated only evanescently
- B $V_0 < 2mc^2$ only: a step below the total energy can be crossed
- C $V_0 < mc^2$ and $V_0 > 2mc^2$
- D $V_0 < mc^2$ and $V_0 > 3mc^2$

Answer

For a spin- $\frac{1}{2}$ particle, the time-reversal operator in the basis of S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ is $\Theta = \exp(-i\pi S_y/\hbar)K = -i\sigma_y K$, where K takes the complex conjugate of every component in that basis. Θ is antiunitary: $\Theta(a|u\rangle + b|v\rangle) = a^*\Theta|u\rangle + b^*\Theta|v\rangle$ and $\langle\Theta u|\Theta v\rangle = \langle v|u\rangle$. The Pauli matrices are $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $S = (\hbar/2)\sigma$.

(a) DETERMINE

[2]

Determine $\Theta|\uparrow\rangle$ and $\Theta|\downarrow\rangle$, and hence Θ^2 . Show that replacing Θ by $e^{i\delta}\Theta$, for any real δ , leaves Θ^2 unchanged.

(b) SHOW THAT

[1]

Show that $\Theta\sigma\Theta^{-1} = -\sigma$, so that $\Theta S\Theta^{-1} = -S$.

(c) JUSTIFY

[2]

A particle, or an atom, whose only degeneracy is the $2j + 1$ values of m in a rotation-invariant theory is said to have a permanent electric dipole moment d if $\langle j, m|d|j, m\rangle \neq 0$. Using the Wigner-Eckart theorem within the multiplet, justify the claim that such a moment requires both parity and time-reversal invariance to fail.

(d) EXPLAIN

[1]

The $n = 2$ level of hydrogen shows a linear Stark effect, a shift proportional to the applied field, as though the atom had a permanent dipole moment. Explain why this does not contradict (c).

(e) SHOW THAT

[2]

For a Dirac spinor in the Dirac representation, $\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$ for $i = 1, 2, 3$ in 2×2 blocks, and time reversal acts as $T = i\gamma^1\gamma^3K$. Show that $T^2 = -1$.

A spinless particle of mass m and charge $-e$ is bound by a point nucleus of charge Ze , treated as the fixed source of the static potential energy $V(r) = -Z\alpha\hbar c/r$. Covariant notation uses $x^\mu = (ct, r)$, the metric $g = \text{diag}(1, -1, -1, -1)$ and $p^\mu = i\hbar\partial^\mu$, so that $p^0 = (i\hbar/c)\partial/\partial t$ and $p = -i\hbar\nabla$; minimal coupling replaces the energy E by $E - V$, and a stationary state has $\psi \propto e^{-iEt/\hbar}$. Schrödinger's hydrogen problem, $-(\hbar^2/2M)u'' + [\hbar^2\lambda(\lambda + 1)/(2Mr^2) - k/r]u = \epsilon u$, has bound states $\epsilon = -Mk^2/[2\hbar^2(n_r + \lambda + 1)^2]$, $n_r = 0, 1, 2, \dots$, for any real $\lambda > -\frac{1}{2}$, taking the solution regular at the origin, $u \propto r^{\lambda+1}$. Take $\alpha = 1/137.036$.

- (a) **DERIVE** [2]
Derive the free Klein-Gordon equation from $p_\mu p^\mu = m^2 c^2$, and show that with the Coulomb potential included a stationary state obeys $[(E - V)^2 - m^2 c^4]\psi = -\hbar^2 c^2 \nabla^2 \psi$.
- (b) **DETERMINE** [3]
With $\psi = [u(r)/r]Y_l^m(\theta, \phi)$, the equation becomes $-\hbar^2 c^2 u'' + [\hbar^2 c^2 l(l+1) - (Z\alpha\hbar c)^2]u/r^2 - 2EZ\alpha\hbar c u/r = (E^2 - m^2 c^4)u$. By mapping it onto Schrödinger's problem, determine the exact bound-state energies, writing $n = n_r + l + 1$.
- (c) **SHOW THAT** [2]
Show that, to order $(Z\alpha)^4$, $E = mc^2 - mc^2(Z\alpha)^2/(2n^2) - [mc^2(Z\alpha)^4/(2n^4)][n/(l + 1/2) - 3/4]$.
- (d) **CALCULATE** [2]
The charged-pion mass is determined from the X-rays of the $5g \rightarrow 4f$ transition in pionic nitrogen, a π^- bound to ^{14}N ($Z = 7$), where the strong interaction is negligible for these orbits. Treat the pion as moving in the static field of the nucleus with its mass replaced by the reduced mass μ , using $m_\pi c^2 = 139.570$ MeV and a nuclear rest energy of 13040.20 MeV. Calculate the transition energy to order $(Z\alpha)^2$, and the order- $(Z\alpha)^4$ correction to it, in eV.
- (e) **SKETCH** [2]
On one energy-level diagram, sketch the $n = 2$ levels of a spinless particle of the electron's mass bound to a proton, from (c) with $Z = 1$, beside the $n = 2$ levels of hydrogen given by the Dirac equation, $E = mc^2 - mc^2\alpha^2/(2n^2) - [mc^2\alpha^4/(2n^4)][n/(j + 1/2) - 3/4]$. Mark each level's shift from the unperturbed $n = 2$ level in units of $mc^2\alpha^4/128$, and label the hydrogen levels with the terms of the fine-structure Hamiltonian that move them away from the Klein-Gordon levels.

In the Dirac representation the free Dirac Hamiltonian is $H = c\alpha \cdot p + \beta mc^2$, with $\beta = [[1, 0], [0, -1]]$ and $\alpha_i = [[0, \sigma_i], [\sigma_i, 0]]$ in 2×2 blocks, and $\bar{\psi} = \psi^\dagger \beta$. An electron moves along $+z$ with momentum $p = (3/4)mc$. Positive-energy spinors are normalised to $u^\dagger u = 2E/c$, and the matrix that carries a rest spinor into the spinor of an electron moving along $+z$ with Lorentz factor γ is $S = \cosh(\eta/2) + \alpha_z \sinh(\eta/2)$, where $\cosh \eta = \gamma$. Take $mc^2 = 0.511$ MeV, $\hbar = 1.055 \times 10^{-34}$ J s, $\hbar c = 197.3$ MeV fm and $1 \text{ MeV} = 1.602 \times 10^{-13}$ J.

(a) DERIVE

[3]

From the Dirac equation $Hu = Eu$, derive the positive-energy spinor $u(p, \uparrow)$, with spin up along z , in the form $N(1, 0, b, 0)$, finding E , b and N .

(b) SHOW THAT

[2]

Show that S applied to the rest spinor $\sqrt{2mc}(1, 0, 0, 0)$ gives the spinor of (a), and that S is not unitary, in a way consistent with $\bar{u}u$ staying at $2mc$ while $u^\dagger u$ grows to $2E/c$.

(c) CALCULATE

[2]

Calculate $\langle c\alpha_z \rangle = u^\dagger c\alpha_z u / u^\dagger u$ for this spinor, and compare it with the group velocity $\partial E / \partial p$.

(d) EXPLAIN

[1]

Every eigenvalue of $c\alpha_z$ is $+c$ or $-c$. Explain how this is consistent with (c).

(e) ESTIMATE

[2]

A wave packet built from both positive- and negative-energy solutions near this momentum shows Zitterbewegung. Estimate the angular frequency and the amplitude of the trembling, stating your assumptions, and say what happens to it in a packet built from positive-energy solutions alone.

An electron of mass m is bound to a point nucleus of charge Ze , with $Z < 137$, so that $V(r) = -Z\alpha\hbar c/r$. In a central field the Dirac Hamiltonian commutes with J , with the Dirac parity operator and with $K = \beta(\Sigma \cdot L + \hbar)$, whose eigenvalue is written $-\kappa\hbar$, so that $\kappa = -(j + \frac{1}{2})$ for $j = l + \frac{1}{2}$ and $\kappa = +(j + \frac{1}{2})$ for $j = l - \frac{1}{2}$, l being the orbital angular momentum of the upper component. Writing the upper component as $[G(r)/r]$ times a spinor harmonic of orbital angular momentum l , and the lower as $i[F(r)/r]$ times one of orbital angular momentum l' , the radial equations are $\hbar c(dG/dr + \kappa G/r) = (E + mc^2 - V)F$ and $\hbar c(dF/dr - \kappa F/r) = -(E - mc^2 - V)G$. Take $\alpha = 1/137.036$ and $mc^2 = 511.0$ keV.

(a) STATE

[2]

State κ for each state of the $n = 2$ shell, $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$, together with l' and the parity of each state.

(b) DERIVE

[3]

For a solution with no radial nodes, try $G = A r^\gamma e^{(-\lambda r)}$ and $F = B r^\gamma e^{(-\lambda r)}$, with $\lambda > 0$. Derive from the radial equations γ , the ratio B/A , and λ in terms of E .

(c) SHOW THAT

[2]

Show that $E = mc^2\gamma/|\kappa|$, and that such a nodeless solution exists only for $\kappa < 0$.

(d) CALCULATE

[2]

For hydrogen-like lead, Pb^{81+} ($Z = 82$), calculate the $1s$ binding energy $mc^2 - E$, comparing it with the value to order $(Z\alpha)^4$, $mc^2[(Z\alpha)^2/2 + (Z\alpha)^4/8]$, and the fraction of the $1s$ probability carried by the small component.

(e) PREDICT

[2]

Predict how the $1s$ solution behaves as Z approaches 137 and beyond, for a point nucleus, and say what a nucleus of finite size changes.

Identical nuclei, each in its ground state of spin s , scatter in their centre-of-mass frame through the Coulomb interaction alone, which does not act on spin; every run is far enough below the Coulomb barrier that the nuclear force can be neglected. A detector of angular half-width 0.50° centred on $\theta = 90^\circ$ counts N_{90} nuclei, and N_R is the number it would have counted in the same run if $|f(\theta)|^2$ and $|f(\pi - \theta)|^2$ added without interference, computed from the Rutherford cross-section and the measured beam and target. For the exact Coulomb amplitude $f(\theta)$, the interference term is proportional to $\cos[\eta \ln \tan^2(\theta/2)]$, where $\eta = Z^2\alpha/\beta$, βc is the relative speed, $\beta^2 = 2E_{\text{cm}}/(\mu c^2)$ and μ is the reduced mass. Take $\alpha = 1/137.036$ and $1 \text{ u} = 931.5 \text{ MeV}/c^2$, treat N_{90} as a Poisson count and N_R as exact.

Counts near 90° for six pairs of identical nuclei

Pair	Z	E_{cm} (MeV)	N_{90} (counts)	N_R (counts)
$^{12}\text{C} + ^{12}\text{C}$	6	3.00	9958	5000
$^{13}\text{C} + ^{13}\text{C}$	6	3.00	4035	8000
$^{14}\text{N} + ^{14}\text{N}$	7	3.50	12046	9000
$^{10}\text{B} + ^{10}\text{B}$	5	2.50	13650	12000
$^{11}\text{B} + ^{11}\text{B}$	5	2.50	7462	10000
$^{17}\text{O} + ^{17}\text{O}$	8	4.00	8371	10000

(a) **SHOW THAT**

[2]

The $(2s + 1)^2$ spin states of the pair divide into $(s + 1)(2s + 1)$ that are symmetric under exchange of the two spins and $s(2s + 1)$ that are antisymmetric. Show that for an unpolarised beam and target $d\sigma/d\Omega = |f(\theta)|^2 + |f(\pi - \theta)|^2 + [2(-1)^{(2s)/(2s + 1)}\text{Re}[f(\theta)f^*(\pi - \theta)]]$, and hence that $R = N_{90}/N_R$ should equal $1 + (-1)^{(2s)/(2s + 1)}$.

(b) **CALCULATE**

[2]

Calculate R and its standard uncertainty for each pair.

(c) **DETERMINE**

[3]

Determine the spin of each nucleus, and the significance with which the nearest alternative value is excluded. Say which assignments are secure at three standard deviations.

(d) **EVALUATE** [2]

Near 90° , $\ln \tan^2(\theta/2) \approx 2(\theta - 90^\circ)$, with the angle in radians, so the detector averages $\cos[2\eta(\theta - 90^\circ)]$ over its acceptance. Evaluate whether this averaging biases any spin assignment, using the pair with the largest η .

(e) **EXPLAIN** [1]

Explain why the measurement is made at 90° rather than at another angle.

14
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-06.01 · UAQM-06.02 · UAQM-06.05 · UAQM-06.06 · UAQM-06.07 · UAQM-06.08 · UAQM-06.09

[16]

An electron of charge $q = -e$ and mass m moves in static potentials with Dirac Hamiltonian $H = c\alpha \cdot \pi + \beta mc^2 + V(r)$, where $\pi = p - qA$, $p = -i\hbar\nabla$ and V is the electrostatic potential energy. In the Dirac representation write $\psi = (\phi, \chi)$, with ϕ the large and χ the small two-component spinor, and $E = mc^2 + E'$. You may use $(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$ for vector operators a and b that commute with σ , and $[p, f(r)] = -i\hbar\nabla f$. Take $\alpha = 1/137.036$ and $mc^2 = 0.5110$ MeV.

(a) **DERIVE** [3]

Derive the two coupled equations for ϕ and χ . For $|E' - V| \ll mc^2$, eliminate χ to leading order and derive the Pauli Hamiltonian, showing that its spin term is $-(q/m)S \cdot B$, so that $g = 2$.

(b) **ESTIMATE** [2]

Estimate the ratio $|\chi|/|\phi|$ and the fraction of the probability carried by χ in the ground state of hydrogen, stating your assumptions.

(c) **EXPLAIN** [2]

Now set $A = 0$. Keeping the next term in χ gives $E'\phi = [p^2/(2m) + V - (1/(4m^2c^2))(\sigma \cdot p)(E' - V)(\sigma \cdot p)]\phi$. Explain why this cannot be read as a Schrödinger equation for ϕ with a Hermitian Hamiltonian at order $(v/c)^2$.

(d) **SHOW THAT** [4]

Show that $\Phi = [1 + p^2/(8m^2c^2)]\phi$ is normalised to order $(v/c)^2$, and that it obeys $E'\Phi = H_{FW}\Phi$ with $H_{FW} = p^2/(2m) + V - p^4/(8m^3c^2) + (\hbar/(4m^2c^2))\sigma \cdot (\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$.

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(e) **DETERMINE** [3]

For a central potential $V(r)$, and then for $V = -Z\alpha\hbar c/r$, determine the spin-orbit term in the form $\xi(r)S \cdot L$ and the Darwin term, and evaluate the Darwin shift of the 1s level of hydrogen in eV, given $|\psi_{100}(0)|^2 = 1/(\pi a_0^3)$ with $a_0 = \hbar/(\alpha mc)$.

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(f) **COMPARE** [2]

In a semiclassical picture the electron, moving with velocity v through the electric field $\mathcal{E} = \nabla V/e$, sees in its rest frame the magnetic field $B' = -v \times \mathcal{E}/c^2$. Compare the spin-orbit coefficient that $-\mu \cdot B'$ gives, with $\mu = -g(e/2m)S$, with the one in (e), account for the difference, and say what changes if $g = 2(1 + a)$.

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University Advanced Quantum Mechanics · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 **[1]**

CORE
COURSE

For a system of total angular momentum J , the time-reversal operator satisfies $\Theta^2 = (-1)^{2J}$, whatever phase is chosen for it. Options A to C place an ion at a crystal site with no point symmetry at all, so the crystal field removes every degeneracy that symmetry does not protect; option D is a free atom. In which case does time-reversal invariance guarantee that every level of the ground multiplet stays at least twofold degenerate?

KEY C **Er^{3+} , whose ground multiplet has $J = 15/2$, in the crystal field and a uniform static electric field of $1.0 \times 10^7 \text{ V/m}$**

A takes Kramers' theorem to protect every ion. Holmium's $J = 8$ is an integer, so $\Theta^2 = +1$, and a crystal field with no point symmetry can split the 17 states of the multiplet into 17 singlets, each mapped into itself by Θ ; whatever pairing a non-Kramers ion shows comes from the symmetry of its site, which this site lacks. B has the right ion and the wrong field: a magnetic field is odd under time reversal, so a Hamiltonian containing $-\mu \cdot B$ no longer commutes with Θ , and each Kramers doublet splits linearly in B , which is how such doublets are seen in electron paramagnetic resonance. D counts the electron's spin alone, but the atom also contains the proton: its total angular momentum F is 0 or 1, both integers, so $\Theta^2 = +1$, and the $F = 0$ ground level is already a non-degenerate singlet. C is correct: $J = 15/2$ gives $\Theta^2 = -1$, and an electric field is even under T , so H still commutes with Θ and each state is orthogonal to its time-reversed partner at the same energy. The field can shift the eight Kramers doublets but never split them, however strong it is.

U1

2 MULTIPLE CHOICE · ROUTINE · UAQM-02.06 **[1]**

CORE
COURSE

Two electrons scatter in their centre-of-mass frame through a spin-independent interaction whose direct amplitude is $f(\theta)$. Beam and target are unpolarised, and a detector at $\theta = 90^\circ$ counts an electron whichever of the two it is. In units of $|f(90^\circ)|^2$, what differential cross-section does the detector measure?

KEY B 1

A is the result for beam and target fully polarised parallel: the pair is then in the spin triplet, whose spatial amplitude $f(\theta) - f(\pi - \theta)$ vanishes at 90° ; but an unpolarised pair is in the singlet a quarter of the time. C is what two distinguishable particles would give, $|f(\theta)|^2 + |f(\pi - \theta)|^2 = 2|f|^2$ at 90° , the classical count of either particle reaching the detector with no interference term. D is the result for identical spinless bosons, $|f(\theta) + f(\pi - \theta)|^2 = 4|f|^2$, and would hold for electrons only if they were always in the singlet. B is correct: averaging over spins weights the singlet, whose spatial amplitude is symmetric, by $1/4$ and the triplet, antisymmetric, by $3/4$, so $d\sigma/d\Omega = (1/4)|f + f|^2 + (3/4)|f - f|^2$ with $f = f(\pi - \theta)$. At 90° , $f = f$, giving $(1/4) \times 4|f|^2 + 0 = |f|^2$, half the distinguishable-particle value.

U2

3

CORE
COURSE

MULTIPLE CHOICE · RECALL · UAQM-03.07 · UAQM-03.08

[1]

For a spinless particle bound to a point nucleus of charge Ze , the exact Klein-Gordon energies are $E = mc^2[1 + (Z\alpha)^2/(n - l - 1/2 + s)^2]^{1/2}$, with $s = \sqrt{(l + 1/2)^2 - (Z\alpha)^2}$ and $\alpha = 1/137.04$. Above about what Z does the formula first fail, giving a complex energy for the $1s$ state?

KEY A 69

B is the limit of the Dirac equation's point-nucleus $1s$ state, where $\sqrt{1 - (Z\alpha)^2}$ becomes imaginary at $Z\alpha = 1$. The Klein-Gordon limit is lower because its effective centrifugal coefficient, $l(l + 1) - (Z\alpha)^2$, turns attractive enough to produce fall to the centre once $(Z\alpha)^2$ exceeds $(l + 1/2)^2$, not κ^2 . C is the charge at which, for a nucleus of finite size, the Dirac $1s$ level reaches $-mc^2$ and an empty level would be filled by pair creation, a different question from Unit 8. D is the limit for the $2p$ states, $Z\alpha = 3/2$, since s stays real up to $Z\alpha = l + 1/2$. A is correct: s is real only while $Z\alpha < l + 1/2$, and for $l = 0$ that means $Z\alpha < 1/2$, $Z < 68.5$; beyond it the solutions oscillate infinitely often near $r = 0$ and the point-Coulomb problem has no ground state.

U1

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-04.07 · UAQM-04.08

[1]

A free Dirac particle has $H = c\alpha \cdot p + \beta mc^2$, where α_i and β are the 4×4 Dirac matrices, $\Sigma = \text{diag}(\sigma, \sigma)$ is the spin matrix and $L = r \times p$. Which statement is correct?

KEY D $[H, \Sigma \cdot p] = 0$, so helicity is conserved, although Σ_z on its own is not

A carries a classical argument into a Hamiltonian it does not fit: $[H, L] = c[\alpha \cdot p, L] = -i\hbar c(\alpha \times p)$, which is not zero, so orbital angular momentum alone is not conserved. B fails in the same way in the spin sector: $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$ gives $[H, \Sigma] = 2ic(\alpha \times p)$, and the two commutators cancel only in $J = L + (\hbar/2)\Sigma$, which is why the equation itself describes a particle of spin $1/2$. C overlooks that β anticommutes with every α_i , so $[H, \beta] = -2c\beta(\alpha \cdot p)$; the kinetic term continually mixes upper and lower components, which is how a moving electron acquires its small component. D is correct: the same relation gives $[\alpha \cdot p, \Sigma \cdot p] = 2i\epsilon_{ijk}p_i p_j \alpha_k = 0$ by antisymmetry, and $\Sigma \cdot p$ commutes with β , so helicity is conserved, whereas Σ_z alone changes whenever p has a component transverse to z .

U1

5

CORE
COURSE

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[1]

Under parity a Dirac field transforms as $\psi(r, t) \rightarrow \gamma^0 \psi(-r, t)$, with $\bar{\psi} = \psi^\dagger \gamma^0$ and $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. Which bilinear is unchanged by every proper Lorentz transformation but changes sign under parity?

KEY C $\bar{\psi} \gamma^5 \psi$

Under parity $\psi \rightarrow \gamma^0 \psi$ and $\bar{\psi} \rightarrow \bar{\psi} \gamma^0$, so a bilinear $\bar{\psi} \Gamma \psi$ becomes $\bar{\psi} \gamma^0 \Gamma \gamma^0 \psi$ at the reflected point. A is a Lorentz scalar, but $\gamma^0 \cdot 1 \cdot \gamma^0 = 1$ makes it even under parity: it is the true scalar. B is even under parity too, since $\gamma^0 \gamma^0 \gamma^0 = \gamma^0$, and it is not a scalar at all but the time component of the vector current, the density $\psi^\dagger \psi$, which a boost mixes with the spatial current. D does change sign, because $\gamma^0 (\gamma^0 \gamma^5) \gamma^0 = -\gamma^0 \gamma^5$, but it is the time component of the axial-vector current, and a boost mixes it with $\bar{\psi} \gamma^i \gamma^5 \psi$; it behaves as a pseudoscalar only under rotations. C is correct: γ^5 anticommutes with every γ^μ and so commutes with every generator $\sigma^{\mu\nu}$, giving $\bar{\psi} \gamma^5 \psi \rightarrow \bar{\psi} S^{-1} \gamma^5 S \psi = \bar{\psi} \gamma^5 \psi$ under a proper transformation, while $\gamma^0 \gamma^5 \gamma^0 = -\gamma^5$ makes it odd under parity: a pseudoscalar.

U1

6

CORE
COURSE

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[1]

An electron is held in a Penning trap whose magnetic field is 4.70 T. Its cyclotron frequency is $f_c = eB/(2\pi m)$, with $e/m = 1.75882 \times 10^{11} \text{ C kg}^{-1}$. In the exact Dirac Landau levels, where $g = 2$, the level with n quanta of cyclotron motion and spin up along B coincides with the level with $n + 1$ quanta and spin down. Taking $g = 2(1 + a)$, with $a = 1.15965 \times 10^{-3}$, and neglecting relativistic shifts and the trap's electric field, at what frequency must a drive be applied to connect these two levels?

KEY **A 152.6 MHz**

B doubles the anomaly. The spin-flip energy of the electron is $(g/2)\hbar\omega_c$, and going from n quanta to $n + 1$ removes $\hbar\omega_c$ of it, leaving $(g/2 - 1)\hbar\omega_c = a\hbar\omega_c$, not $(g - 2)\hbar\omega_c$. C is the spin-flip frequency itself, $f_s = (g/2)f_c = (1 + a)f_c = 131.72 \text{ GHz}$, which connects n with spin up to n with spin down; the degenerate pair also differs by a cyclotron quantum, and that quantum cancels all of f_s except the anomaly. D takes the shift of one level for the separation of the pair: with $g = 2(1 + a)$ the spin-up level rises by $a\hbar\omega_c/2$ and the spin-down level falls by as much, so they separate by $a\hbar\omega_c$. A is correct: $f_c = (1.75882 \times 10^{11} \text{ C kg}^{-1} \times 4.70 \text{ T})/(2\pi) = 131.56 \text{ GHz}$, and $f_a = af_c = 152.6 \text{ MHz}$. Because the trap measures f_a and f_c together, it yields $a = f_a/f_c$ directly, which is why $g - 2$ is measured rather than g .

U3

7

CORE
COURSE

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[1]

In the exact Dirac-Coulomb spectrum of a hydrogen-like ion with a point nucleus and $Z < 137$, which statement is correct?

KEY **B $3P_{3/2}$ and $3D_{3/2}$ coincide exactly, because the energies depend only on n and on $|\kappa| = j + \frac{1}{2}$.**

A mistakes a radiative correction for a relativistic one. The exact energies, $E = mc^2[1 + (Z\alpha)^2/(n_r + \gamma)^2]^{-1/2}$ with $\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}$, contain κ only through κ^2 , so the degeneracy survives to all orders in $Z\alpha$, and the Lamb shift needs the quantised radiation field. C carries the Schrödinger degeneracy over the wrong way: in Dirac's spectrum levels of the same l and different j split, which is the fine structure, here with $3D_{5/2}$ above $3D_{3/2}$. D states the opposite of the theorem: $\kappa = -1$ and $\kappa = +1$ give the same κ^2 and, for $n = 2$, the same $n_r = 1$, so the two states coincide. B is correct: $3P_{3/2}$ has $\kappa = -2$ and $3D_{3/2}$ has $\kappa = +2$, both with $n_r = 1$, so they share one energy although they have opposite parity.

U1

8

CORE
COURSE

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[1]

An electron of total energy $E = 2mc^2$, so kinetic energy mc^2 , meets an electrostatic step on which its potential energy rises by V_0 . Beyond the step its momentum $\hbar q$ obeys $c^2\hbar^2 q^2 = (E - V_0)^2 - m^2 c^4$. For which step heights is the transmitted wave oscillatory rather than evanescent?

KEY **D $V_0 < mc^2$ and $V_0 > 3mc^2$**

A is the Schrödinger answer: a step lower than the kinetic energy is crossed and a higher one only penetrated. For the Dirac equation that holds only up to $V_0 = E + mc^2$. B compares the step with the total energy instead of the kinetic energy, and misses the gap in which $|E - V_0| < mc^2$: for $mc^2 < V_0 < 3mc^2$, q is imaginary. C uses the threshold $2mc^2$, which is the Klein threshold for an electron at rest, $E = mc^2$; for this electron the step must exceed $E + mc^2 = 3mc^2$. D is correct: q is real when $|E - V_0| > mc^2$, that is for $V_0 < E - mc^2 = mc^2$, the ordinary transmitted electron, or for $V_0 > E + mc^2 = 3mc^2$, the Klein regime, in which $E - V_0$ lies below $-mc^2$ and hole theory reads the transmitted wave as positrons created at the step.

U2

For a spin- $\frac{1}{2}$ particle, the time-reversal operator in the basis of S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ is $\Theta = \exp(-i\pi S_y/\hbar)K = -i\sigma_y K$, where K takes the complex conjugate of every component in that basis. Θ is antiunitary: $\Theta(a|u\rangle + b|v\rangle) = a^*\Theta|u\rangle + b^*\Theta|v\rangle$ and $\langle\Theta u|\Theta v\rangle = \langle v|u\rangle$. The Pauli matrices are $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $S = (\hbar/2)\sigma$.

(a) DETERMINE · U2

[2]

Determine $\Theta|\uparrow\rangle$ and $\Theta|\downarrow\rangle$, and hence Θ^2 . Show that replacing Θ by $e^{i\delta}\Theta$, for any real δ , leaves Θ^2 unchanged.

1 $-i\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is real, so $\Theta|\uparrow\rangle = |\downarrow\rangle$ and $\Theta|\downarrow\rangle = -|\uparrow\rangle$; then $\Theta^2|\uparrow\rangle = \Theta|\downarrow\rangle = -|\uparrow\rangle$ and $\Theta^2|\downarrow\rangle = -\Theta|\uparrow\rangle = -|\downarrow\rangle$, so $\Theta^2 = -1$

2 $(e^{i\delta}\Theta)(e^{i\delta}\Theta) = e^{i\delta}e^{i\delta}\Theta^2 = \Theta^2$, because the antilinear Θ conjugates the phase it passes through: the sign of Θ^2 belongs to the spin, not to a convention

Candidates who treat Θ as linear find $(e^{i\delta}\Theta)\Theta^2 = e^{i\delta}\Theta^2$, conclude that the sign of Θ^2 can be chosen at will, and so lose the whole basis of Kramers' theorem.

(b) SHOW THAT · U2

[1]

Show that $\Theta\sigma\Theta^{-1} = -\sigma$, so that $\Theta S\Theta^{-1} = -S$.

1 with $\Theta^{-1} = K(i\sigma_y)$, $\Theta\sigma\Theta^{-1} = \sigma_y\sigma^*\sigma_y$: σ_x and σ_z are real and anticommute with σ_y , giving $\sigma_y\sigma_x\sigma_y = -\sigma_x$ and $\sigma_y\sigma_z\sigma_y = -\sigma_z$, while $\sigma_y^* = -\sigma_y$ gives $\sigma_y(-\sigma_y)\sigma_y = -\sigma_y$

(c) JUSTIFY · U1

[2]

A particle, or an atom, whose only degeneracy is the $2j + 1$ values of m in a rotation-invariant theory is said to have a permanent electric dipole moment d if $\langle j, m|d|j, m\rangle \neq 0$. Using the Wigner-Eckart theorem within the multiplet, justify the claim that such a moment requires both parity and time-reversal invariance to fail.

1 within one multiplet the Wigner-Eckart theorem makes every vector operator proportional to J : $\langle j, m'|d|j, m\rangle = k\langle j, m'|J|j, m\rangle$, with k a single real constant because d and J are Hermitian, so a permanent moment means $k \neq 0$

2 d is odd under P and even under T , while J is even under P and odd under T ; if P is a symmetry it maps the non-degenerate multiplet into itself, and $PdP^{-1} = -d$ with $PJP^{-1} = J$ forces $k = -k$, and if T is a symmetry, $\Theta d\Theta^{-1} = d$ with $\Theta J\Theta^{-1} = -J$ forces $k = -k$ again, so $k \neq 0$ needs both to fail

Answers that show only that d is odd under parity have proved that P must fail and have said nothing about T . The argument needs the proportionality to J , which carries the T -odd signature into the expectation value.

(d) EXPLAIN · U1

[1]

The $n = 2$ level of hydrogen shows a linear Stark effect, a shift proportional to the applied field, as though the atom had a permanent dipole moment. Explain why this does not contradict (c).

1 the linear effect needs states of opposite parity at one energy, $2s$ and $2p$, so the level is not the non-degenerate multiplet that (c) assumes; the field mixes them into states that carry a dipole, and once the field is weak beside the $2S_{1/2}$ - $2P_{1/2}$ splitting the shift becomes quadratic, as it must for a non-degenerate level

(e) SHOW THAT · U2

[2]

For a Dirac spinor in the Dirac representation, $\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$ for $i = 1, 2, 3$ in 2×2 blocks, and time reversal acts as $T = i\gamma^1\gamma^3K$. Show that $T^2 = -1$.

1 γ^1 and γ^3 contain only σ_x and σ_z , so they are real, and $K(i\gamma^1\gamma^3) = (-i\gamma^1\gamma^3)K$; hence $T^2 = (i\gamma^1\gamma^3)(-i\gamma^1\gamma^3)K^2 = \gamma^1\gamma^3\gamma^1\gamma^3$

2 $\gamma^3\gamma^1 = -\gamma^1\gamma^3$ and $(\gamma^i)^2 = g^{ii} = -1$, so $\gamma^1\gamma^3\gamma^1\gamma^3 = -(\gamma^1)^2(\gamma^3)^2 = -(-1)(-1) = -1$: the four-component spinor, like the two-component one in (a), has $T^2 = -1$, and a Dirac particle's levels in any T -invariant field are Kramers doublets

Forgetting that K acts on the i in front of the second factor gives $T^2 = i^2\gamma^1\gamma^3\gamma^1\gamma^3 = +1$, the wrong sign for a spin- $\frac{1}{2}$ particle.

A spinless particle of mass m and charge $-e$ is bound by a point nucleus of charge Ze , treated as the fixed source of the static potential energy $V(r) = -Z\alpha\hbar c/r$. Covariant notation uses $x^\mu = (ct, \mathbf{r})$, the metric $g = \text{diag}(1, -1, -1, -1)$ and $p^\mu = i\hbar\partial^\mu$, so that $p^0 = (i\hbar/c)\partial/\partial t$ and $\mathbf{p} = -i\hbar\nabla$; minimal coupling replaces the energy E by $E - V$, and a stationary state has $\psi \propto e^{-(iEt/\hbar)}$. Schrödinger's hydrogen problem, $-(\hbar^2/2M)u'' + [\hbar^2\lambda(\lambda+1)/(2Mr^2) - k/r]u = \epsilon u$, has bound states $\epsilon = -Mk^2/[2\hbar^2(n_r + \lambda + 1)^2]$, $n_r = 0, 1, 2, \dots$, for any real $\lambda > -1/2$, taking the solution regular at the origin, $u \propto r^{-(\lambda+1)}$. Take $\alpha = 1/137.036$.

(a) DERIVE · U2

[2]

Derive the free Klein-Gordon equation from $p_\mu p^\mu = m^2 c^2$, and show that with the Coulomb potential included a stationary state obeys $[(E - V)^2 - m^2 c^4]\psi = -\hbar^2 c^2 \nabla^2 \psi$.

1 $p_\mu p^\mu = (p^0)^2 - \mathbf{p} \cdot \mathbf{p}$, with $p^0 = (i\hbar/c)\partial/\partial t$ and $\mathbf{p} = -i\hbar\nabla$, gives $-(\hbar^2/c^2)\partial^2\psi/\partial t^2 + \hbar^2\nabla^2\psi = m^2 c^2 \psi$, that is $(1/c^2)\partial^2\psi/\partial t^2 - \nabla^2\psi + (mc/\hbar)^2\psi = 0$, Lorentz invariant because $p_\mu p^\mu$ is a scalar

2 for $\psi \propto e^{-(iEt/\hbar)}$, p^0 acting on ψ gives E ; minimal coupling replaces it by $E - V$, so $(E - V)^2 \psi - c^2 \mathbf{p}^2 \psi = m^2 c^4 \psi$, that is $[(E - V)^2 - m^2 c^4]\psi = -\hbar^2 c^2 \nabla^2 \psi$

The sign of the metric decides the sign of the ∇^2 term. Writing $(p^0)^2 + p^2$ builds a Euclidean invariant, and the equation that follows has no plane-wave solutions with real energy above mc^2 .

(b) DETERMINE · U2

[3]

With $\psi = [u(r)/r]Y_l^m(\theta, \phi)$, the equation becomes $-\hbar^2 c^2 u'' + [\hbar^2 c^2 l(l+1) - (Z\alpha\hbar c)^2]u/r^2 - 2EZ\alpha\hbar c u/r = (E^2 - m^2 c^4)u$. By mapping it onto Schrödinger's problem, determine the exact bound-state energies, writing $n = n_r + l + 1$.

1 dividing by $2E$ gives Schrödinger's form with $M = E/c^2$, $k = Z\alpha\hbar c$, $\epsilon = (E^2 - m^2 c^4)/(2E)$ and $\lambda(\lambda+1) = l(l+1) - (Z\alpha)^2$, so $\lambda = -1/2 + \sqrt{(l+1/2)^2 - (Z\alpha)^2}$, the root that tends to l as $Z\alpha \rightarrow 0$

2 then $(E^2 - m^2 c^4)/(2E) = -(E/c^2)(Z\alpha\hbar c)^2/(2\hbar^2 N^2) = -E(Z\alpha)^2/(2N^2)$, with $N = n_r + \lambda + 1 = n - l - 1/2 + \sqrt{(l+1/2)^2 - (Z\alpha)^2}$

3 so $E^2[1 + (Z\alpha)^2/N^2] = m^2 c^4$ and $E = mc^2[1 + (Z\alpha)^2/N^2]^{(-1/2)}$, which depends on l through N , unlike the Schrödinger levels

The mapping fails for candidates who identify M with m : the effective mass, E/c^2 , and the effective energy, $(E^2 - m^2 c^4)/(2E)$, both contain E , and only when both are written that way does E^2 factor out. Taking the other root of $\lambda(\lambda+1)$ gives a solution that is singular at the origin and does not reduce to l .

(c) SHOW THAT · U2

[2]

Show that, to order $(Z\alpha)^4$, $E = mc^2 - mc^2(Z\alpha)^2/(2n^2) - [mc^2(Z\alpha)^4/(2n^4)][n/(l+1/2) - 3/4]$.

1 $N = n - \delta$ with $\delta = (l+1/2) - \sqrt{(l+1/2)^2 - (Z\alpha)^2} \approx (Z\alpha)^2/(2l+1)$, so $(Z\alpha)^2/N^2 \approx (Z\alpha)^2/n^2 + 2(Z\alpha)^4/[n^3(2l+1)]$

2 with $(1+x)^{(-1/2)} \approx 1 - x/2 + 3x^2/8$ and $x^2 \approx (Z\alpha)^4/n^4$, $E \approx mc^2[1 - (Z\alpha)^2/(2n^2) - (Z\alpha)^4/(n^3(2l+1)) + 3(Z\alpha)^4/(8n^4)]$, and $(Z\alpha)^4/(n^3(2l+1)) = [(Z\alpha)^4/(2n^4)]n/(l+1/2)$, which gives the result

The $(Z\alpha)^4$ term has two sources, the shift δ in N and the second-order term of the inverse square root. Dropping $3x^2/8$ loses the $-3/4$, and the correction then no longer equals the expectation value of $-p^4/(8m^3c^2)$, the check that ties the exact result to its non-relativistic limit.

(d) CALCULATE · U3**[2]**

The charged-pion mass is determined from the X-rays of the $5g \rightarrow 4f$ transition in pionic nitrogen, a π^- bound to ^{14}N ($Z = 7$), where the strong interaction is negligible for these orbits. Treat the pion as moving in the static field of the nucleus with its mass replaced by the reduced mass μ , using $m_\pi c^2 = 139.570$ MeV and a nuclear rest energy of 13040.20 MeV. Calculate the transition energy to order $(Z\alpha)^2$, and the order- $(Z\alpha)^4$ correction to it, in eV.

- $\mu c^2 = (139.570 \times 13040.20/13179.77)$ MeV = 138.092 MeV and $Z\alpha = 0.051081$, so to order $(Z\alpha)^2$ the line is $[\mu c^2(Z\alpha)^2/2](1/16 - 1/25) = 4053.66$ eV
- the $(Z\alpha)^4$ terms are -0.272 eV for $5g$ ($n = 5, l = 4$) and -0.721 eV for $4f$ ($n = 4, l = 3$), so the correction is $+0.450$ eV and the line lies at 4054.11 eV; at 1.1×10^{-4} of the energy, a pion mass quoted to a few parts per million must include it

Using m_π in place of μ moves the line by about 1.1%, 43 eV, a hundred times the relativistic correction the part asks for. The $4f$ term dominates the correction because it falls as $1/n^4$ and is larger at smaller l .

(e) SKETCH · U1**[2]**

On one energy-level diagram, sketch the $n = 2$ levels of a spinless particle of the electron's mass bound to a proton, from (c) with $Z = 1$, beside the $n = 2$ levels of hydrogen given by the Dirac equation, $E = mc^2 - mc^2\alpha^2/(2n^2) - [mc^2\alpha^4/(2n^4)][n/(j + 1/2) - 3/4]$. Mark each level's shift from the unperturbed $n = 2$ level in units of $mc^2\alpha^4/128$, and label the hydrogen levels with the terms of the fine-structure Hamiltonian that move them away from the Klein-Gordon levels.

- Klein-Gordon: $2s$ at -13 units and $2p$ at $-7/3$ units, $2p$ above $2s$ by $32/3$ units, $mc^2\alpha^4/12$ or 29.2 GHz; hydrogen: $2S_{1/2}$ and $2P_{1/2}$ drawn as one level at -5 units and $2P_{3/2}$ at -1 unit, every level below the unperturbed one
- the differences labelled: the Darwin term raises $2s$ by 8 units to $2S_{1/2}$; spin-orbit coupling raises $2p$ by $4/3$ units to $2P_{3/2}$ and lowers it by $8/3$ units to $2P_{1/2}$; the Klein-Gordon levels carry only the $-p^4/(8m^3c^2)$ term, so their pattern is right only for a spinless particle, as in a pionic atom

The expected wrong sketch puts the Klein-Gordon $2s$ and $2p$ levels together, carrying over Schrödinger's degeneracy in l . The exact Klein-Gordon levels depend on l through N ; it is the Dirac equation, through the spin-orbit and Darwin terms, that restores a degeneracy, now in j .

11
CORE
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[10]

In the Dirac representation the free Dirac Hamiltonian is $H = \alpha \cdot p + \beta mc^2$, with $\beta = [[1, 0], [0, -1]]$ and $\alpha_i = [[0, \sigma_i], [\sigma_i, 0]]$ in 2×2 blocks, and $\bar{\psi} = \psi^\dagger \beta$. An electron moves along $+z$ with momentum $p = (3/4)mc$. Positive-energy spinors are normalised to $u^\dagger u = 2E/c$, and the matrix that carries a rest spinor into the spinor of an electron moving along $+z$ with Lorentz factor γ is $S = \cosh(\eta/2) + \alpha_z \sinh(\eta/2)$, where $\cosh \eta = \gamma$. Take $mc^2 = 0.511$ MeV, $\hbar = 1.055 \times 10^{-34}$ J s, $\hbar c = 197.3$ MeV fm and $1 \text{ MeV} = 1.602 \times 10^{-13}$ J.

(a) DERIVE · U2**[3]**

From the Dirac equation $Hu = Eu$, derive the positive-energy spinor $u(p, \uparrow)$, with spin up along z , in the form $N(1, 0, b, 0)$, finding E , b and N .

- the anticommutation relations make $H^2 = (p^2c^2 + m^2c^4)I$, so the positive energy is $E = mc^2\sqrt{1 + 9/16} = (5/4)mc^2$
- writing $u = (\phi, \chi)$, the lower block of $Hu = Eu$ reads $c\sigma \cdot p \phi - mc^2\chi = E\chi$, so $\chi = [cp\sigma_z/(E + mc^2)]\phi$, and with $\phi = (1, 0)$, $b = (3/4)/(9/4) = 1/3$
- $N^2(1 + 1/9) = 2E/c = (5/2)mc$, so $N^2 = (9/4)mc$ and $u = \sqrt{(mc)}(3/2, 0, 1/2, 0)$

The slip to catch is the sign in the lower block: writing $E - mc^2$ in the denominator gives $b = 3$, a spinor dominated by its lower component, which is the pattern of a negative-energy solution, not of an electron at $0.6c$.

- (b) **SHOW THAT** · U2 [2]
- Show that S applied to the rest spinor $\sqrt{(2mc)}(1, 0, 0, 0)$ gives the spinor of (a), and that S is not unitary, in a way consistent with $\bar{u}u$ staying at $2mc$ while $u\bar{u}$ grows to $2E/c$.
- $\gamma = 5/4$, so $\cosh(\eta/2) = \sqrt{(\gamma + 1)/2} = 3/(2\sqrt{2})$ and $\sinh(\eta/2) = \sqrt{(\gamma - 1)/2} = 1/(2\sqrt{2})$; since $\alpha_z(1, 0, 0, 0) = (0, 0, 1, 0)$, $S\sqrt{(2mc)}(1, 0, 0, 0) = \sqrt{(2mc)}(3/(2\sqrt{2}), 0, 1/(2\sqrt{2}), 0) = \sqrt{(mc)}(3/2, 0, 1/2, 0)$, the spinor of (a)
 - S is Hermitian, so $S^\dagger S = S^2 = \cosh \eta + \alpha_z \sinh \eta \neq I$, and $u\bar{u} = 2mc \cosh \eta = (5/2)mc = 2E/c$ because α_z has no diagonal element in the rest spinor; but $\beta S \beta = S^{-1}$, so $S^\dagger \beta S = \beta$ and $\bar{u}u = 2mc$ is unchanged
- A boost matrix that preserved $u\bar{u}$ would leave the density the same in every frame, though $\psi^\dagger \psi$ is the time component of a four-vector and must grow by γ . Candidates who assume S unitary, as the rotation matrices are, cannot reach $2E/c$.*
- (c) **CALCULATE** · U3 [2]
- Calculate $\langle \alpha_z \rangle = u^\dagger \alpha_z u / u\bar{u}$ for this spinor, and compare it with the group velocity $\partial E / \partial p$.
- $\alpha_z u = \sqrt{(mc)}(1/2, 0, 3/2, 0)$, so $u^\dagger \alpha_z u = mc(3/4 + 3/4) = (3/2)mc$ and $\langle \alpha_z \rangle = [(3/2)/(5/2)]c = 0.6c$
 - $\partial E / \partial p = pc^2/E = [(3/4)/(5/4)]c = 0.6c$, the same: in a positive-energy plane wave the velocity operator $\alpha = (i/\hbar)[H, r]$ has the group velocity as its expectation value
- Candidates who take the velocity to be p/m get $0.75c$, the non-relativistic answer, and miss that α already contains the relativistic kinematics.*
- (d) **EXPLAIN** · U1 [1]
- Every eigenvalue of α_z is $+c$ or $-c$. Explain how this is consistent with (c).
- $\alpha_z^2 = I$ gives eigenvalues ± 1 , but α_z does not commute with H , since it anticommutes with β , so its eigenstates are superpositions of positive- and negative-energy solutions; the electron of (c) is not in an α_z eigenstate, and $0.6c$ is the average of α_z over a state of one sign of energy
- The weak answer says the electron moves at c and averages to $0.6c$ by reversing direction, as if the eigenvalues were velocities a positive-energy electron could have. The eigenstates of α_z are not states of one sign of energy, and the resolution needs the negative-energy components of Unit 8.*
- (e) **ESTIMATE** · U4 [2]
- A wave packet built from both positive- and negative-energy solutions near this momentum shows Zitterbewegung. Estimate the angular frequency and the amplitude of the trembling, stating your assumptions, and say what happens to it in a packet built from positive-energy solutions alone.
- assuming the packet is narrow in momentum about p , the interference between components at $+E$ and $-E$ oscillates at $\omega = 2E/\hbar = 2 \times 0.639 \times 1.602 \times 10^{-13} \text{ J} / (1.055 \times 10^{-34} \text{ J s}) = 1.9 \times 10^{21} \text{ s}^{-1}$, with amplitude of order $\hbar c / (2E) = 197.3 \text{ MeV fm} / 1.28 \text{ MeV} \approx 1.5 \times 10^{-13} \text{ m}$, close to $\hbar / (2mc) = 1.9 \times 10^{-13} \text{ m}$, a fraction of the Compton wavelength
 - the trembling is an interference between the two signs of energy, so it vanishes identically in a packet of positive energy only, where $\langle r \rangle$ moves uniformly at the group velocity of (c); confining a packet within about $\hbar / (mc)$ forces in negative-energy components, and the trembling with them
- A frequency of $E/\hbar$, the rate at which one plane wave's phase turns, misses that the effect is a beat between $+E$ and $-E$. Candidates who describe the trembling as an observable motion of every free electron forget that it exists only in packets containing both signs of energy.*

An electron of mass m is bound to a point nucleus of charge Ze , with $Z < 137$, so that $V(r) = -Z\alpha\hbar c/r$. In a central field the Dirac Hamiltonian commutes with J , with the Dirac parity operator and with $K = \beta(\Sigma \cdot L + \hbar)$, whose eigenvalue is written $-\kappa\hbar$, so that $\kappa = -(j + \frac{1}{2})$ for $j = l + \frac{1}{2}$ and $\kappa = +(j + \frac{1}{2})$ for $j = l - \frac{1}{2}$, l being the orbital angular momentum of the upper component. Writing the upper component as $[G(r)/r]$ times a spinor harmonic of orbital angular momentum l , and the lower as $i[F(r)/r]$ times one of orbital angular momentum l' , the radial equations are $\hbar c(dG/dr + \kappa G/r) = (E + mc^2 - V)F$ and $\hbar c(dF/dr - \kappa F/r) = -(E - mc^2 - V)G$. Take $\alpha = 1/137.036$ and $mc^2 = 511.0$ keV.

(a) STATE · U1

[2]

State κ for each state of the $n = 2$ shell, $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$, together with l' and the parity of each state.

- 1 $2S_{1/2}$: $\kappa = -1$; $2P_{1/2}$: $\kappa = +1$; $2P_{3/2}$: $\kappa = -2$
- 2 $l' = 2j - l$: 1 for $2S_{1/2}$, 0 for $2P_{1/2}$ and 2 for $2P_{3/2}$; the parity is $(-1)^{l'}$ of the upper component, even for $2S_{1/2}$ and odd for both $2P$ states, the β in the Dirac parity operator compensating the opposite $(-1)^{l'}$ of the lower component

K's eigenvalue is $-\kappa\hbar$, so the states with $j = l + \frac{1}{2}$, $1s$ and $2P_{3/2}$ among them, have negative κ ; candidates who drop the minus sign assign the wrong states to the nodeless solutions of (c). Reading the parity from l' instead makes $2S_{1/2}$ odd and $2P_{1/2}$ even, which would forbid $2P_{1/2} \rightarrow 1S_{1/2}$ as an electric-dipole transition, although it is one of the two components of Lyman- α .

(b) DERIVE · U2

[3]

For a solution with no radial nodes, try $G = Ar^\gamma e^{(-\lambda r)}$ and $F = Br^\gamma e^{(-\lambda r)}$, with $\lambda > 0$. Derive from the radial equations γ , the ratio B/A , and λ in terms of E .

- 1 the $1/r$ terms of the first equation give $\hbar c(\gamma + \kappa)A = Z\alpha\hbar cB$, so $B/A = (\gamma + \kappa)/(Z\alpha)$; those of the second give $(\gamma - \kappa)B = -Z\alpha A$, so $B/A = -Z\alpha/(\gamma - \kappa)$
- 2 equating the two ratios, $\gamma^2 - \kappa^2 = -(Z\alpha)^2$, so $\gamma = \sqrt{(\kappa^2 - (Z\alpha)^2)}$, the positive root, since the negative one gives G and F that diverge at the origin, as $r^{(-|\kappa|)}$ when $Z\alpha$ is small, and cannot be normalised; for $|\kappa| = 1$, γ is less than 1
- 3 the r -independent terms give $-\hbar c\lambda A = (E + mc^2)B$ and $-\hbar c\lambda B = -(E - mc^2)A$; multiplying them, $(\hbar c\lambda)^2 = m^2c^4 - E^2$, so $\lambda = \sqrt{(m^2c^4 - E^2)/(\hbar c)}$, real for a bound state with $|E| < mc^2$

Candidates who match powers of r without separating the $1/r$ terms from the constant terms obtain one tangled relation between E and γ . The structure to see is that the singular terms fix γ independently of E , as the centrifugal term fixes the power at the origin in Schrödinger's problem.

(c) SHOW THAT · U2

[2]

Show that $E = mc^2\gamma/|\kappa|$, and that such a nodeless solution exists only for $\kappa < 0$.

- 1 from the constant terms $B/A = -\hbar c\lambda/(E + mc^2) = -\sqrt{(mc^2 - E)/(mc^2 + E)}$, which is negative; with $B/A = (\gamma + \kappa)/(Z\alpha)$ this needs $\gamma + \kappa < 0$, impossible for $\kappa > 0$ because $\gamma > 0$, so $\kappa = -|\kappa|$ and only states with $j = l + \frac{1}{2}$, $1S_{1/2}$, $2P_{3/2}$, $3D_{5/2}$ and so on, can be nodeless
- 2 squaring, $(|\kappa| - \gamma)^2/(Z\alpha)^2 = (mc^2 - E)/(mc^2 + E)$; since $(Z\alpha)^2 = (|\kappa| - \gamma)(|\kappa| + \gamma)$ the left side is $(|\kappa| - \gamma)/(|\kappa| + \gamma)$, so $E/mc^2 = \gamma/|\kappa| = \sqrt{1 - (Z\alpha)^2/\kappa^2}$

Both lines are needed. A candidate who squares at once loses the sign of B/A , and with it the reason that no $2S_{1/2}$ or $2P_{1/2}$ state is a single term: those states need a radial node, as the $n_r = 1$ that the general formula gives them shows.

(d) CALCULATE · U3 [2]

For hydrogen-like lead, Pb^{81+} ($Z = 82$), calculate the $1s$ binding energy $mc^2 - E$, comparing it with the value to order $(Z\alpha)^4$, $mc^2[(Z\alpha)^2/2 + (Z\alpha)^4/8]$, and the fraction of the $1s$ probability carried by the small component.

1 $Z\alpha = 0.59838$ and $\gamma = \sqrt{1 - 0.35806} = 0.80121$, so $mc^2 - E = 511.0 \text{ keV} \times 0.19879 = 101.6 \text{ keV}$, against 99.7 keV to order $(Z\alpha)^4$ and 91.5 keV non-relativistically

2 G and F share the radial factor, so the small component carries $B^2/(A^2 + B^2) = (1 - \gamma)/2 = 0.099$, about 10% of the probability, against $\alpha^2/4 = 1.3 \times 10^{-5}$ in hydrogen

The order- $(Z\alpha)^4$ expansion misses 1.9 keV, 2% of the binding, at $Z = 82$. Candidates who then estimate the small-component share as $(Z\alpha)^2/4 = 0.090$ underestimate it by a tenth for the same reason.

(e) PREDICT · U1 [2]

Predict how the $1s$ solution behaves as Z approaches 137 and beyond, for a point nucleus, and say what a nucleus of finite size changes.

1 as $Z\alpha \rightarrow 1$, $\gamma \rightarrow 0$ and $E = mc^2\gamma \rightarrow 0$, so the $1s$ level reaches zero energy, not $-mc^2$, while the density near the origin, $\propto r^{2\gamma-2}$, grows ever more singular; for $Z\alpha > 1$, γ is imaginary, the solutions oscillate as $r^{2\gamma}(\pm i|y|)$ with no limit at the origin, and the point-nucleus problem has no ground state, the Dirac analogue of the Klein-Gordon fall to the centre at $Z\alpha = 1/2$

2 a finite nucleus removes the singularity, so the $1s$ level continues smoothly past $Z = 137$ and reaches the top of the negative-energy continuum, $E = -mc^2$, near $Z \approx 172$; beyond that the one-particle theory fails, and an empty $1s$ level would be filled by spontaneous pair creation

The common prediction is that the $1s$ level reaches $-mc^2$ at $Z = 137$. The point-nucleus formula ends at $E = 0$ there; only because a finite nucleus keeps the solution regular can the level go on falling, and it takes about 35 more units of charge to reach $-mc^2$, which is why $Z \approx 172$, not 137, is the threshold for spontaneous positron emission.

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CORE
COURSE

DATA ANALYSIS · DEMANDING · UAQM-02.05 · UAQM-02.06 · UAQM-02.07 · UAQM-02.08

[10]

Identical nuclei, each in its ground state of spin s , scatter in their centre-of-mass frame through the Coulomb interaction alone, which does not act on spin; every run is far enough below the Coulomb barrier that the nuclear force can be neglected. A detector of angular half-width 0.50° centred on $\theta = 90^\circ$ counts N_{90} nuclei, and N_R is the number it would have counted in the same run if $|f(\theta)|^2$ and $|f(\pi - \theta)|^2$ added without interference, computed from the Rutherford cross-section and the measured beam and target. For the exact Coulomb amplitude $f(\theta)$, the interference term is proportional to $\cos[\eta \ln \tan^2(\theta/2)]$, where $\eta = Z^2\alpha/\beta$, βc is the relative speed, $\beta^2 = 2E_{\text{cm}}/(\mu c^2)$ and μ is the reduced mass. Take $\alpha = 1/137.036$ and $1 \text{ u} = 931.5 \text{ MeV}/c^2$, treat N_{90} as a Poisson count and N_R as exact.

(a) SHOW THAT · U2 [2]

The $(2s + 1)^2$ spin states of the pair divide into $(s + 1)(2s + 1)$ that are symmetric under exchange of the two spins and $s(2s + 1)$ that are antisymmetric. Show that for an unpolarised beam and target $d\sigma/d\Omega = |f(\theta)|^2 + |f(\pi - \theta)|^2 + [2(-1)^{(2s)}/(2s + 1)]\text{Re}[f(\theta)f^*(\pi - \theta)]$, and hence that $R = N_{90}/N_R$ should equal $1 + (-1)^{(2s)}/(2s + 1)$.

1 the whole state must be symmetric for integer s and antisymmetric for half-integer s , so for bosons the symmetric spin states, a fraction $(s + 1)/(2s + 1)$, go with $|f + f'|^2$ and the antisymmetric fraction $s/(2s + 1)$ with $|f - f'|^2$, where $f' = f(\pi - \theta)$, and for fermions the two fractions are exchanged; the cross terms $\pm 2\text{Re}(ff^*)$ then combine with weight $[(s + 1) - s]/(2s + 1) = 1/(2s + 1)$ and sign $(-1)^{(2s)}$

2 at $\theta = 90^\circ$, $f(\pi - \theta) = f(\theta)$, phase included, so $\text{Re}(ff^*) = |f|^2$ and $d\sigma/d\Omega = 2|f|^2[1 + (-1)^{(2s)}/(2s + 1)]$, while the non-interfering sum is $2|f|^2$, giving $R = 1 + (-1)^{(2s)}/(2s + 1)$

Candidates who weight the symmetric and antisymmetric spin spaces equally lose the $1/(2s + 1)$. The checks are $s = 1/2$, where the weights must reduce to the $1/4$ and $3/4$ of two electrons, and $s = 0$, where R must be 2.

(b) CALCULATE · U4 [2]

Calculate R and its standard uncertainty for each pair.

1 $R = N_{90}/N_R$: 1.9916 for ^{12}C , 0.5044 for ^{13}C , 1.3384 for ^{14}N , 1.1375 for ^{10}B , 0.7462 for ^{11}B and 0.8371 for ^{17}O

2 $\sigma_R = \sqrt{N_{90}/N_R}$: 0.0200, 0.0079, 0.0122, 0.0097, 0.0086 and 0.0091 respectively

$\sigma_R = \sqrt{N_{90}/N_R} = \sqrt{(R/N_R)}$, so it grows with R and falls with the non-interfering count N_R : ^{12}C , with the largest R and the smallest N_R , has the largest σ_R , while ^{13}C , with the fewest counts N_{90} , has the smallest. Candidates who quote one uncertainty for the whole table, or rank the rows by N_{90} alone, cannot judge the high-spin rows in (c).

(c) DETERMINE · U4 [3]

Determine the spin of each nucleus, and the significance with which the nearest alternative value is excluded. Say which assignments are secure at three standard deviations.

1 $R > 1$ marks bosons and $R < 1$ fermions, and $|R - 1| = 1/(2s + 1)$: ^{12}C gives 1.992 ± 0.020 , consistent with $s = 0$, with $s = 1$ ($R = 1.333$) excluded at 33σ , and so a check on the normalisation of N_R ; ^{13}C gives $2s + 1 = 2.0$, $s = 1/2$, with $s = 3/2$ ($R = 0.750$) excluded at 31σ

2 ^{14}N : $2s + 1 = 2.95$, $s = 1$, with $s = 2$ ($R = 1.200$) excluded at 11σ ; ^{11}B : $2s + 1 = 3.94$, $s = 3/2$, with $s = 5/2$ ($R = 0.833$) excluded at 10σ ; ^{17}O : $2s + 1 = 6.1$, $s = 5/2$, with $s = 7/2$ ($R = 0.875$) excluded at 4.1σ and $s = 3/2$ at 9.5σ

3 ^{10}B : $2s + 1 = 7.3$, so $s = 3$ ($R = 1.1429$, 0.6σ away) is favoured and $s = 2$ is excluded at 6.4σ , but $s = 4$ ($R = 1.1111$) is only 2.7σ away, so this assignment is not secure at 3σ ; separating 3 from 4 at 5σ needs $\sigma_R \approx 0.0318/5 = 0.0064$, about 2.3 times as many counts

The spacing between successive spins, $1/(2s + 1) - 1/(2s + 3)$, falls roughly as $1/(2s^2)$, so a precision that settles $s = 1$ by a wide margin cannot settle $s = 3$. Candidates who assign each spin from the nearest value without testing the alternatives cannot say which assignments the data support.

(d) EVALUATE · U4 [2]

Near 90° , $\ln \tan^2(\theta/2) \approx 2(\theta - 90^\circ)$, with the angle in radians, so the detector averages $\cos[2\eta(\theta - 90^\circ)]$ over its acceptance. Evaluate whether this averaging biases any spin assignment, using the pair with the largest η .

1 the largest η is for $^{17}\text{O} + ^{17}\text{O}$: $\mu c^2 = 8.5 \times 931.5 \text{ MeV} = 7918 \text{ MeV}$, $\beta = \sqrt{8.00/7918} = 0.0318$ and $\eta = 64/(137.036 \times 0.0318) = 14.7$; averaging over $\pm 0.50^\circ = \pm 0.00873$ rad multiplies the interference term by $\sin x/x$ with $x = 2\eta \times 0.00873 = 0.256$, that is by 0.989

2 the expected R for $s = 5/2$ moves from 0.8333 to 0.8352, a shift of 0.0018, that is 0.2σ and about 4% of the 0.042 gap to the $s = 7/2$ value; the fractional loss of interference, $1 - \sin x/x \approx x^2/6$, grows as η^2 , so every other pair is moved by under 2% of the gap to its neighbouring spin and by at most 0.22σ , and the averaging biases no assignment; a detector three times wider would move ^{17}O by 1.7σ and begin to pull the high-spin values towards 1

Candidates often dismiss the acceptance without a number, or test it on the pair with the smallest η . The fractional loss grows as η^2 , and $\eta = Z^2\alpha/\beta$ is dominated by Z^2 , since β hardly differs between these runs, which is why the highest- Z pair is the one to check. Comparing raw shifts in R across pairs misleads: a spin-0 pair has the largest interference term to lose, but also the widest gap to the next spin.

(e) EXPLAIN · U1

[1]

Explain why the measurement is made at 90° rather than at another angle.

- 1 only at 90° does $f(\pi - \theta)$ equal $f(\theta)$ exactly, so the Coulomb phases cancel and the interference term has its full size whatever η and E_{cm} are; at any other angle it carries $\cos[\eta \ln \tan^2(\theta/2)]$, which near 90° oscillates with a period of $180^\circ/\eta$, between 12° and 32° for these pairs, and the spin would be entangled with η

Answers that say 90° is where the cross-section is largest miss the point: for fermions it is a minimum. What matters is that the interference phase vanishes there.

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CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-06.01 · UAQM-06.02 · UAQM-06.05 · UAQM-06.06 · UAQM-06.07 · UAQM-06.08 · UAQM-06.09

[16]

An electron of charge $q = -e$ and mass m moves in static potentials with Dirac Hamiltonian $H = c\alpha \cdot \pi + \beta mc^2 + V(r)$, where $\pi = p - qA$, $p = -i\hbar\nabla$ and V is the electrostatic potential energy. In the Dirac representation write $\psi = (\phi, \chi)$, with ϕ the large and χ the small two-component spinor, and $E = mc^2 + E'$. You may use $(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$ for vector operators a and b that commute with σ , and $[p, f(r)] = -i\hbar\nabla f$. Take $\alpha = 1/137.036$ and $mc^2 = 0.5110$ MeV.

(a) DERIVE · U2

[3]

Derive the two coupled equations for ϕ and χ . For $|E' - V| \ll mc^2$, eliminate χ to leading order and derive the Pauli Hamiltonian, showing that its spin term is $-(q/m)S \cdot B$, so that $g = 2$.

- 1 the blocks of $H\psi = E\psi$ are $(E' - V)\phi = c\sigma \cdot \pi \chi$ and $(E' + 2mc^2 - V)\chi = c\sigma \cdot \pi \phi$, so $\chi = [c/(2mc^2 + E' - V)]\sigma \cdot \pi \phi \approx [\sigma \cdot \pi/(2mc)]\phi$, smaller than ϕ by about $v/(2c)$
- 2 substituting, $E'\phi = [(\sigma \cdot \pi)^2/(2m) + V]\phi$, and $(\sigma \cdot \pi)^2 = \pi^2 + i\sigma \cdot (\pi \times \pi)$, the cross product surviving because the components of π do not commute
- 3 $[\pi_x, \pi_y] = iq\hbar(\partial A_y/\partial x - \partial A_x/\partial y) = iq\hbar B_z$ and cyclically, so $\pi \times \pi = iq\hbar B$ and $(\sigma \cdot \pi)^2 = \pi^2 - q\hbar \sigma \cdot B$; with $S = (\hbar/2)\sigma$, $H_P = \pi^2/(2m) - (q/m)S \cdot B + V$, a moment $(q/m)S = g(q/2m)S$ with $g = 2$, fixed by minimal coupling rather than put in by hand

The error to catch is treating π like p and writing $(\sigma \cdot \pi)^2 = \pi^2$. That is right for a free particle and loses the whole magnetic moment, which lives entirely in the non-commuting components of $p - qA$.

(b) ESTIMATE · U4

[2]

Estimate the ratio $|\chi|/|\phi|$ and the fraction of the probability carried by χ in the ground state of hydrogen, stating your assumptions.

- 1 assuming the typical momentum is $\hbar/a_0 = \alpha mc$, so that $v/c \approx \alpha$, $|\chi|/|\phi| \approx p/(2mc) \approx \alpha/2 = 3.6 \times 10^{-3}$
- 2 the probability in χ is then about $(\alpha/2)^2 = 1.3 \times 10^{-5}$, in agreement with the exact Dirac 1s value $(1 - \sqrt{1 - \alpha^2})/2 = 1.33 \times 10^{-5}$; the norm of ϕ alone therefore falls short of 1 by about 10^{-5} , the same relative order as the fine structure, which is why the next order needs care

Estimates that quote $v/c \approx \alpha$ without saying where it comes from have not stated an assumption, and those that take $|\chi|/|\phi| \approx \alpha$ rather than $\alpha/2$ overestimate the share by a factor of four.

(c) EXPLAIN · U1

[2]

Now set $A = 0$. Keeping the next term in χ gives $E'\phi = [p^2/(2m) + V - (1/(4m^2c^2))(\sigma \cdot p)(E' - V)(\sigma \cdot p)]\phi$. Explain why this cannot be read as a Schrödinger equation for ϕ with a Hermitian Hamiltonian at order $(v/c)^2$.

- 1 the bracket contains the eigenvalue E' , so each state obeys a different operator and the equation is not an eigenvalue problem for one fixed Hamiltonian; removing E' with the lowest-order relation $(E' - V)\phi \approx [p^2/(2m)]\phi$ means commuting $E' - V$ through $\sigma \cdot p$, and the commutator leaves the term $(i\hbar/(4m^2c^2))\nabla V \cdot p$, which is not Hermitian, although $(\sigma \cdot p)V(\sigma \cdot p)$ itself is
- 2 the reason is that probability is shared: Dirac eigenstates satisfy $\int (\phi_n^\dagger \phi_m + \chi_n^\dagger \chi_m) d^3r = \delta_{nm}$, so the large components of different states are neither normalised nor orthogonal, $\int \phi_n^\dagger \phi_n d^3r \approx 1 - \langle p^2 \rangle / (4m^2c^2)$, and no Hermitian operator can have them as its eigenfunctions until ϕ is renormalised

Candidates notice the stray E' and replace it by $p^2/(2m)$, which hides the problem rather than solving it. Calling $(\sigma \cdot p)V(\sigma \cdot p)$ non-Hermitian is wrong, since it is its own adjoint: the non-Hermiticity is a symptom of using a wavefunction that is not normalised, and the cure has to act on ϕ itself.

(d) SHOW THAT · U2

[4]

Show that $\Phi = [1 + p^2/(8m^2c^2)]\phi$ is normalised to order $(v/c)^2$, and that it obeys $E'\Phi = H_{FW}\Phi$ with $H_{FW} = p^2/(2m) + V - p^4/(8m^3c^2) + (\hbar/(4m^2c^2))\sigma \cdot (\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$.

- 1 $\int \chi^\dagger \chi d^3r \approx \int \phi^\dagger [p^2/(4m^2c^2)]\phi d^3r$, so $\int (\phi^\dagger \phi + \chi^\dagger \chi) d^3r = \int \phi^\dagger [1 + p^2/(4m^2c^2)]\phi d^3r = \int \Phi^\dagger \Phi d^3r$ to this order, since $[1 + p^2/(8m^2c^2)]^2 \approx 1 + p^2/(4m^2c^2)$ and p^2 is Hermitian
- 2 substituting $\phi = [1 - p^2/(8m^2c^2)]\Phi$ and multiplying on the left by $1 - p^2/(8m^2c^2)$, the left side becomes $E'[1 - p^2/(4m^2c^2)]\Phi$, whose $E'p^2$ term cancels the $-E'p^2/(4m^2c^2)$ inside $(\sigma \cdot p)(E' - V)(\sigma \cdot p)$, leaving $E'\Phi = [p^2/(2m) + V - p^4/(8m^3c^2) - (1/(8m^2c^2))(p^2V + Vp^2) + (1/(4m^2c^2))(\sigma \cdot p)V(\sigma \cdot p)]\Phi$
- 3 $(\sigma \cdot p)V = V\sigma \cdot p - i\hbar\sigma \cdot \nabla V$, so $(\sigma \cdot p)V(\sigma \cdot p) = Vp^2 - i\hbar\nabla V \cdot p + \hbar\sigma \cdot (\nabla V \times p)$ by the given identity, and $p^2V = Vp^2 - \hbar^2\nabla^2 V - 2i\hbar\nabla V \cdot p$, so $p^2V + Vp^2 = 2Vp^2 - \hbar^2\nabla^2 V - 2i\hbar\nabla V \cdot p$
- 4 combining, the Vp^2 terms cancel and so do the two $\nabla V \cdot p$ terms, neither Hermitian on its own, $(1/(4m^2c^2))(-i\hbar\nabla V \cdot p) - (1/(8m^2c^2))(-2i\hbar\nabla V \cdot p) = 0$, leaving $(\hbar/(4m^2c^2))\sigma \cdot (\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$: H_{FW} , Hermitian, with every term traced to a step

Most attempts fail at the bookkeeping of the second line: the E' inside the bracket must be cancelled by the renormalisation, not replaced by $p^2/(2m)$, and p^2 and V must be kept in the symmetric order the anticommutator gives. Because the result is printed, the last mark needs the two $-i\hbar\nabla V \cdot p$ terms shown cancelling.

(e) DETERMINE · U2

[3]

For a central potential $V(r)$, and then for $V = -Z\alpha\hbar c/r$, determine the spin-orbit term in the form $\xi(r)S \cdot L$ and the Darwin term, and evaluate the Darwin shift of the 1s level of hydrogen in eV, given $|\psi_{100}(0)|^2 = 1/(\pi a_0^3)$ with $a_0 = \hbar/(\alpha mc)$.

- 1 $\nabla V = (dV/dr)r/r$, so $\sigma \cdot (\nabla V \times p) = (1/r)(dV/dr)\sigma \cdot L$, and with $\sigma = 2S/\hbar$ the spin-orbit term is $[1/(2m^2c^2)](1/r)(dV/dr)S \cdot L$
- 2 for $V = -Z\alpha\hbar c/r$, $\nabla^2 V = -Z\alpha\hbar c\nabla^2(1/r) = 4\pi Z\alpha\hbar c\delta^3(r)$, so the Darwin term is $[\pi\hbar^2/(2m^2c^2)]Z\alpha\hbar c\delta^3(r)$, which acts only on s states
- 3 for hydrogen 1s, $(H_D) = [\pi\hbar^2\alpha\hbar c/(2m^2c^2)]/(\pi a_0^3) = \alpha^4 mc^2/2 = 7.25 \times 10^{-4} \text{ eV}$

Leaving the result as $\nabla^2 V$ hides the physics: the Coulomb Laplacian is a delta function, so the Darwin term shifts only s levels, which is how it takes the place of the spin-orbit term at $l = 0$. A sign slip in $\nabla^2(1/r) = -4\pi\delta^3(r)$ gives a Darwin term that lowers the s levels.

In a semiclassical picture the electron, moving with velocity v through the electric field $\mathcal{E} = \nabla V/e$, sees in its rest frame the magnetic field $B' = -v \times \mathcal{E}/c^2$. Compare the spin-orbit coefficient that $-\mu \cdot B'$ gives, with $\mu = -g(e/2m)S$, with the one in (e), account for the difference, and say what changes if $g = 2(1 + a)$.

- 1 $B' = [1/(emc^2)](1/r)(dV/dr)L$, so $-\mu \cdot B' = [g/(2m^2c^2)](1/r)(dV/dr)S \cdot L$, which for $g = 2$ has coefficient $1/(m^2c^2)$, twice the Dirac result
- 2 the rest frame of an accelerated electron rotates relative to the laboratory, the Thomas precession, which adds $-[1/(2m^2c^2)](1/r)(dV/dr)S \cdot L$; the total, $[(g - 1)/(2m^2c^2)](1/r)(dV/dr)S \cdot L$, equals the Dirac result for $g = 2$, where it emerges with no frame argument at all, and becomes $(1 + 2a)/(2m^2c^2)$ for an anomalous moment, since the kinematic part does not depend on g

The factor of two is where answers split: some insert the Thomas half without justification, others declare the rest-frame argument simply wrong. The point is that the reduction, done in the laboratory frame, contains both the magnetic precession and the kinematic one, and only the magnetic one scales with g .

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 20 MARKS · 27%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 36 MARKS · 49%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

7 MARKS · 9%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

11 MARKS · 15%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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