UNIVERSITY · GIOPHYSICS UNIVERSITY PHYSICS · PHYSICS I–V, QUANTUM MECHANICS I–II, ADVANCED QM AND QFT I

University Physics

A midterm and a final for each of the nine University Physics courses, from calculus-based mechanics to quantum field theory, with a mark-by-mark answer to every part and questions graded from entry points up to the derivations that separate the top grade.

PAPERS	QUESTIONS	TOTAL MARKS	FIGURES
18	292	1713	0

EVERY PAPER IN THIS FILE

1 **University Physics I · Midterm Examination** — 13 questions · 73 marks · 90 minutes
Calculus-based mechanics through energy: units, uncertainty and dimensional reasoning; motion in one and two dimensions; force models built from system boundaries and connector constraints; work done by variable forces; and potential-energy landscapes analysed with an explicit system and reference

2 **University Physics I · Final Examination** — 18 questions · 116 marks · 150 minutes
The whole of calculus-based mechanics, weighted to the second half of the course: momentum and collisions, rotational kinematics and dynamics, angular momentum and precession, static equilibrium and elasticity, gravitation and orbits, and oscillations, with fluids as an extension. Kinematics,

3 **University Physics II · Midterm Examination** — 13 questions · 72 marks · 90 minutes
Covers Units 1 through 10: electric forces: charge and Coulomb's law, electric fields and Gauss's law, electric potential, capacitance and dielectrics, DC and multi-loop circuits with RC transients, and the Lorentz force on moving charges. Most marks sit on setting up a field, circuit or force model before

4 **University Physics II · Final Examination** — 19 questions · 117 marks · 150 minutes
The whole of University Physics II, weighted towards the first half, and every answer is expected to name the system, the conservation law and the model it relies on before any algebra. Units 1 through 10: electric forces, fields and Gauss's law, electric potential, capacitance and dielectrics, DC and

5 **University Physics III · Midterm Examination** — 14 questions · 74 marks · 90 minutes
Covers Units 1 to 10: conservation and induction, inductance and AC circuits, Maxwell's equations and electromagnetic waves, geometric and physical optics, and the photon and matter waves. Most marks are for building a model from a named law, stating its conditions, and then using it. Units 1 through

6 **University Physics III · Final Examination** — 19 questions · 118 marks · 150 minutes
The whole of University Physics III, with the units return through Maxwell's equations and light, then weighted towards its second half. Every answer is expected to name the law, state the conditions under which it applies, and check the result against the approximation that made it valid.

7 **University Physics IV · Midterm Examination** — 13 questions · 73 marks · 90 minutes
Covers Units 1 to 10: blackbody radiation, the photoelectric effect, the Bohr model, matter waves and the Schrödinger equation in one dimension, the hydrogen atom, multi-electron atoms and spectroscopy, nuclear structure and decay, fission and fusion, particle classification, the Standard Model,

8 **University Physics IV · Final Examination** — 19 questions · 116 marks · 150 minutes
The whole of University Physics IV, with the units return through blackbody radiation, the photoelectric effect, the Bohr model, matter waves, and the Schrödinger equation solved for the square well and the harmonic oscillator.

9 **University Physics V · Midterm Examination** — 14 questions · 72 marks · 90 minutes
Examines Units 1 to 10: states and Hermitian operators in Dirac notation, the postulates of quantum mechanics, Born's rule and unitary evolution, the box, the finite well and tunnelling, the oscillator's ladder, commutators and the Robertson bound, and angular momentum and spin one-half.

10 **University Physics V · Final Examination** — 19 questions · 120 marks · 150 minutes
The whole of University Physics V, weighted towards its second half: the hydrogen atom, multi-electron atoms and spectroscopy, nuclear structure and decay, fission and fusion, particle classification, the Standard Model,

11 **University Quantum Mechanics I · Midterm Examination** — 14 questions · 74 marks · 90 minutes
The first eight units on one paper: the experiments that forced a wavefunction, conservation-law audits and quantum information, with the state-space, spin and uncertainty, the free particle, the infinite and finite square wells, and tunnelling at steps and barriers. Every probability is an integral of $|\psi|^2$, and

12 **University Quantum Mechanics I · Final Examination** — 19 questions · 118 marks · 150 minutes
The whole of Quantum Mechanics I, weighted towards the second half of the course: the harmonic oscillator by series and by ladder operators, Dirac notation and two-state systems, angular momentum and spin-½, and the

13 **University Quantum Mechanics II · Midterm Examination** — 14 questions · 74 marks · 90 minutes
The first eight units of Quantum Mechanics II on one paper: the spherical Schrödinger equation in three dimensions through to hydrogen. The harmonics and the hydrogen radial functions derived rather than quoted, spin and charge in magnetic fields, composite systems and density operators,

14 **University Quantum Mechanics II · Final Examination** — 19 questions · 118 marks · 150 minutes
The whole of Quantum Mechanics II, weighted towards the second half of the course: external fields and hyperfine structure, the variational principle, WKB, time-dependent perturbation theory and radiation, Berry's phase, and

15 **University Advanced Quantum Mechanics · Midterm Examination** — 14 questions · 74 marks · 90 minutes
The first eight units of Advanced Quantum Mechanics on one paper: the interaction picture and the Dyson series, Wick's theorem with Feynman diagrams, the Lamb shift, open systems and the density matrix, and the Thomas–Fermi model. Every answer is expected to name the

16 **University Advanced Quantum Mechanics · Final Examination** — 18 questions · 114 marks · 150 minutes
The whole of Advanced Quantum Mechanics, weighted towards the second half: quantised fields and spin-statistics, second quantisation, the radiation field and spontaneous emission, Bethe's Lamb shift, open systems and the density matrix, and the Thomas–Fermi model. Every answer is expected to

17 **University Quantum Field Theory I · Midterm Examination** — 14 questions · 73 marks · 90 minutes
The first eight units of Quantum Field Theory I on one paper: the Klein–Gordon equation and its exact Coulomb levels, Dirac's free spinors and the hydrogen spectrum, the Lagrangian route to the Dirac equation, and the discrete symmetries C, P and T. Every answer is expected to name the

18 **University Quantum Field Theory I · Final Examination** — 19 questions · 117 marks · 150 minutes
The whole of Quantum Field Theory I, with the units return through the Klein–Gordon equation, Dirac's free spinors and the hydrogen spectrum, the Lagrangian route to the Dirac equation, and the discrete symmetries C, P and T. Every answer is expected to name the

ASSESSMENT

Each paper is examined by: a midterm on the first half of the course, with a mark-by-mark answer to every part and questions graded from entry points up to the derivations that separate the top grade; and a final on the whole course, weighted towards the second half. Every paper combines short multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them, and your own formula sheet is assumed.

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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> GioPhysics University Physics course maps — https://giophysics.com/university-physics/

University Physics I · Midterm Examination

Calculus-based mechanics through energy: units, uncertainty and dimensional reasoning; motion in one and two dimensions; force models built from system boundaries and connector constraints; work done by variable forces; and potential-energy landscapes analysed with an explicit system and reference level. Every written question is set so that the model has to be built before anything can be substituted.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**73**	**13**	**8**

SECTION	**UPI · Midterm · University Physics I — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Recall 1 · Routine 5 · Demanding 6 · Discriminating 1** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UPI-01 · UPI-02 · UPI-03 · UPI-04 · UPI-05 · UPI-06 · UPI-07

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Physics I · Midterm Examination

Answer all questions. 73 marks in 90 minutes.

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1 [1]

CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.02

A heavy, flexible rope of length L hangs freely from a hook. A small sideways flick at its lower end sends a pulse up the rope. A student proposes that the time t for the pulse to reach the hook depends only on L, the gravitational acceleration g (in m/s²) and the rope's mass per unit length μ (in kg/m), in the form t = C L^a g^b μ^c, where C is a dimensionless constant. Which exponents does dimensional analysis require?

A a = ½, b = −½, c = 0

B a = ½, b = −½, c = ½

C a = −½, b = ½, c = 0

D a = ½, b = ½, c = 0

Answer ☐

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2 [1]

CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.03

To find the average speed of a trolley, a student measures the distance between two marks on a bench as (1.250 ± 0.005) m and times the trolley between the marks as (0.50 ± 0.02) s. Which result is reported correctly?

A (2.500 ± 0.025) m/s

B (2.50 ± 0.25) m/s

C (2.50 ± 0.01) m/s

D (2.5 ± 0.1) m/s

Answer ☐

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3 [1]

CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-02.01 · UPI-02.07

A cart moves along a straight track with velocity v(t) = (6.0 − 2.0t) m/s, where t is in seconds, for 0 ≤ t ≤ 5.0 s. What distance does the cart travel in this interval?

A 5.0 m

B 9.0 m

C 13 m

D 30 m

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-03.03 **[1]**

A survey drone flies at 12.0 m/s relative to the air. A steady wind blows towards the east at 5.0 m/s relative to the ground. The pilot points the drone so that its path over the ground runs due north. What is the drone's speed relative to the ground?

A 7.0 m/s

B 10.9 m/s

C 13.0 m/s

D 17.0 m/s

Answer

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UPI-04.04 **[1]**

A small magnet is held motionless on a vertical steel door. The door attracts it horizontally, and friction stops it sliding down. Which force is the Newton's-third-law partner of the upward friction force that the door exerts on the magnet?

A The weight of the magnet: the downward gravitational force of the Earth on the magnet

B The normal force that the door exerts on the magnet

C The downward friction force that the magnet exerts on the door

D The magnetic attraction that the door exerts on the magnet

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-05.04 **[1]**

A 62.0 kg passenger stands on bathroom scales in a lift. The lift is moving downwards and slowing down at 1.40 m/s². What do the scales read? Take g = 9.81 m/s².

A 695 N

B 608 N

C 521 N

D 86.8 N

Answer

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-03.05 · UPI-05.05 **[1]**

A child on a playground swing is at the highest point of her swing, momentarily at rest, with the taut ropes at an angle to the vertical. Treat the child as a particle on a light rope of fixed length. What is the direction of her acceleration at that instant?

A She has no acceleration, because her velocity is zero.

B Vertically downwards, with magnitude g.

C Along the rope, towards the pivot.

D Perpendicular to the rope, along the arc towards the lowest point.

Answer

8

CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-07.05 · UPI-07.06 **[1]**

A particle moves along the x-axis under a single conservative force. Its potential energy is $U(x) = (3.0\ \text{J/m}^2)x^2 - (1.0\ \text{J/m}^3)x^3$, which has a minimum $U = 0$ at $x = 0$ and a maximum $U = 4.0$ J at $x = 2.0$ m. The particle is released from rest at $x = -0.80$ m. Which statement describes its subsequent motion?

A It oscillates between $x = -0.80$ m and $x = +0.80$ m, because turning points lie equal distances either side of a minimum.

B It oscillates between $x = -0.80$ m and a turning point that lies between $x = +0.80$ m and $x = +2.0$ m.

C It passes $x = +2.0$ m and moves off towards large positive x.

D It moves to $x = 0$, where U is least, and remains at rest there.

Answer

9 CORE COURSE

STRUCTURED · DEMANDING · UPI-03.02 · UPI-03.07 · UPI-03.08 **[12]**

A campus fountain has its nozzle at ground level. The designer wants each jet of water to reach the highest point of its path exactly at the centre of a decorative ring, which is H = 1.80 m above the nozzle and d = 3.00 m from it horizontally. A low wall 0.60 m high stands 5.40 m horizontally from the nozzle, in the vertical plane of the jet, on the far side of the ring. Treat each parcel of water as a projectile launched from the nozzle at speed v_0 and angle θ above the horizontal, with no air resistance. Take g = 9.81 m/s².

(a) SHOW THAT **[3]**

Show that the launch angle must satisfy tan θ = 2H/d.

(b) CALCULATE **[3]**

Calculate the launch angle θ and the launch speed v_0.

(c) EXPLAIN **[2]**

Explain why the launch angle in (a) does not depend on g, whereas the launch speed in (b) does.

(d) DETERMINE **[2]**

Determine the height of the jet as it passes the position of the wall, and state whether the jet clears the wall.

(e) PREDICT **[2]**

In service, a partly blocked filter reduces the launch speed by 2.0 % while the launch angle stays the same. Predict whether the jet still clears the wall, supporting your prediction with a calculation.

10 CORE COURSE

STRUCTURED · DEMANDING · UPI-04.05 · UPI-04.07 · UPI-04.08 · UPI-05.02 · UPI-05.03 **[13]**

Block A, of mass $m_A = 4.00$ kg, rests on a horizontal bench. A light inextensible cord is tied to A and runs horizontally to a light, frictionless pulley fixed at the edge of the bench. From that pulley the cord hangs vertically downwards, passes under a second light, frictionless pulley that is free to move, and rises vertically to a hook fixed to the underside of the bench top, so the two vertical sections of cord are parallel. Load B, of mass $m_B = 6.00$ kg, hangs from the axle of the movable pulley. The coefficients of friction between A and the bench are $\mu_s = 0.50$ (static) and $\mu_k = 0.35$ (kinetic). The system is released from rest. Take $g = 9.81\ \text{m/s}^2$.

(a) EXPLAIN **[2]**

Explain why the magnitude of B's acceleration is half the magnitude of A's acceleration.

(b) DETERMINE **[3]**

Determine the smallest mass of B that would set A in motion from rest, and hence confirm that A moves when the 6.00 kg load is used.

(c) DETERMINE **[5]**

Determine the acceleration of A, the acceleration of B and the tension in the cord while the system is moving.

(d) EVALUATE **[3]**

A student treats A and B together as a single system and writes $m_B g - \mu_k m_A g = (m_A + m_B)a$. Evaluate this equation as a model of the motion.

11 CORE COURSE

STRUCTURED · DEMANDING · UPI-06.02 · UPI-06.03 · UPI-07.01 · UPI-07.02 · UPI-07.07 **[11]**

A loaded truck of mass $m = 1.80 \times 10^4$ kg loses its brakes on a mountain road and enters an emergency arrester bed at $v_0 = 25.0$ m/s. The bed is a straight lane of loose gravel that rises uphill at a constant angle $\theta = 5.0°$ to the horizontal. The gravel gets deeper along the bed, and it exerts a resistive force on the truck, parallel to the slope and opposite to the motion, of magnitude $F = \mu(s)mg\cos\theta$, where s is the distance travelled along the bed and $\mu(s) = \mu_0 + \beta s$, with $\mu_0 = 0.20$ and $\beta = 5.0 \times 10^{-3}$ m^{-1}. Treat the truck as a particle and ignore air resistance. Take $g = 9.81$ m/s^2.

(a) DERIVE **[3]**

Starting from the definition of work, derive an expression for the work done on the truck by the gravel as the truck travels a distance s along the bed.

(b) SHOW THAT **[2]**

Show that the stopping distance s satisfies $\tfrac{1}{2}v_0^2 = gs\sin\theta + g\cos\theta(\mu_0 s + \tfrac{1}{2}\beta s^2)$.

(c) CALCULATE **[2]**

Calculate the stopping distance.

(d) DETERMINE **[2]**

Determine the percentage of the truck's initial kinetic energy that is stored as gravitational potential energy and the percentage that is dissipated in the gravel.

(e) EVALUATE **[2]**

A highway engineer argues that a heavier truck needs a longer arrester bed. Evaluate this argument, first within the model and then for a real bed.

12 CORE COURSE

DATA ANALYSIS · DEMANDING · UPI-01.08 · UPI-01.09 · UPI-05.06 · UPI-06.06 **[12]**

A cyclist rides a bicycle fitted with a power meter along a level road on a windless day. At each of six steady speeds v she records the power P that she delivers to the rear wheel. A model of the total resistance on the rider and bicycle is $F = F_r + kv^2$, where F_r is a constant rolling resistance and kv^2 is aerodynamic drag, with k constant. Take $g = 9.81$ m/s².

Steady-speed readings on a level road

Trial	Speed v (m/s)	Power P (W)
1	4.0	29
2	6.0	68
3	8.0	133
4	10.0	243
5	12.0	390
6	14.0	608

(a) EXPLAIN **[2]**

Explain why, at a steady speed on a level road, the model predicts that $P = (F_r + kv^2)v$.

(b) DETERMINE **[4]**

By processing the data so that the model predicts a straight line, determine F_r and k.

(c) COMPARE **[2]**

A second student proposes that rolling resistance is negligible, so that $P = kv^3$. Compare how well this model and the two-term model account for the data.

(d) ESTIMATE **[2]**

Estimate the coefficient of rolling resistance $C_r = F_r/N$, where N is the normal force of the road on the bicycle and rider, stating the assumption you make.

(e) DESCRIBE **[2]**

Describe two changes to the procedure that would reduce the uncertainty in the value of F_r.

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13 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-01.07 · UPI-07.03 · UPI-07.04 · UPI-07.05 · UPI-07.06 **[17]**

A snap-action switch is modelled as a slider of mass m that moves without friction along a straight horizontal rail, taken as the x-axis. A light spring of stiffness k and natural length L_0 joins the slider to a fixed pivot at perpendicular distance a from the rail, and the foot of that perpendicular is at x = 0. The pivot is closer to the rail than the spring's natural length, so $a < L_0$. Take the elastic potential energy of the spring to be zero at its natural length. Gravity and the normal force of the rail are perpendicular to the rail and do no work on the slider.

(a) DERIVE **[2]**

Starting from the elastic potential energy of a spring, $\frac{1}{2}k \times (\text{extension})^2$, derive an expression for the potential energy U(x) of the system when the slider is at position x.

(b) DERIVE **[3]**

Using the relation between a conservative force and its potential-energy function, derive an expression for the component F_x(x) of the force on the slider along the rail.

(c) DETERMINE **[2]**

Determine all the equilibrium positions of the slider.

(d) JUSTIFY **[2]**

Justify, for each equilibrium position, whether it is stable or unstable.

(e) SKETCH **[3]**

Sketch U(x) against x over a range that includes all the equilibrium positions, marking each position and the value of U there.

(f) CALCULATE **[3]**

For a switch with k = 250 N/m, L_0 = 50.0 mm, a = 40.0 mm and m = 20.0 g, calculate the positions of the two stable states and the minimum speed the slider must have as it passes through one stable state if it is to reach the other.

(g) PREDICT **[2]**

The pivot is moved further from the rail until $a \geq L_0$. Predict how the switch now behaves, reasoning from the energy landscape.

MARK SCHEME

University Physics I · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.02 **[1]**

A heavy, flexible rope of length L hangs freely from a hook. A small sideways flick at its lower end sends a pulse up the rope. A student proposes that the time t for the pulse to reach the hook depends only on L, the gravitational acceleration g (in m/s²) and the rope's mass per unit length μ (in kg/m), in the form t = C L^a g^b μ^c, where C is a dimensionless constant. Which exponents does dimensional analysis require?

KEY **A** **a = ½, b = −½, c = 0**

Only μ contains the kilogram, so nothing else can cancel it and c must be 0; matching seconds then needs b = −½, and matching metres needs a = ½, giving t = C√(L/g). B is tempting because a heavier rope feels as though it ought to change the time, but a non-zero c leaves kg^½ uncancelled — and physically the tension at each height grows in proportion to μ, so the mass per unit length drops out of the pulse speed. C is what a candidate obtains by writing the dimensions of g as m·s² instead of m·s⁻²: matching seconds then gives b = ½ and matching metres gives a = −½, and the resulting √(g/L) has the dimensions of a frequency, s⁻¹, not of a time. D is √(gL), which has the dimensions of a speed: it is what a candidate writes who remembers that wave speeds on hanging ropes look like √(gL) and forgets that the question asks for a time.

U2

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.03 **[1]**

To find the average speed of a trolley, a student measures the distance between two marks on a bench as (1.250 ± 0.005) m and times the trolley between the marks as (0.50 ± 0.02) s. Which result is reported correctly?

KEY **D** **(2.5 ± 0.1) m/s**

For a quotient the fractional uncertainties combine: 0.005/1.250 is 0.4 % and 0.02/0.50 is 4 %, so the speed carries about 4 % of 2.5 m/s, which is ±0.1 m/s whether the two are added (0.11 m/s) or combined in quadrature (0.10 m/s); with a one-figure uncertainty the value is quoted to the same decimal place, 2.5 m/s. A adds the two absolute uncertainties, 0.005 m and 0.02 s, which have different units and cannot be added at all. B divides one uncertainty by the other, as though the uncertainty of a ratio were the ratio of the uncertainties. C propagates only the distance's 0.4 %, the tempting choice because the distance is written to more decimal places — but the timing uncertainty is ten times larger as a fraction and dominates the result.

U4

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-02.01 · UPI-02.07 **[1]**

A cart moves along a straight track with velocity v(t) = (6.0 − 2.0t) m/s, where t is in seconds, for 0 ≤ t ≤ 5.0 s. What distance does the cart travel in this interval?

KEY **C** **13 m**

The velocity changes sign at t = 3.0 s, so the distance is the integral of the speed |v|: ∫₀³ (6.0 − 2.0t) dt = 9.0 m forwards, and ∫₃⁵ (6.0 − 2.0t) dt = −4.0 m, which is 4.0 m backwards, for a total of 13 m. A is the displacement, ∫₀⁵ v dt = 5.0 m, in which the return leg cancels part of the outward one. B counts only the outward leg — the answer of a candidate who finds the turning point and stops there. D multiplies the initial speed by the whole interval, as though the velocity were constant.

U3

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-03.03 **[1]**

A survey drone flies at 12.0 m/s relative to the air. A steady wind blows towards the east at 5.0 m/s relative to the ground. The pilot points the drone so that its path over the ground runs due north. What is the drone's speed relative to the ground?

KEY **B** **10.9 m/s**

The ground velocity is the vector sum of the velocity relative to the air and the wind velocity. To cancel the eastward wind the drone must head partly west, so its 12.0 m/s airspeed is the hypotenuse of a right triangle whose east–west leg is 5.0 m/s, and the northward ground speed is $\sqrt{(12.0^2 - 5.0^2)} = 10.9$ m/s. C treats 12.0 m/s as the northward leg and adds the wind at right angles, which is the ground speed if the drone points due north and is allowed to drift east — not the path the pilot is holding. A and D subtract and add the speeds as though the wind blew along the line of flight.

U3

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UPI-04.04 **[1]**

A small magnet is held motionless on a vertical steel door. The door attracts it horizontally, and friction stops it sliding down. Which force is the Newton's-third-law partner of the upward friction force that the door exerts on the magnet?

KEY **C** **The downward friction force that the magnet exerts on the door**

A third-law pair is one interaction seen from each side: the same type of force, acting on the two different objects involved. The door's friction on the magnet therefore partners the magnet's friction on the door, which points down. A is the tempting answer because the weight is equal and opposite to the friction force here — but both of those forces act on the magnet, and they balance because the magnet is in equilibrium, which is the first law at work, not the third; the weight's own partner is the magnet's gravitational pull on the Earth. B is tempting because the normal force and the friction are the two components of the single contact force the door exerts, so they feel like a matched pair; but both act on the magnet and they are perpendicular, not opposite. D is tempting because the magnetic attraction is what presses the magnet against the door and so makes the friction possible; but it too acts on the magnet, is horizontal, and belongs to the magnetic interaction, whose partner is the magnet's pull on the door.

U1

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-05.04 **[1]**

A 62.0 kg passenger stands on bathroom scales in a lift. The lift is moving downwards and slowing down at 1.40 m/s². What do the scales read? Take g = 9.81 m/s².

KEY **A** **695 N**

Slowing down while moving downwards means the acceleration points upwards. Taking up as positive, $N - mg = ma$, so $N = m(g + a) = 62.0 \times (9.81 + 1.40) = 695$ N. C is $m(g - a)$, the reading for a lift that is speeding up on the way down — the answer of a candidate who takes the direction of the acceleration from the direction of motion. B is the reading at rest or at constant velocity, and D is ma alone, as though the scales measured only the net force.

U3

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-03.05 · UPI-05.05 **[1]**

A child on a playground swing is at the highest point of her swing, momentarily at rest, with the taut ropes at an angle to the vertical. Treat the child as a particle on a light rope of fixed length. What is the direction of her acceleration at that instant?

KEY **D** **Perpendicular to the rope, along the arc towards the lowest point.**

The radial component of acceleration is v^2/r, which is zero when $v = 0$, so the tension exactly balances the component of the weight along the rope, $T = mg\cos\theta$. What is left is the component $mg\sin\theta$ perpendicular to the rope, so the acceleration is $g\sin\theta$ along the tangent to the arc, towards the bottom of the swing. A confuses zero velocity with zero acceleration — she is at rest for an instant only. B would be right if the rope went slack; with the rope taut the tension is not zero, so gravity is not the only force. C is the reflex that circular motion means acceleration towards the centre, which holds only when the speed is not changing — here the speed is changing and the radial part has vanished.

U1

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-07.05 · UPI-07.06 **[1]**

A particle moves along the x-axis under a single conservative force. Its potential energy is $U(x) = (3.0\ \text{J/m}^2)x^2 - (1.0\ \text{J/m}^3)x^3$, which has a minimum $U = 0$ at $x = 0$ and a maximum $U = 4.0$ J at $x = 2.0$ m. The particle is released from rest at $x = -0.80$ m. Which statement describes its subsequent motion?

KEY **B** **It oscillates between $x = -0.80$ m and a turning point that lies between $x = +0.80$ m and $x = +2.0$ m.**

The total energy is fixed at $U(-0.80\ \text{m}) = 1.92 + 0.51 = 2.43$ J. At $x = +0.80$ m, $U = 1.92 - 0.51 = 1.41$ J, which is less than 2.43 J, so the particle still has kinetic energy there and has not reached a turning point; and because 2.43 J is below the 4.0 J maximum it turns back before $x = 2.0$ m (at about $x = 1.15$ m). A assumes the well is symmetric, but the x^3 term makes it shallower on the positive side. C tempts a candidate who sees the particle cross $x = 0$ with all 2.43 J as kinetic energy and supposes nothing can stop it, but U rises again beyond $x = 0$, and escaping past $x = 2.0$ m would need a total energy greater than 4.0 J. D confuses the position of least potential energy, where the speed is greatest, with a place where the particle stops; with no dissipation it passes through $x = 0$ at its maximum speed.

U1

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9 CORE COURSE

STRUCTURED · DEMANDING · UPI-03.02 · UPI-03.07 · UPI-03.08 **[12]**

A campus fountain has its nozzle at ground level. The designer wants each jet of water to reach the highest point of its path exactly at the centre of a decorative ring, which is $H = 1.80$ m above the nozzle and $d = 3.00$ m from it horizontally. A low wall 0.60 m high stands 5.40 m horizontally from the nozzle, in the vertical plane of the jet, on the far side of the ring. Treat each parcel of water as a projectile launched from the nozzle at speed v_0 and angle θ above the horizontal, with no air resistance. Take $g = 9.81\ \text{m/s}^2$.

(a) SHOW THAT · U2 **[3]**

Show that the launch angle must satisfy $\tan\theta = 2H/d$.

1. At the highest point the vertical velocity is zero, so the time to reach it is $t_1 = v_0\sin\theta/g$ and the height is $H = (v_0\sin\theta)^2/(2g)$, equivalently $H = \tfrac{1}{2}gt_1^2$
2. The horizontal velocity is constant, so $d = (v_0\cos\theta)t_1$
3. $\tan\theta = (v_0\sin\theta)/(v_0\cos\theta) = gt_1/(d/t_1) = gt_1^2/d = 2H/d$

   Candidates who aim the nozzle straight at the ring write $\tan\theta = H/d$, treating the jet as though it travelled in a straight line. The factor of 2 is the whole content of the result, which is why it is given: the part rewards the route, not the answer.

(b) CALCULATE · U3 **[3]**

Calculate the launch angle θ and the launch speed v_0.

1 $\tan\theta = 2 \times 1.80/3.00 = 1.20$, so $\theta = 50.2°$

2 Vertical component $v_0 \sin\theta = \sqrt{(2gH)} = \sqrt{(2 \times 9.81 \times 1.80)} = 5.94$ m/s; time to the top $t_1 = 5.94/9.81 = 0.606$ s, so horizontal component $v_0 \cos\theta = 3.00/0.606 = 4.95$ m/s

3 $v_0 = \sqrt{(5.94^2 + 4.95^2)} = 7.74$ m/s

The usual slip is to obtain 5.94 m/s from $\sqrt{(2gH)}$ and report it as the launch speed. It is only the vertical component; the horizontal component is fixed separately by the requirement that the top of the path lies 3.00 m away.

(c) EXPLAIN · U1 **[2]**

Explain why the launch angle in (a) does not depend on g, whereas the launch speed in (b) does.

1 The path of a projectile is a parabola, and a parabola that starts at the nozzle and has its vertex at the point (d, H) is fixed by those two points alone, so its initial slope $\tan\theta = 2H/d$ is set by geometry

2 g sets only how quickly the water travels along that parabola: a larger g needs larger velocity components for the same path ($v_0 \sin\theta = \sqrt{(2gH)}$, so $v_0 \propto \sqrt{g}$), leaving the direction unchanged

The part catches candidates who offer 'g cancels in the algebra' as an explanation. The credit is for saying what g does and does not control — the time scale of the motion, not the shape of the path.

(d) DETERMINE · U3 **[2]**

Determine the height of the jet as it passes the position of the wall, and state whether the jet clears the wall.

1 Using the symmetry of the parabola about $x = d$, $y = H[1 - ((x - d)/d)^2] = 1.80 \times [1 - (2.40/3.00)^2] = 0.648$ m (or, with unrounded components, $t = 5.40/4.952 = 1.090$ s and $y = 5.943t - 4.905t^2 = 0.648$ m; accept 0.64 m to 0.65 m from rounded intermediate values)

2 0.648 m is greater than 0.60 m, so the jet clears the wall, but only by about 0.05 m (4.8 cm)

Candidates who note that the jet lands at $2d = 6.00$ m and conclude that anything nearer is safe have ignored the height of the wall. The size of the margin, not the yes or no, is what part (e) depends on.

(e) PREDICT · U3 **[2]**

In service, a partly blocked filter reduces the launch speed by 2.0 % while the launch angle stays the same. Predict whether the jet still clears the wall, supporting your prediction with a calculation.

1 The trajectory is $y = x\tan\theta - gx^2/(2v_0^2\cos^2\theta)$; at $x = 5.40$ m the first term is 6.48 m and the gravitational drop term is 5.83 m, which scales as $1/v_0^2$ and so grows by a factor $1/0.98^2 = 1.041$ to 6.07 m

2 $y = 6.48 - 6.07 = 0.41$ m, below the 0.60 m wall, so the jet now strikes the wall: a 2 % loss of speed lowers the jet at the wall by about 0.24 m

Most candidates expect a 2 % change in speed to change the height by about 2 %. At the wall the drop term nearly cancels the rise term, so their small difference is extremely sensitive to v_0 — the design weakness the question is set to expose.

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10 CORE COURSE

STRUCTURED · DEMANDING · UPI-04.05 · UPI-04.07 · UPI-04.08 · UPI-05.02 · UPI-05.03 **[13]**

Block A, of mass $m_A = 4.00$ kg, rests on a horizontal bench. A light inextensible cord is tied to A and runs horizontally to a light, frictionless pulley fixed at the edge of the bench. From that pulley the cord hangs vertically downwards, passes under a second light, frictionless pulley that is free to move, and rises vertically to a hook fixed to the underside of the bench top, so the two vertical sections of cord are parallel. Load B, of mass $m_B = 6.00$ kg, hangs from the axle of the movable pulley. The coefficients of friction between A and the bench are $\mu_s = 0.50$ (static) and $\mu_k = 0.35$ (kinetic). The system is released from rest. Take $g = 9.81$ m/s².

(a) EXPLAIN · U1 **[2]**

Explain why the magnitude of B's acceleration is half the magnitude of A's acceleration.

1. If A moves a distance s towards the edge, the horizontal section of cord shortens by s, so, because the cord is inextensible, the combined length of the two vertical sections increases by s
2. That extra length is shared equally by the two vertical sections that support the movable pulley, so B descends s/2; differentiating twice with respect to time gives $a_B = a_A/2$

   Candidates who assume that everything on one cord shares one acceleration are applying a rule that holds only for a cord over fixed pulleys. The relation has to be derived from the length of the cord, and a candidate who skips that step carries the wrong acceleration into every later part.

(b) DETERMINE · U2 **[3]**

Determine the smallest mass of B that would set A in motion from rest, and hence confirm that A moves when the 6.00 kg load is used.

1. With the system at rest, the light movable pulley is in equilibrium under two upward tensions and B's weight, so $2T = m_B g$
2. A stays at rest only if the friction it needs, which equals T, does not exceed $\mu_s m_A g = 0.50 \times 4.00 \times 9.81 = 19.6$ N
3. Motion starts if $m_B g/2 > \mu_s m_A g$, that is $m_B > 2\mu_s m_A = 4.00$ kg; 6.00 kg exceeds this (it would need $T = 29.4$ N), so A moves

   The common error is to compare B's whole weight, 58.9 N, with the maximum static friction. The movable pulley halves the pull the cord exerts on A, and a candidate who misses it predicts motion for any load above 2.00 kg.

(c) DETERMINE · U3 **[5]**

Determine the acceleration of A, the acceleration of B and the tension in the cord while the system is moving.

1. For A, horizontally: $T - \mu_k m_A g = m_A a_A$, where $\mu_k m_A g = 0.35 \times 4.00 \times 9.81 = 13.7$ N
2. For B together with the movable pulley, vertically: $m_B g - 2T = m_B a_B$, with $a_B = a_A/2$
3. Eliminating T: $a_A = g(m_B - 2\mu_k m_A)/(2m_A + m_B/2) = 9.81 \times 3.20/11.00 = 2.85$ m/s²
4. $a_B = a_A/2 = 1.43$ m/s², downwards
5. $T = \mu_k m_A g + m_A a_A = 13.7$ N + 11.4 N = 25.1 N (check on B with unrounded values: 58.86 N − 2 × 25.15 N = 8.56 N, and 6.00 kg × 1.427 m/s² = 8.56 N)

   Two errors dominate: writing $m_B g - T = m_B a$ for B, which leaves out one of the two sections of cord holding the pulley up, and giving both bodies the same acceleration. Each produces a plausible-looking number, which is why substituting back into B's equation is worth doing.

(d) EVALUATE · U1 **[3]**

A student treats A and B together as a single system and writes m_B g − μ_k m_A g = (m_A + m_B)a. Evaluate this equation as a model of the motion.

1. The equation gives both bodies one acceleration a, but a_B = a_A/2, and A accelerates horizontally while B accelerates vertically, so (m_A + m_B)a is not the rate of change of the system's momentum
2. It treats the tension as internal and lets it cancel, but the hook and the fixed pulley lie outside the system and exert forces on the cord, so external forces are missing from the left-hand side (equally creditworthy: the cord pulls A with T but supports B with 2T, so the tension forces do not cancel)
3. It predicts a = 9.81 × (6.00 − 1.40)/10.00 = 4.51 m/s², far above the correct 2.85 m/s², so the equation is not a valid model of this arrangement

A system boundary is a free choice, but Newton's second law for a system then needs every external force on it and the acceleration of every part. Candidates who defend the equation because 'tension is internal' have not asked what the hook is doing.

11

CORE COURSE

STRUCTURED · DEMANDING · UPI-06.02 · UPI-06.03 · UPI-07.01 · UPI-07.02 · UPI-07.07 **[11]**

A loaded truck of mass m = 1.80 × 10⁴ kg loses its brakes on a mountain road and enters an emergency arrester bed at v_0 = 25.0 m/s. The bed is a straight lane of loose gravel that rises uphill at a constant angle θ = 5.0° to the horizontal. The gravel gets deeper along the bed, and it exerts a resistive force on the truck, parallel to the slope and opposite to the motion, of magnitude F = μ(s)mg cos θ, where s is the distance travelled along the bed and μ(s) = μ_0 + βs, with μ_0 = 0.20 and β = 5.0 × 10⁻³ m⁻¹. Treat the truck as a particle and ignore air resistance. Take g = 9.81 m/s².

(a) DERIVE · U2 **[3]**

Starting from the definition of work, derive an expression for the work done on the truck by the gravel as the truck travels a distance s along the bed.

1. The force varies with position, so W = $\int_0^s$ F·ds′ with the force antiparallel to the displacement: W = −$\int_0^s$ (μ_0 + βs′)mg cos θ ds′
2. Integrating term by term: W = −mg cos θ[μ_0s′ + ½βs′²] evaluated from 0 to s
3. W_gravel = −mg cos θ(μ_0s + ½βs²)

Candidates who multiply the force at the entry point, or at the far end, by the distance have used the constant-force rule on a force that nearly triples along the bed. Setting up the integral, with its sign, is the point of the part.

(b) SHOW THAT · U2 **[2]**

Show that the stopping distance s satisfies ½$v_0$² = gs sin θ + g cos θ(μ_0s + ½βs²).

1. Gravity does work −mgs sin θ as the truck rises a height s sin θ, and the normal force does no work because it is perpendicular to the displacement
2. The work–kinetic energy theorem, W_net = 0 − ½m$v_0$², gives −mgs sin θ − mg cos θ(μ_0s + ½βs²) = −½m$v_0$², and dividing by m gives the stated result

Two errors recur: adding work by the normal force to the net work, and giving gravity's work the wrong sign so that it adds kinetic energy on an uphill slope. The stated result exposes both, since neither version reproduces it.

(c) CALCULATE · U3 **[2]**

Calculate the stopping distance.

1. With g sin θ = 0.855 m/s² and g cos θ = 9.77 m/s², the equation becomes 0.0244s² + 2.81s − 312.5 = 0, with s in metres
2. The physical, positive root is s = 69.4 m

The quadratic also has a root at −184 m, which candidates sometimes report as 184 m. It has no meaning here: s is measured forwards from the entry point, and the model only describes the truck while it is moving forwards.

(d) DETERMINE · U3 **[2]**

Determine the percentage of the truck's initial kinetic energy that is stored as gravitational potential energy and the percentage that is dissipated in the gravel.

1. Gravitational potential energy gained = $mgs\sin\theta = 1.80 \times 10^4 \times 0.855 \times 69.4 = 1.07 \times 10^6$ J, which is 19.0 % of the initial kinetic energy $\tfrac{1}{2}mv_0^2 = 5.63 \times 10^6$ J
2. The remaining 4.56×10^6 J, 81.0 %, is dissipated by the gravel's nonconservative force as thermal energy and deformation of the bed

   The part separates the conservative interaction, whose work is stored and could be returned, from the dissipative one. Candidates who say that all the kinetic energy is dissipated have treated the whole slope as friction.

(e) EVALUATE · U1 **[2]**

A highway engineer argues that a heavier truck needs a longer arrester bed. Evaluate this argument, first within the model and then for a real bed.

1. Within the model both forces along the slope, the gravel's resistance and the weight component, are proportional to m, so m cancels from the work–energy equation and the stopping distance is independent of mass: the model does not support the argument
2. A real bed has effects that do not scale with m — air resistance, set by the truck's size and speed, slows a heavier truck less and would lengthen its stop, while a heavier truck sinks deeper and meets a larger μ, which would shorten it — so the argument could hold only through such an unmodelled effect, and it must be tested rather than assumed

   Candidates tend either to accept the argument on intuition ('more momentum to stop') or to reject it outright because 'mass cancels'. The credit goes to the answer that sees the cancellation as a consequence of the model's assumptions and names what would break it.

12

CORE COURSE

DATA ANALYSIS · DEMANDING · UPI-01.08 · UPI-01.09 · UPI-05.06 · UPI-06.06 **[12]**

A cyclist rides a bicycle fitted with a power meter along a level road on a windless day. At each of six steady speeds v she records the power P that she delivers to the rear wheel. A model of the total resistance on the rider and bicycle is $F = F_r + kv^2$, where F_r is a constant rolling resistance and kv^2 is aerodynamic drag, with k constant. Take $g = 9.81$ m/s².

(a) EXPLAIN · U1 **[2]**

Explain why, at a steady speed on a level road, the model predicts that $P = (F_r + kv^2)v$.

1. At constant velocity the acceleration is zero, so by Newton's second law the forward driving force equals the total resistive force $F_r + kv^2$
2. The power delivered by a force is $P = F \cdot v$; the driving force is parallel to the velocity, so $P = (F_r + kv^2)v$

   Candidates often look for a change in kinetic energy to account for the power. At steady speed there is none: all of the power goes into doing work against the resistance.

(b) DETERMINE · U4 **[4]**

By processing the data so that the model predicts a straight line, determine F_r and k.

1 Rearranges the model to $P/v = F_r + kv^2$, so a graph of P/v against v^2 should be a straight line with gradient k and intercept F_r

2 Calculates $P/v = 7.25$, 11.3, 16.6, 24.3, 32.5 and 43.4 N for $v^2 = 16$, 36, 64, 100, 144 and 196 m^2/s^2

3 Gradient of a best-fit line through all six points: $k = 0.200$ kg/m (accept 0.19 to 0.21 kg/m)

4 Intercept: $F_r = 4.0$ N (accept 3.5 to 4.5 N)

Plotting P against v gives a curve from which neither constant can be read, and plotting P against v^3 discards the rolling term; dividing by v is the step being rewarded. A gradient taken from two neighbouring points rather than from a line through all six is the usual source of values outside the accepted range — neighbouring pairs give intercepts anywhere from 2.2 N to 5.7 N.

(c) COMPARE · U4 **[2]**

A second student proposes that rolling resistance is negligible, so that $P = kv^3$. Compare how well this model and the two-term model account for the data.

1 In the pure-drag model P/v^3 should be constant, but it falls steadily from 0.453 kg/m at 4.0 m/s to 0.222 kg/m at 14.0 m/s: a systematic trend, not random scatter

2 In the two-term model the values of P/v lie close to a straight line, with residuals within about ±0.35 N that are scattered in sign with no systematic trend, and the intercept of 4.0 N is clearly not zero — at 4.0 m/s it is more than half of the 7.25 N total resistance — so the rolling term is needed, above all at low speed

Judged only on the fastest trials, where drag dominates, $P = kv^3$ looks adequate. The comparison has to be made where the two models differ most, at low speed, and it has to rest on a trend rather than on one point.

(d) ESTIMATE · U4 **[2]**

Estimate the coefficient of rolling resistance $C_r = F_r/N$, where N is the normal force of the road on the bicycle and rider, stating the assumption you make.

1 States a plausible combined mass for rider and bicycle, for example 80 kg (any value from 60 kg to 100 kg), so that on a level road $N = mg \approx 80 \times 9.81 = 785$ N

2 $C_r = 4.0\ \text{N}/785\ \text{N} \approx 0.005$ (0.004 to 0.007 across the range of masses allowed)

The mass is omitted deliberately. Candidates who stop because a datum is missing, rather than stating a reasonable value and carrying on, have misread what an estimate asks of them.

(e) DESCRIBE · U4 **[2]**

Describe two changes to the procedure that would reduce the uncertainty in the value of F_r.

1 Add readings at lower speeds, for example 2.0 m/s to 4.0 m/s, where rolling resistance is the larger share of the total, so that the line is fixed close to its intercept rather than extrapolated to it

2 Ride each speed on the same stretch of road in both directions, repeating and averaging, so that a slight gradient or breeze, which the model assumes absent, cancels instead of biasing the power readings

'Take more readings' or 'use a better power meter' earns nothing unless the answer says where the readings should be taken and why that helps. F_r is an intercept, so its uncertainty is governed by the low-speed end of the line.

13 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-01.07 · UPI-07.03 · UPI-07.04 · UPI-07.05 · UPI-07.06 **[17]**

A snap-action switch is modelled as a slider of mass m that moves without friction along a straight horizontal rail, taken as the x-axis. A light spring of stiffness k and natural length L_0 joins the slider to a fixed pivot at perpendicular distance a from the rail, and the foot of that perpendicular is at x = 0. The pivot is closer to the rail than the spring's natural length, so $a < L_0$. Take the elastic potential energy of the spring to be zero at its natural length. Gravity and the normal force of the rail are perpendicular to the rail and do no work on the slider.

(a) DERIVE · U2 **[2]**

Starting from the elastic potential energy of a spring, $\frac{1}{2}k \times (\text{extension})^2$, derive an expression for the potential energy U(x) of the system when the slider is at position x.

1. The spring runs from the pivot to the slider, so its length is $\sqrt{(x^2 + a^2)}$ and its extension is $\sqrt{(x^2 + a^2)} - L_0$ (negative when the spring is compressed)
2. Elastic potential energy $\frac{1}{2}k(\text{extension})^2$ gives $U(x) = \frac{1}{2}k(\sqrt{(x^2 + a^2)} - L_0)^2$

   Candidates who write $U = \frac{1}{2}kx^2$ have taken the slider's position as the spring's extension. The spring is not aligned with the rail, and its extension has to come from the geometry.

(b) DERIVE · U2 **[3]**

Using the relation between a conservative force and its potential-energy function, derive an expression for the component $F_x(x)$ of the force on the slider along the rail.

1. For a conservative force in one dimension, $F_x = -dU/dx$
2. By the chain rule, $dU/dx = k(\sqrt{(x^2 + a^2)} - L_0) \times x/\sqrt{(x^2 + a^2)}$
3. $F_x = -kx(1 - L_0/\sqrt{(x^2 + a^2)})$

   The chain-rule factor $x/\sqrt{(x^2 + a^2)}$ is the projection of the spring force onto the rail. Candidates who drop it obtain a force of $-k(a - L_0)$ at x = 0, which the left–right symmetry of the arrangement rules out.

(c) DETERMINE · U2 **[2]**

Determine all the equilibrium positions of the slider.

1. $F_x = 0$ requires either x = 0 or $\sqrt{(x^2 + a^2)} = L_0$
2. The equilibria are x = 0 and $x = \pm\sqrt{(L_0^2 - a^2)}$, the outer pair existing because $a < L_0$

   Candidates commonly find only the symmetric pair, where the spring is at its natural length, and miss x = 0, where the spring is compressed but pushes straight across the rail and so exerts no force along it.

(d) JUSTIFY · U1 **[2]**

Justify, for each equilibrium position, whether it is stable or unstable.

1. At $x = \pm\sqrt{(L_0^2 - a^2)}$, U = 0, which is the least value U can take because it is $\frac{1}{2}k$ times a square, so these are minima of U and the equilibria are stable
2. At x = 0, $U = \frac{1}{2}k(L_0 - a)^2 > 0$ is a maximum between the two minima; equivalently, for small x, $F_x \approx kx(L_0/a - 1)$ has the same sign as x and pushes the slider away, so x = 0 is unstable

   An equilibrium with a compressed spring feels as though it ought to be stable, because springs restore. Here the compressed spring pushes the slider sideways as soon as it is displaced, and the judgement has to rest on the shape of U or the sign of F_x, not on intuition about springs.

(e) SKETCH · U1 **[3]**

Sketch U(x) against x over a range that includes all the equilibrium positions, marking each position and the value of U there.

1. A curve symmetric about x = 0 with two minima at $x = \pm\sqrt{L_0^2 - a^2}$, where it touches U = 0 without crossing below the axis
2. A smooth local maximum (zero slope, no cusp) at x = 0, marked at height $\tfrac{1}{2}k(L_0 - a)^2$
3. U rising without limit on both sides beyond the minima, growing roughly as $\tfrac{1}{2}kx^2$ once |x| is much larger than L_0, so that the whole curve is a double well that never falls below zero

Sketches that dip below the axis, or put a cusp at x = 0, show a candidate drawing a generic double well from memory instead of the function derived in (a). Every feature asked for can be read from U(x) and F_x(x) without further calculation.

(f) CALCULATE · U3 **[3]**

For a switch with k = 250 N/m, L_0 = 50.0 mm, a = 40.0 mm and m = 20.0 g, calculate the positions of the two stable states and the minimum speed the slider must have as it passes through one stable state if it is to reach the other.

1. Stable states at $x = \pm\sqrt{0.0500^2 - 0.0400^2}$ m = ±0.0300 m, that is ±30.0 mm
2. Height of the energy barrier: $U(0) - 0 = \tfrac{1}{2} \times 250 \times (0.0500 - 0.0400)^2 = 1.25 \times 10^{-2}$ J
3. Conservation of mechanical energy, $\tfrac{1}{2}mv^2 = 1.25 \times 10^{-2}$ J, gives $v = \sqrt{2 \times 1.25 \times 10^{-2}/0.0200} = 1.12$ m/s (strictly, any speed greater than this)

The speed is set by the height of the barrier, not by the distance between the states. Candidates who use $\tfrac{1}{2}kx^2$ with x = 30.0 mm or x = 60.0 mm obtain 3.35 m/s or 6.71 m/s, treating the slider's displacement as the spring's extension again.

(g) PREDICT · U1 **[2]**

The pivot is moved further from the rail until $a \geq L_0$. Predict how the switch now behaves, reasoning from the energy landscape.

1. For $a \geq L_0$ the spring's length $\sqrt{x^2 + a^2}$ is never less than a, so the spring is never compressed; $\sqrt{x^2 + a^2} = L_0$ has no solution when $a > L_0$ and only x = 0 when $a = L_0$, so x = 0 is the only equilibrium: as a rises towards L_0 the outer minima at $\pm\sqrt{L_0^2 - a^2}$ move in to x = 0 and the barrier $\tfrac{1}{2}k(L_0 - a)^2$ shrinks to zero
2. U now rises steadily with |x| from a single minimum at x = 0 (a very flat one when $a = L_0$), so x = 0 is stable: a displaced slider is always pushed back towards the centre and oscillates about it, so the switch loses its two latched states and no longer snaps

This is the limit check on the whole model: the positions from (c) and the barrier from (d) and (f) must vanish continuously as a approaches L_0. Candidates who predict that the switch will 'snap harder' have not followed the landscape through the limit.

University Physics I · Final Examination

The whole of calculus-based mechanics, weighted to the second half of the course: momentum and collisions, rotational kinematics and dynamics, angular momentum and precession, static equilibrium and elasticity, gravitation and orbits, and oscillations, with fluids as an extension. Kinematics, Newton's laws, work and energy are sampled from the first half, and every answer is expected to name the system, the conservation law and the approximation it rests on.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**116**	**18**	**10**

SECTION — **UPI · Final · University Physics I — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

ROUTES — Core course only — 17 questions, 115 marks. Core course and Extension — the whole paper, 18 questions, 116 marks. Every question is tagged in the left margin.

DEMAND — **Routine 9 · Demanding 6 · Discriminating 3**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPI-01 · UPI-02 · UPI-03 · UPI-04 · UPI-05 · UPI-06 · UPI-07 · UPI-08 · UPI-09 · UPI-10 · UPI-11 · UPI-12 · UPI-13 · UPI-14 · UPI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Physics I · Final Examination

Answer all questions. 116 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.02 **[1]**

Dry sand drains through a circular hole of diameter D in the flat base of a hopper. Experiments show that the mass flow rate $\dot{m}$, in kg/s, does not depend on the depth of sand above the hole. Assume that $\dot{m}$ depends only on the bulk density ρ of the sand, the gravitational field strength g and the diameter D, and use dimensional analysis. If the diameter of the hole is doubled, by what factor does the mass flow rate increase?

A 2.8

B 4.0

C 5.7

D 8.0

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.06 · UPI-03.01 **[1]**

A particle moves in the xy-plane with position vector $\mathbf{r}(t) = (2.0t^2 - 8.0t)\,\hat{\imath} + (6.0t - 1.0t^2)\,\hat{\jmath}$, where r is in metres and t is in seconds. At what time is the speed of the particle least?

A 2.0 s

B 2.2 s

C 2.5 s

D 3.0 s

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-04.08 · UPI-05.01 **[1]**

A 12 kg crate rests on a loading ramp inclined at 20° to the horizontal. The coefficients of friction between the crate and the ramp are $\mu_s = 0.60$ and $\mu_k = 0.45$. Take $g = 9.81\ \text{m/s}^2$. What is the magnitude of the friction force that the ramp exerts on the crate?

A 40 N

B 50 N

C 66 N

D 71 N

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-07.04 · UPI-07.05 **[1]**

An atom adsorbed on a crystal surface can move only along a straight row of binding sites. Along the row its potential energy is $U(x) = -U_0 \cos(2\pi x/a)$, where U_0 is a positive constant and a is the spacing of the sites, so that the sites lie at $x = 0, \pm a, \pm 2a$ and so on. For positions between $x = 0$ and $x = a/2$, where is the force that the surface exerts on the atom along the row greatest in magnitude, and in which direction does that force point there?

A At $x = a/2$, pointing towards $x = 0$

B At $x = a/4$, pointing away from $x = 0$

C At $x = a/2$, pointing away from $x = 0$

D At $x = a/4$, pointing towards $x = 0$

Answer

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-08.01 · UPI-08.02 **[1]**

A 0.170 kg ice-hockey puck slides at 6.00 m/s directly towards a player, who strikes it straight back along the same line. The force of the stick on the puck, directed away from the player, rises linearly from zero to 800 N in 4.0 ms, stays at 800 N for 3.0 ms, then falls linearly to zero in a further 3.0 ms. Friction with the ice is negligible. What is the speed of the puck just after the strike?

A 17.5 m/s

B 24.6 m/s

C 30.6 m/s

D 36.6 m/s

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-09.05 · UPI-09.07 **[1]**

A bicycle wheel of radius 0.35 m rolls without slipping along a level road, its centre moving at 7.0 m/s. Relative to the road, what are the speed of the point at the top of the rim and the speed of a point on the rim level with the axle?

A 14 m/s at the top; 9.9 m/s level with the axle

B 14 m/s at the top; 7.0 m/s level with the axle

C 7.0 m/s at the top; 7.0 m/s level with the axle

D 14 m/s at the top; 14 m/s level with the axle

Answer

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-01.06 · UPI-11.05 **[1]**

A spinning bicycle wheel is mounted at one end of a light horizontal axle whose other end rests on a frictionless pivot at the origin. The axle lies along the x-axis with the wheel's centre at x = +0.150 m, and the z-axis points vertically upwards. The wheel has mass 2.00 kg and moment of inertia 0.120 kg m² about the axle, and it spins at 40.0 rad/s with its spin angular momentum pointing in the +x direction. Take g = 9.81 m/s². In the fast-spin approximation, at what rate does the axle precess, and in which sense does it turn when viewed from above, looking down the z-axis?

A 0.61 rad/s, clockwise

B 1.6 rad/s, counterclockwise

C 1.6 rad/s, clockwise

D 0.61 rad/s, counterclockwise

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-13.02 · UPI-13.05 **[1]**

Transit and radial-velocity observations show that a rocky exoplanet has 4.5 times the mass of Earth and 1.5 times its radius. Treat both planets as spherically symmetric. Compared with the values for Earth, how do the gravitational field strength at the planet's surface and the escape speed from its surface change?

A Field strength doubles; escape speed increases by a factor of 1.7

B Field strength doubles; escape speed increases by a factor of 1.4

C Field strength increases by a factor of 3.0; escape speed increases by a factor of 1.7

D Field strength doubles; escape speed increases by a factor of 3.0

Answer

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-14.03 · UPI-14.04 **[1]**

A block attached to a horizontal spring slides on a frictionless surface in simple harmonic motion with an amplitude of 0.12 m and a period of 0.80 s. What is the speed of the block when its displacement from equilibrium is 0.060 m?

A 0.13 m/s

B 0.47 m/s

C 0.82 m/s

D 0.94 m/s

Answer

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10 MULTIPLE CHOICE · ROUTINE · UPI-15.06 · UPI-15.07 **[1]**

EXTENSION

Water of density 1000 kg/m³ flows steadily through a horizontal pipe whose internal diameter narrows from 0.040 m to 0.020 m. The speed of the water in the wide section is 1.5 m/s. Treating the water as an ideal incompressible fluid, by how much is the pressure in the narrow section lower than the pressure in the wide section?

A 3.4 kPa

B 10 kPa

C 17 kPa

D 18 kPa

Answer

11 CORE COURSE

STRUCTURED · ROUTINE · UPI-09.04 · UPI-10.03 · UPI-10.06 · UPI-10.07 **[12]**

A laboratory test rig stores braking energy in a flywheel: a uniform steel annular disc of inner radius $R_1 = 0.100$ m, outer radius $R_2 = 0.300$ m and thickness 0.0500 m, which spins about its symmetry axis. The density of the steel is 7.80×10^3 kg/m³. In normal operation the flywheel spins at $\omega_0 = 314$ rad/s.

(a) SHOW THAT **[3]**

Starting from the definition $I = \int r^2 \, dm$, show that the moment of inertia of a uniform annular disc of mass M about its symmetry axis is $I = \tfrac{1}{2}M(R_1^2 + R_2^2)$.

(b) CALCULATE **[3]**

Calculate the mass of the flywheel, its moment of inertia, and the kinetic energy it stores at ω_0.

(c) DETERMINE **[4]**

When the flywheel is disconnected at ω_0 it coasts to rest in 1.50×10^3 s under a constant frictional torque from its bearings. Determine the frictional torque, the number of revolutions the flywheel makes while coasting, and the rate at which friction dissipates energy at the instant the flywheel is disconnected.

(d) COMPARE **[2]**

Compare the energy this flywheel stores at ω_0 with the energy stored by a uniform solid disc of the same mass and the same outer radius spinning at the same angular speed, and explain the difference in terms of how the mass is distributed.

12 CORE COURSE

STRUCTURED · DEMANDING · UPI-02.07 · UPI-04.08 · UPI-06.03 · UPI-06.06 **[12]**

An electric car of mass 1500 kg, driver included, starts from rest on a level road. Its motor controller limits the power delivered to the driven wheels to P = 60.0 kW. Ignore air resistance and rolling resistance throughout, and take g = 9.81 m/s².

(a) SHOW THAT **[2]**

Suppose the full power P is delivered from the instant the car starts. Show that the speed of the car at time t is $v(t) = \sqrt{(2Pt/m)}$.

(b) CALCULATE **[2]**

Using this model, calculate the time the car takes to reach 25.0 m/s and its acceleration at that instant.

(c) DETERMINE **[3]**

Determine, by integrating v(t), the distance the car travels in reaching 25.0 m/s under this model.

(d) EVALUATE **[3]**

The coefficient of static friction between the tyres and the road is 0.80, and all four wheels are driven, so the car's whole weight is available for traction. Evaluate the constant-power model close to t = 0: find the speed below which the car cannot use its full power, and state what limits its motion below that speed.

(e) SKETCH **[2]**

The real car accelerates at the traction limit until it can use its full power, then continues at constant power. Sketch a graph of its speed against time from t = 0 until it reaches 25.0 m/s, and mark the point where the two stages join.

13 CORE COURSE

STRUCTURED · DEMANDING · UPI-12.01 · UPI-12.04 · UPI-12.08 · UPI-12.10 **[13]**

A uniform horizontal steel beam of length 3.00 m and mass 45.0 kg projects from a vertical wall, to which one end is attached by a frictionless hinge. A steel cable runs from the outer end of the beam to a point on the wall directly above the hinge, making an angle of 30.0° with the beam. A shop sign of mass 80.0 kg hangs from the beam 2.40 m from the wall. The cable is 3.46 m long and 4.00 mm in diameter, and its steel has Young modulus 2.00×10^{11} Pa and yield strength 2.50×10^{8} Pa. The weight of the cable is negligible. Take $g = 9.81\ \text{m/s}^2$.

(a) EXPLAIN **[2]**

Explain why taking torques about the hinge is the most efficient first step in finding the tension in the cable.

(b) CALCULATE **[2]**

Calculate the tension in the cable.

(c) DETERMINE **[3]**

Determine the magnitude and direction of the force exerted by the hinge on the beam.

(d) CALCULATE **[2]**

Calculate the tensile stress in the cable and the extension of the cable.

(e) EVALUATE **[2]**

The building code requires every load-bearing cable to have a safety factor of at least 2.0 against yield. Evaluate whether this cable complies, and recommend the smallest cable diameter that would.

(f) DESCRIBE **[2]**

Describe what the stress–strain curve for the cable steel would show if the cable were loaded beyond its yield point and then unloaded, and what this would mean for the sign.

14 CORE COURSE

STRUCTURED · DEMANDING · UPI-07.07 · UPI-08.02 · UPI-08.03 · UPI-08.05 **[14]**

Car A, of mass 1200 kg and travelling due east, and car B, of mass 1800 kg and travelling due north, collide in the middle of an intersection and lock together. Skid marks show that the wreck then slid in a straight line for 8.00 m, directed 35.0° north of east, before stopping. The coefficient of kinetic friction between the tyres and the road is 0.65, and g = 9.81 m/s². The speed limit at the intersection is 13.9 m/s (50 km/h).

(a) JUSTIFY **[2]**

Justify treating the two cars as an isolated system for the duration of the collision, even though friction from the road acts on them.

(b) CALCULATE **[2]**

Calculate the speed of the wreck immediately after the collision.

(c) DETERMINE **[3]**

Determine the speed of each car immediately before the collision.

(d) CALCULATE **[2]**

Calculate the fraction of the cars' kinetic energy that was dissipated in the collision itself.

(e) ESTIMATE **[2]**

Estimate the magnitude of the average force exerted on a 75 kg occupant of car B, held by a seatbelt, during the collision. State the assumptions you make.

(f) EVALUATE **[3]**

The driver of car A claims not to have exceeded the speed limit. The investigator's values are uncertain: μ_k lies between 0.55 and 0.75, and the direction of the skid is 35° ± 3° north of east. Evaluate the driver's claim.

15 CORE COURSE

STRUCTURED · DISCRIMINATING · UPI-08.03 · UPI-10.04 · UPI-11.03 · UPI-11.04 **[13]**

A uniform rod of mass M = 2.00 kg and length L = 1.20 m hangs vertically at rest from a frictionless horizontal pivot through its upper end. A lump of clay of mass m = 0.400 kg, moving horizontally at 6.00 m/s perpendicular to the pivot axis, strikes the lower end of the rod and sticks to it. Treat the clay as a particle and take g = 9.81 m/s². The moment of inertia of a uniform rod about a perpendicular axis through its centre is $ML^2/12$.

(a) STATE **[1]**

State the condition under which the angular momentum of a system about a fixed point is conserved.

(b) CALCULATE **[2]**

Calculate the moment of inertia of the rod and clay together about the pivot, immediately after the collision.

(c) EXPLAIN **[2]**

Explain why the linear momentum of the rod–clay system is not conserved in the collision, whereas its angular momentum about the pivot is.

(d) DETERMINE **[3]**

Determine the angular speed of the rod immediately after the collision and the greatest angle from the vertical through which it then swings.

(e) DETERMINE **[2]**

Determine the horizontal impulse exerted by the pivot on the rod during the collision, including its direction.

(f) DERIVE **[3]**

Derive, from conservation of angular momentum about the pivot and the impulse–momentum theorem, the distance x below the pivot at which the clay must strike for the pivot to exert no horizontal impulse during the collision. Show that the result does not depend on the mass of the clay.

16 CORE COURSE

DATA ANALYSIS · DEMANDING · UPI-01.09 · UPI-14.04 · UPI-14.07 **[12]**

A glider of mass 0.250 kg on a level air track is held between two springs and oscillates with a period of 0.800 s. A card sail fixed to the glider increases the air drag. A motion sensor records the amplitude of the oscillation, the greatest displacement from equilibrium, at the end of every fifth oscillation, with an uncertainty of ±0.3 mm in each reading. Two damping models are proposed. Model V (viscous drag): the resistive force is −bv, so the amplitude decays as A = A₀e^(−bt/(2m)). Model F (sliding friction): a resistive force of constant magnitude acts, which reduces the amplitude by the same amount in every oscillation.

Amplitude of the glider's oscillation, recorded at the end of every fifth oscillation

Reading	Time t (s)	Amplitude A (mm)
1	0.0	80.0
2	4.0	57.6
3	8.0	40.8
4	12.0	29.6
5	16.0	20.9
6	20.0	15.3
7	24.0	10.8

(a) DESCRIBE **[2]**

Describe the pattern that each model predicts in amplitudes recorded at equal intervals of time.

(b) JUSTIFY **[3]**

Justify, using calculations from the table, which model the data support.

(c) DETERMINE **[3]**

Determine the damping constant b by a method that uses all of the readings.

(d) CALCULATE **[2]**

Calculate the fraction of the glider's mechanical energy that is lost during one oscillation.

(e) PREDICT **[2]**

A 0.250 kg brass block is fixed to the glider, which keeps the same sail and the same springs. Predict the time for the amplitude to fall to half its initial value, before and after the block is added, and give the reason for the change.

17

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION: MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-11.06 · UPI-13.04 · UPI-13.05 · UPI-13.06 **[15]**

A communications satellite is raised from a circular parking orbit of radius $r_1 = 6.78 \times 10^6$ m to a circular orbit of radius $r_2 = 4.22 \times 10^7$ m by a two-burn transfer. A short engine burn at radius r_1 places it on an elliptical transfer orbit whose nearest point to Earth's centre (perigee) is at r_1 and whose farthest point (apogee) is at r_2. A second short burn at apogee makes the orbit circular. For Earth, $GM = 3.986 \times 10^{14}$ m³/s². Treat both burns as instantaneous, treat Earth as a spherically symmetric mass, and ignore every other body.

(a) DERIVE **[2]**

Derive, from Newton's law of gravitation and Newton's second law, an expression for the speed v of a satellite in a circular orbit of radius r.

(b) EXPLAIN **[2]**

Explain why the satellite's angular momentum about Earth's centre is constant along the transfer orbit, and why its magnitude at perigee and at apogee is simply mrv.

(c) DERIVE **[4]**

Using conservation of angular momentum and conservation of mechanical energy between perigee and apogee, derive an expression for the speed v_p at perigee on the transfer orbit, in terms of GM, r_1 and r_2.

(d) SHOW THAT **[2]**

Show that your expression reduces to the circular-orbit speed at r_1 when $r_2 = r_1$, and to the escape speed from r_1 as $r_2 \to \infty$, and state what each limit means physically.

(e) CALCULATE **[3]**

Calculate the change in speed required at each burn, and the total.

(f) CALCULATE **[2]**

Calculate the time the satellite spends on the transfer orbit, from perigee to apogee.

18

CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION: FROM A CONSERVATION LAW TO THE EQUATION OF MOTION · DISCRIMINATING · UPI-09.07 · UPI-10.08 · UPI-14.02 · UPI-14.06 **[15]**

A seismic isolation bearing is modelled as a uniform solid cylinder of mass m and radius r that rolls without slipping on the inside of a fixed, concave cylindrical track of radius R, where R > r and both axes are horizontal and parallel. The moment of inertia of the cylinder about its own axis is $\tfrac{1}{2}mr^2$. Let φ be the angle between the vertical and the line from the track's axis of curvature to the cylinder's axis, so that $\varphi = 0$ at the lowest position. Take $g = 9.81\ \text{m/s}^2$, and ignore rolling resistance and air resistance.

(a) EXPLAIN **[2]**

Explain why the speed of the cylinder's axis is $(R - r)|d\varphi/dt|$, and use the rolling condition to obtain the cylinder's angular speed ω about its own axis in terms of $d\varphi/dt$.

(b) DERIVE **[5]**

Derive, from conservation of mechanical energy, the equation of motion for φ.

(c) SHOW THAT **[2]**

Show that, for small oscillations, the motion is simple harmonic with period $T = 2\pi\sqrt{(3(R - r)/(2g))}$.

(d) COMPARE **[2]**

Compare this period with that of a small block sliding without friction whose centre of mass moves on the same circle of radius $R - r$, and explain the difference.

(e) CALCULATE **[2]**

The engineers want an isolation period of 2.50 s. Calculate the value of $R - r$ required.

(f) PREDICT **[2]**

Predict the period of the bearing if the solid cylinder is replaced by a thin-walled tube of the same mass and outer radius, whose moment of inertia about its own axis is mr^2, with $R - r$ unchanged, and give the reason.

MARK SCHEME

University Physics I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.02 **[1]**

Dry sand drains through a circular hole of diameter D in the flat base of a hopper. Experiments show that the mass flow rate $\dot{m}$, in kg/s, does not depend on the depth of sand above the hole. Assume that $\dot{m}$ depends only on the bulk density ρ of the sand, the gravitational field strength g and the diameter D, and use dimensional analysis. If the diameter of the hole is doubled, by what factor does the mass flow rate increase?

KEY **C** **5.7**

Writing $\dot{m} = k\rho^a g^b D^c$ and matching dimensions, kg s^{-1} = (kg m^{-3})^a(m s^{-2})^{b}m^c gives a = 1 from mass, b = ½ from time and c = 3 − ½ = 5/2 from length, so $\dot{m} \propto \rho g$^(1/2)D^(5/2) and doubling D multiplies $\dot{m}$ by 2^(5/2) = 5.7. B is the answer of a candidate who assumes the flow rate scales with the area of the hole, which silently assumes the sand leaves at a speed independent of D; the dimensional result shows that the exit speed must scale as √(gD). D comes from forgetting that g carries a dimension of length, so that the length balance reads c − 3 = 0 and c = 3. A, 2^(3/2), combines an exit speed √(gD) with an opening proportional to D rather than D², as if the hole were a long slot of width D instead of a circle.

U2

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-01.06 · UPI-03.01 **[1]**

A particle moves in the xy-plane with position vector r(t) = (2.0t² − 8.0t) î + (6.0t − 1.0t²) ĵ, where r is in metres and t is in seconds. At what time is the speed of the particle least?

KEY **B** **2.2 s**

The velocity is v = (4.0t − 8.0) î + (6.0 − 2.0t) ĵ and the acceleration is the constant a = 4.0 î − 2.0 ĵ. Because d(v · v)/dt = 2v · a, the speed is stationary when v · a = 20t − 44 = 0, which gives t = 2.2 s, where the speed is 1.8 m/s. A is the instant at which the x-component of velocity vanishes and D the instant at which the y-component vanishes; each is tempting because in one-dimensional motion the speed is least where the velocity is zero, but here the other component is still non-zero, and the speeds at those instants are 2.0 m/s and 4.0 m/s. C simply averages those two instants, which has no physical basis: where the minimum falls is set by the components of the acceleration, not by splitting the difference.

U2

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-04.08 · UPI-05.01 **[1]**

A 12 kg crate rests on a loading ramp inclined at 20° to the horizontal. The coefficients of friction between the crate and the ramp are μ_s = 0.60 and μ_k = 0.45. Take g = 9.81 m/s². What is the magnitude of the friction force that the ramp exerts on the crate?

KEY **A** **40 N**

Because tan 20° = 0.36 is less than μ_s, the crate stays at rest, and static friction takes whatever value equilibrium requires: along the ramp f = mg sin 20° = 40 N. C is $\mu_s N = \mu_s mg \cos 20°$ = 66 N, which is only the largest value static friction can reach; candidates who treat $\mu_s N$ as the friction force rather than as a ceiling choose it. B is $\mu_k mg \cos 20°$ = 50 N and applies the kinetic coefficient to a crate that is not sliding. D is $\mu_s mg$ = 71 N, which compounds the ceiling error by taking the normal force on an incline to be the whole weight.

U1

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-07.04 · UPI-07.05 **[1]**

An atom adsorbed on a crystal surface can move only along a straight row of binding sites. Along the row its potential energy is $U(x) = -U_0 \cos(2\pi x/a)$, where U_0 is a positive constant and a is the spacing of the sites, so that the sites lie at x = 0, ±a, ±2a and so on. For positions between x = 0 and x = a/2, where is the force that the surface exerts on the atom along the row greatest in magnitude, and in which direction does that force point there?

KEY **D** **At x = a/4, pointing towards x = 0**

Along the row the force is $F = -dU/dx = -(2\pi U_0/a)\sin(2\pi x/a)$. Between x = 0 and x = a/2 the sine is positive, so F is negative: the surface pulls the atom back towards the site at x = 0. The magnitude of F is greatest where the sine equals 1, at x = a/4, which is where U changes most steeply with position; U is zero there, midway between its least value $-U_0$ at the site and its greatest value $+U_0$ at x = a/2. B has the right position and the wrong direction: it follows from writing $F = +dU/dx$, which would push the atom up the slope of U, away from the site where its energy is least. A and C tempt candidates who equate a large potential energy with a large force and so pick the top of the barrier at x = a/2. But U has zero slope at its maximum, so the force there is zero and the atom is in unstable equilibrium, whichever direction is attached to it. A pairs that position with the direction the force has everywhere else in the range, and C pairs it with the reversed direction that the sign error in B produces.

U1

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-08.01 · UPI-08.02 **[1]**

A 0.170 kg ice-hockey puck slides at 6.00 m/s directly towards a player, who strikes it straight back along the same line. The force of the stick on the puck, directed away from the player, rises linearly from zero to 800 N in 4.0 ms, stays at 800 N for 3.0 ms, then falls linearly to zero in a further 3.0 ms. Friction with the ice is negligible. What is the speed of the puck just after the strike?

KEY **B** **24.6 m/s**

The impulse is the area under the force–time profile: 1.6 N s + 2.4 N s + 1.2 N s = 5.2 N s. Taking the direction away from the player as positive, the momentum changes from (0.170)(−6.00) = −1.02 kg m/s to −1.02 + 5.2 = 4.18 kg m/s, so the puck leaves at 24.6 m/s. C is J/m = 30.6 m/s, the answer of a candidate who treats the puck as starting from rest and forgets that part of the impulse is spent reversing it. D adds the incoming speed instead of subtracting it, a sign error in the initial momentum. A treats the pulse as a single triangle of height 800 N spread over the whole 10 ms, which gives an impulse of only 4.0 N s because it discards the plateau.

U3

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-09.05 · UPI-09.07 **[1]**

A bicycle wheel of radius 0.35 m rolls without slipping along a level road, its centre moving at 7.0 m/s. Relative to the road, what are the speed of the point at the top of the rim and the speed of a point on the rim level with the axle?

KEY **A** **14 m/s at the top; 9.9 m/s level with the axle**

The velocity of any point on the rim is the velocity of the centre plus the velocity due to rotation about the centre, of magnitude $\omega R = 7.0$ m/s and tangent to the rim. At the top both contributions point forwards, giving 14 m/s; level with the axle the rotational contribution is vertical, perpendicular to the forward 7.0 m/s, so the speed is $7.0\sqrt{2} = 9.9$ m/s. C gives every rim point the rim speed ωR, which is correct only in the frame moving with the axle. D adds the two speeds as scalars at the side as well as at the top, ignoring that they are perpendicular there. B keeps only the translational part at the side, as though a vertical contribution could not add to the speed.

U1

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPI-01.06 · UPI-11.05 **[1]**

A spinning bicycle wheel is mounted at one end of a light horizontal axle whose other end rests on a frictionless pivot at the origin. The axle lies along the x-axis with the wheel's centre at x = +0.150 m, and the z-axis points vertically upwards. The wheel has mass 2.00 kg and moment of inertia 0.120 kg m² about the axle, and it spins at 40.0 rad/s with its spin angular momentum pointing in the +x direction. Take g = 9.81 m/s². In the fast-spin approximation, at what rate does the axle precess, and in which sense does it turn when viewed from above, looking down the z-axis?

KEY **D** **0.61 rad/s, counterclockwise**

About the pivot the weight exerts a torque $\tau = r \times F = (0.150\ \text{m}\ \hat{\imath}) \times (-19.6\ \text{N}\ \hat{k}) = +2.94\ \text{N m}\ \hat{\jmath}$. That torque changes the angular momentum $L = I\omega\ \hat{\imath} = 4.80\ \text{kg m}^2/\text{s}\ \hat{\imath}$ in its own direction, so L swings from +x towards +y, which is counterclockwise seen from above, at $\Omega = \tau/L = 2.94/4.80 = 0.61$ rad/s. A has the rate right and the sense wrong: it follows from computing F × r instead of r × F, or from letting L swing opposite to the torque, when dL/dt must point along τ. B and C invert the ratio, but $L/\tau = 1.6$ s is a time, not a rate, and it grows as the spin gets faster, whereas faster spin must mean slower precession because the same torque then turns a larger angular momentum.

U1

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-13.02 · UPI-13.05 **[1]**

Transit and radial-velocity observations show that a rocky exoplanet has 4.5 times the mass of Earth and 1.5 times its radius. Treat both planets as spherically symmetric. Compared with the values for Earth, how do the gravitational field strength at the planet's surface and the escape speed from its surface change?

KEY **A** **Field strength doubles; escape speed increases by a factor of 1.7**

The surface field is $g = GM/R^2$, which scales as $4.5/1.5^2 = 2.0$, and the escape speed is $\sqrt{(2GM/R)}$, which scales as $\sqrt{(4.5/1.5)} = \sqrt{3.0} = 1.7$. C uses GM/R for the field strength, which gives 4.5/1.5 = 3.0 and forgets that the inverse-square law squares the radius. D drops the square root from the escape speed: the energy balance $\tfrac{1}{2}mv^2 = GMm/R$ fixes v^2, which grows by a factor of 3.0, not v. B writes the escape speed as $\sqrt{(2gR)}$ and scales g by 2.0 while leaving R unchanged, which gives $\sqrt{2.0} = 1.4$ and misses that the radius has grown by half as well.

U2

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPI-14.03 · UPI-14.04 **[1]**

A block attached to a horizontal spring slides on a frictionless surface in simple harmonic motion with an amplitude of 0.12 m and a period of 0.80 s. What is the speed of the block when its displacement from equilibrium is 0.060 m?

KEY **C** **0.82 m/s**

Energy conservation in simple harmonic motion gives $v = \omega\sqrt{(A^2 - x^2)}$, with $\omega = 2\pi/T = 7.85$ rad/s, so $v = 7.85 \times \sqrt{(0.0144 - 0.0036)} = 0.82$ m/s. D is the maximum speed ωA, which the block has only as it passes through equilibrium. B is half the maximum speed, the answer of a candidate who assumes the speed falls in proportion to the displacement; because the kinetic energy is $\tfrac{1}{2}k(A^2 - x^2)$, three quarters of the energy is still kinetic at half the amplitude, so the speed is $\sqrt{3}/2$ of its maximum. A uses the frequency 1/T in place of the angular frequency 2π/T, which makes it a factor of 2π too small.

U3

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10 MULTIPLE CHOICE · ROUTINE · UPI-15.06 · UPI-15.07 **[1]**

EXTENSION

Water of density 1000 kg/m³ flows steadily through a horizontal pipe whose internal diameter narrows from 0.040 m to 0.020 m. The speed of the water in the wide section is 1.5 m/s. Treating the water as an ideal incompressible fluid, by how much is the pressure in the narrow section lower than the pressure in the wide section?

KEY **C** **17 kPa**

Continuity requires $A_1v_1 = A_2v_2$, and halving the diameter divides the cross-sectional area by four, so $v_2 = 6.0$ m/s. Bernoulli's equation along a horizontal streamline then gives $p_1 - p_2 = \tfrac{1}{2}\rho(v_2^2 - v_1^2) = 500 \times (36 - 2.25) = 1.7 \times 10^4$ Pa. A scales the speed with the diameter instead of with the area, taking $v_2 = 3.0$ m/s. D leaves out the kinetic-energy term for the wide section, which is small here but not negligible. B squares the difference of the speeds, $\tfrac{1}{2}\rho(v_2 - v_1)^2$, which is not a term that appears in Bernoulli's equation.

U3

11

CORE COURSE

STRUCTURED · ROUTINE · UPI-09.04 · UPI-10.03 · UPI-10.06 · UPI-10.07 **[12]**

A laboratory test rig stores braking energy in a flywheel: a uniform steel annular disc of inner radius $R_1 = 0.100$ m, outer radius $R_2 = 0.300$ m and thickness 0.0500 m, which spins about its symmetry axis. The density of the steel is 7.80×10^3 kg/m³. In normal operation the flywheel spins at $\omega_0 = 314$ rad/s.

(a) SHOW THAT · U2 **[3]**

Starting from the definition $I = \int r^2 \, dm$, show that the moment of inertia of a uniform annular disc of mass M about its symmetry axis is $I = \tfrac{1}{2}M(R_1^2 + R_2^2)$.

1. Divides the disc into thin rings of radius r and width dr, each of mass $dm = \sigma(2\pi r\, dr)$, where $\sigma = M/[\pi(R_2^2 - R_1^2)]$ is the mass per unit area
2. Writes $I = \int r^2\sigma(2\pi r)\, dr$ from R_1 to $R_2 = (\pi\sigma/2)(R_2^4 - R_1^4)$
3. Factorises $R_2^4 - R_1^4 = (R_2^2 - R_1^2)(R_2^2 + R_1^2)$ and substitutes for σ to obtain $I = \tfrac{1}{2}M(R_1^2 + R_2^2)$

(b) CALCULATE · U3 **[3]**

Calculate the mass of the flywheel, its moment of inertia, and the kinetic energy it stores at ω_0.

1. $M = \rho\pi(R_2^2 - R_1^2) \times \text{thickness} = 7800 \times \pi \times 0.0800 \times 0.0500 = 98.0$ kg
2. $I = \tfrac{1}{2} \times 98.0 \times (0.0100 + 0.0900) = 4.90$ kg m²
3. $K = \tfrac{1}{2}I\omega_0^2 = \tfrac{1}{2} \times 4.90 \times 314^2 = 2.42 \times 10^5$ J

(c) DETERMINE · U3 **[4]**

When the flywheel is disconnected at ω_0 it coasts to rest in 1.50×10^3 s under a constant frictional torque from its bearings. Determine the frictional torque, the number of revolutions the flywheel makes while coasting, and the rate at which friction dissipates energy at the instant the flywheel is disconnected.

1. Constant angular deceleration $\alpha = \omega_0/t = 314/1500 = 0.209$ rad/s²
2. Frictional torque $\tau = I\alpha = 4.90 \times 0.2093 = 1.03$ N m (accept 1.02 N m from rounded intermediate values)
3. Angle turned $\theta = \tfrac{1}{2}\omega_0 t = 2.36 \times 10^5$ rad, which is $\theta/(2\pi) = 3.75 \times 10^4$ revolutions
4. Initial dissipation rate $P = \tau\omega_0 = 1.026 \times 314 = 322$ W (accept 320 W to 323 W; consistent with $\tau\theta = 2.42 \times 10^5$ J, the whole of the stored energy)

(d) COMPARE · U1 **[2]**

Compare the energy this flywheel stores at ω_0 with the energy stored by a uniform solid disc of the same mass and the same outer radius spinning at the same angular speed, and explain the difference in terms of how the mass is distributed.

1. Solid disc: $I = \tfrac{1}{2}MR_2^2 = 4.41$ kg m², which stores $\tfrac{1}{2}I\omega_0^2 = 2.17 \times 10^5$ J, about 10% less than the annular flywheel
2. The annulus has all of its mass at 0.100 m or more from the axis, where each kilogram contributes a larger r^2 to I, so for the same mass and the same ω it has the larger moment of inertia and stores more kinetic energy

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12

CORE COURSE

STRUCTURED · DEMANDING · UPI-02.07 · UPI-04.08 · UPI-06.03 · UPI-06.06 **[12]**

An electric car of mass 1500 kg, driver included, starts from rest on a level road. Its motor controller limits the power delivered to the driven wheels to $P = 60.0$ kW. Ignore air resistance and rolling resistance throughout, and take $g = 9.81$ m/s².

(a) SHOW THAT · U2 **[2]**

Suppose the full power P is delivered from the instant the car starts. Show that the speed of the car at time t is $v(t) = \sqrt{(2Pt/m)}$.

1. Applies the work–kinetic energy theorem: the work done on the car in time t is $\int_0^t P\,dt = Pt$, and this equals the gain in kinetic energy $\tfrac{1}{2}mv^2$ from rest
2. Rearranges $Pt = \tfrac{1}{2}mv^2$ to $v = \sqrt{(2Pt/m)}$

   Candidates who start from the constant-acceleration equations reach an expression that looks plausible and is wrong. At constant power the driving force P/v falls as the car speeds up, so none of those equations applies.

(b) CALCULATE · U3 **[2]**

Using this model, calculate the time the car takes to reach 25.0 m/s and its acceleration at that instant.

1. $t = mv^2/(2P) = (1500 \times 25.0^2)/(1.20 \times 10^5) = 7.81$ s
2. The driving force is $P/v = 2.40 \times 10^3$ N, so $a = P/(mv) = 1.60$ m/s²

   The usual slip is $a = v/t = 3.20$ m/s². That is the average acceleration over the whole run, twice the instantaneous value at 25.0 m/s.

(c) DETERMINE · U2 **[3]**

Determine, by integrating v(t), the distance the car travels in reaching 25.0 m/s under this model.

1. Sets up $x = \int_0^t \sqrt{(2P/m)}\ t\text{^}(1/2)\,dt$
2. Integrates to $x = (2/3)\sqrt{(2P/m)}\ t\text{^}(3/2)$, which equals $(2/3)vt$
3. $x = (2/3) \times 25.0 \times 7.81 = 130$ m

   Candidates who use $x = \tfrac{1}{2}(u + v)t$, valid only for uniform acceleration, obtain 97.7 m. Here the speed rises steeply at first and then more slowly, so the average speed is two-thirds of the final speed, not half.

(d) EVALUATE · U1 **[3]**

The coefficient of static friction between the tyres and the road is 0.80, and all four wheels are driven, so the car's whole weight is available for traction. Evaluate the constant-power model close to t = 0: find the speed below which the car cannot use its full power, and state what limits its motion below that speed.

1. The model gives $a = P/(mv)$, which grows without limit as $v \to 0$, so it demands an arbitrarily large forward friction force from the road and cannot hold at the start
2. The largest traction force is $\mu_s mg = 0.80 \times 1500 \times 9.81 = 1.18 \times 10^4$ N, so full power requires $P/v \le 1.18 \times 10^4$ N, that is $v \ge P/(\mu_s mg) = 5.10$ m/s
3. Below 5.10 m/s traction, not power, is the limit: the car can accelerate at no more than $\mu_s g = 7.85$ m/s², so the constant-power model is valid only above 5.10 m/s

   The judgement expected is conditional: the model fails at low speed and is sound above 5.10 m/s. A candidate who simply calls the model wrong, or simply accepts it, has not evaluated it.

(e) SKETCH · U1 **[2]**

The real car accelerates at the traction limit until it can use its full power, then continues at constant power. Sketch a graph of its speed against time from t = 0 until it reaches 25.0 m/s, and mark the point where the two stages join.

1. A straight line from the origin with gradient 7.85 m/s², ending at the marked join at about (0.65 s, 5.1 m/s)
2. From the join, a concave-down curve that starts with the same gradient as the line (no corner) and flattens steadily, reaching 25.0 m/s at about 8.1 s, later than the 7.81 s of the full-power model

   Most sketches show a kink at the join. There is none, because 5.10 m/s is by construction the speed at which $P/(mv)$ equals $\mu_s g$. The later arrival should also show: during the traction-limited stage the power rises from zero, so on average only half the available power is used.

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13 CORE COURSE

STRUCTURED · DEMANDING · UPI-12.01 · UPI-12.04 · UPI-12.08 · UPI-12.10 **[13]**

A uniform horizontal steel beam of length 3.00 m and mass 45.0 kg projects from a vertical wall, to which one end is attached by a frictionless hinge. A steel cable runs from the outer end of the beam to a point on the wall directly above the hinge, making an angle of 30.0° with the beam. A shop sign of mass 80.0 kg hangs from the beam 2.40 m from the wall. The cable is 3.46 m long and 4.00 mm in diameter, and its steel has Young modulus 2.00×10^{11} Pa and yield strength 2.50×10^{8} Pa. The weight of the cable is negligible. Take $g = 9.81$ m/s².

(a) EXPLAIN · U1 **[2]**

Explain why taking torques about the hinge is the most efficient first step in finding the tension in the cable.

1. The force exerted by the hinge on the beam is unknown in both magnitude and direction
2. That force acts at the hinge, so it has zero lever arm and zero torque about an axis through the hinge, which leaves the cable tension as the only unknown in the torque equation

Candidates who take torques about the centre of the beam lose the horizontal forces, which act along the beam, but bring the vertical component of the hinge force into the torque equation alongside the tension, so they need the vertical force equation as well before either unknown is found. The route is valid but slower, and it is where most sign errors on this question are made.

(b) CALCULATE · U3 **[2]**

Calculate the tension in the cable.

1. Torques about the hinge: $T \sin 30.0° \times 3.00 \text{ m} = (45.0)(9.81)(1.50) + (80.0)(9.81)(2.40) = 662 + 1884 = 2546$ N m
2. $T = 2546/(3.00 \times \sin 30.0°) = 1.70 \times 10^{3}$ N

The weight of a uniform beam acts at its centre, 1.50 m from the hinge, not at its end. Using cos 30.0° in the lever arm, which gives 980 N, shows that the component of the tension perpendicular to the beam was not identified.

(c) DETERMINE · U3 **[3]**

Determine the magnitude and direction of the force exerted by the hinge on the beam.

1. Horizontal equilibrium: $H = T \cos 30.0° = 1.47 \times 10^{3}$ N, directed away from the wall
2. Vertical equilibrium: $V = (125.0 \text{ kg})(9.81 \text{ m/s}^2) - T \sin 30.0° = 1226.3 \text{ N} - 848.6 \text{ N} = 378$ N, directed upwards
3. Magnitude $\sqrt{(H^2 + V^2)} = 1.52 \times 10^{3}$ N, at 14.4° above the horizontal and pointing away from the wall

The hinge force does not lie along the beam. Candidates who assume it does, as it would for a light strut pinned at both ends, set $V = 0$ and then cannot satisfy vertical equilibrium.

(d) CALCULATE · U3 **[2]**

Calculate the tensile stress in the cable and the extension of the cable.

1. $A = \pi(2.00 \times 10^{-3} \text{ m})^2 = 1.26 \times 10^{-5} \text{ m}^2$, so the stress is $\sigma = T/A = 1.35 \times 10^{8}$ Pa
2. Strain $= \sigma/E = 6.75 \times 10^{-4}$, so the extension is $6.75 \times 10^{-4} \times 3.46 \text{ m} = 2.34 \times 10^{-3}$ m

Using the diameter in place of the radius when finding the area gives a stress four times too small, 3.38×10^{7} Pa, which would then pass the check in part (e) that the cable in fact fails.

(e) EVALUATE · U3 **[2]**

The building code requires every load-bearing cable to have a safety factor of at least 2.0 against yield. Evaluate whether this cable complies, and recommend the smallest cable diameter that would.

1. Safety factor = $(2.50 \times 10^8\ \text{Pa})/(1.35 \times 10^8\ \text{Pa}) = 1.85$, which is below 2.0, so the cable does not comply
2. Compliance requires $\sigma \le 1.25 \times 10^8$ Pa, so $A \ge 1.36 \times 10^{-5}\ \text{m}^2$ and $d \ge 4.16$ mm; recommends the next available size above this, for example 4.5 mm or 5 mm

The judgement must follow from the criterion given. A cable working at 54% of its yield stress will not fail today, and candidates who call it safe enough have ignored the code they were asked to apply.

(f) DESCRIBE · U1 **[2]**

Describe what the stress–strain curve for the cable steel would show if the cable were loaded beyond its yield point and then unloaded, and what this would mean for the sign.

1. The curve is straight only up to the proportional limit; past the elastic limit and the yield point it lies well off that line and the strain increases substantially for little further increase in stress: the steel deforms plastically
2. On unloading, the curve returns along a straight line parallel to the original elastic line and meets the strain axis at a non-zero strain, so the cable keeps a permanent extension and the beam and sign are left sagging

The common error is to say the cable returns to its original length once the load is removed. That holds only below the elastic limit; beyond it the unloading path runs parallel to the loading line rather than retracing the curve.

14

CORE COURSE

STRUCTURED · DEMANDING · UPI-07.07 · UPI-08.02 · UPI-08.03 · UPI-08.05 **[14]**

Car A, of mass 1200 kg and travelling due east, and car B, of mass 1800 kg and travelling due north, collide in the middle of an intersection and lock together. Skid marks show that the wreck then slid in a straight line for 8.00 m, directed 35.0° north of east, before stopping. The coefficient of kinetic friction between the tyres and the road is 0.65, and $g = 9.81\ \text{m/s}^2$. The speed limit at the intersection is 13.9 m/s (50 km/h).

(a) JUSTIFY · U1 **[2]**

Justify treating the two cars as an isolated system for the duration of the collision, even though friction from the road acts on them.

1. The collision lasts a very short time, of order 0.1 s, and during it the forces between the cars are of order 10^5 N, several times the largest friction force the road can exert on the wreck, $\mu_k(3000\ \text{kg})g \approx 1.9 \times 10^4$ N
2. The external impulse from the road over that brief interval is therefore small compared with the momentum of either car, so the total momentum just after the collision equals the total just before

Candidates often assert that friction is negligible, which is plainly false: friction is what stops the wreck. The argument that earns credit compares impulses over the duration of the collision, not forces over the whole event.

(b) CALCULATE · U3 **[2]**

Calculate the speed of the wreck immediately after the collision.

1. The work done by kinetic friction removes all the wreck's kinetic energy: $\tfrac{1}{2}(m_A + m_B)V^2 = \mu_k(m_A + m_B)g\,d$, so $V = \sqrt{2\mu_k g d}$
2. $V = \sqrt{2 \times 0.65 \times 9.81 \times 8.00} = 10.1$ m/s

The mass cancels only if the same mass is used in the kinetic energy and in the friction force. Candidates who use one car's mass in one of them and the combined mass in the other obtain a wrong speed.

(c) DETERMINE · U3 **[3]**

Determine the speed of each car immediately before the collision.

1. Momentum just after: (3000 kg)(10.10 m/s) = 3.030 × 10⁴ kg m/s, with components 2.482 × 10⁴ kg m/s east and 1.738 × 10⁴ kg m/s north
2. The east component is carried only by car A before the collision: v_A = (2.482 × 10⁴)/1200 = 20.7 m/s
3. The north component is carried only by car B: v_B = (1.738 × 10⁴)/1800 = 9.66 m/s (accept 9.67 m/s from a rounded component)

This part catches conservation of speed or of kinetic energy in place of vector momentum, and the pairing of sin 35.0° with car A. Each component equation involves only one car, which is what makes the reconstruction unique.

(d) CALCULATE · U3 **[2]**

Calculate the fraction of the cars' kinetic energy that was dissipated in the collision itself.

1. Just before: ½(1200)(20.7)² + ½(1800)(9.66)² = 2.57 × 10⁵ J + 8.40 × 10⁴ J = 3.41 × 10⁵ J; just after: ½(3000)(10.1)² = 1.53 × 10⁵ J
2. Fraction dissipated = (3.41 × 10⁵ − 1.53 × 10⁵)/(3.41 × 10⁵) = 0.55

Candidates who also subtract the energy removed during the skid count that energy twice. The question concerns only the collision, from just before it to just after it.

(e) ESTIMATE · U4 **[2]**

Estimate the magnitude of the average force exerted on a 75 kg occupant of car B, held by a seatbelt, during the collision. State the assumptions you make.

1. States the assumptions: the occupant moves with car B, whose velocity changes from 9.66 m/s north to the wreck's velocity of 8.27 m/s east and 5.79 m/s north, over a collision time of about 0.1 s
2. |Δv| = √(8.27² + 3.87²) ≈ 9.1 m/s, so F ≈ (75 kg)(9.1 m/s)/(0.1 s) ≈ 7 × 10³ N, roughly nine times the occupant's weight

The change in velocity is a vector difference. Subtracting speeds gives 10.1 − 9.66 ≈ 0.4 m/s and a force about twenty times too small, which should be rejected on physical grounds: a collision that locks two cars together is not a gentle nudge.

(f) EVALUATE · U4 **[3]**

The driver of car A claims not to have exceeded the speed limit. The investigator's values are uncertain: μ_k lies between 0.55 and 0.75, and the direction of the skid is 35° ± 3° north of east. Evaluate the driver's claim.

1. Writes $v_A = (m_A + m_B)\sqrt{(2\mu_k g d)}\cos\theta / m_A$ and identifies that the smallest value comes from the smallest μ_k, 0.55, and the largest angle, 38°
2. $v_{A,\min} = (3000/1200) \times \sqrt{(2 \times 0.55 \times 9.81 \times 8.00)} \times \cos 38° = 18.3$ m/s (66 km/h)
3. Even at the least favourable limits of the evidence v_A exceeds 13.9 m/s by about a third, so the evidence contradicts the driver's claim

An evaluation must test the claim at the edge of the uncertainty, not at the central values. Candidates who recalculate only with $\mu_k = 0.65$ and 35° have not established that the conclusion survives the uncertainty.

15

CORE COURSE

STRUCTURED · DISCRIMINATING · UPI-08.03 · UPI-10.04 · UPI-11.03 · UPI-11.04 **[13]**

A uniform rod of mass M = 2.00 kg and length L = 1.20 m hangs vertically at rest from a frictionless horizontal pivot through its upper end. A lump of clay of mass m = 0.400 kg, moving horizontally at 6.00 m/s perpendicular to the pivot axis, strikes the lower end of the rod and sticks to it. Treat the clay as a particle and take g = 9.81 m/s². The moment of inertia of a uniform rod about a perpendicular axis through its centre is ML²/12.

(a) STATE · U1 **[1]**

State the condition under which the angular momentum of a system about a fixed point is conserved.

1 The net external torque on the system about that point is zero (equivalently, the net external angular impulse about it is zero)

Answering that there must be no external force states a stronger condition than is needed, and one that fails here: the pivot exerts a large external force, yet angular momentum about the pivot is still conserved.

(b) CALCULATE · U3 **[2]**

Calculate the moment of inertia of the rod and clay together about the pivot, immediately after the collision.

1 Parallel-axis theorem for the rod: $I_{rod} = ML^2/12 + M(L/2)^2 = ML^2/3 = 0.960$ kg m²

2 Adds the clay as a particle at distance L: $mL^2 = 0.576$ kg m², giving $I = 1.536$ kg m² (1.54 kg m²)

Using $ML^2/12$, the value about the centre, for a rod turning about its end gives $I = 0.816$ kg m² and an exaggerated swing in part (d).

(c) EXPLAIN · U1 **[2]**

Explain why the linear momentum of the rod–clay system is not conserved in the collision, whereas its angular momentum about the pivot is.

1 The pivot holds the top of the rod in place, so during the collision it exerts a large horizontal impulsive force on the rod; that external force changes the linear momentum of the system

2 The pivot force acts at the pivot and has zero torque about it, and the rod's weight acts along the vertical line through the pivot while the rod hangs straight down, with any gravitational angular impulse negligible over the brief collision; so the net external angular impulse about the pivot is zero and angular momentum about it is conserved

Candidates who conserve linear momentum, $(0.400)(6.00) = (2.40\ \text{kg})v$, reach a quite different answer in part (d). Choosing the conservation law is the substance of the question.

(d) DETERMINE · U3 **[3]**

Determine the angular speed of the rod immediately after the collision and the greatest angle from the vertical through which it then swings.

1 Angular momentum about the pivot: $mvL = (0.400)(6.00)(1.20) = 2.88$ kg m²/s $= I\omega$, so $\omega = 2.88/1.536 = 1.875$ rad/s (1.88 rad/s)

2 Mechanical energy is conserved during the swing: $\tfrac{1}{2}I\omega^2 = 2.70$ J $= (MgL/2 + mgL)(1 - \cos\theta) = (16.5\ \text{J})(1 - \cos\theta)$

3 $1 - \cos\theta = 0.164$, so $\theta = 33.3°$ (accept 33.2° from rounded values)

Energy is conserved during the swing but not during the collision. Candidates who equate the clay's initial 7.20 J to the gain in potential energy predict a swing of 55.7°; only 37.5% of that kinetic energy survives the collision.

(e) DETERMINE · U2 **[2]**

Determine the horizontal impulse exerted by the pivot on the rod during the collision, including its direction.

1 Linear momentum just after is $(ML/2 + mL)\omega = (1.20 + 0.480)(1.875) = 3.15$ kg m/s, compared with $mv = 2.40$ kg m/s just before

2 The pivot's impulse is $3.15 - 2.40 = 0.750$ N s, directed the same way as the clay was moving

The direction surprises most candidates, who expect the pivot to resist the blow. Struck below its centre of percussion, a free rod's upper end would move backwards, so the pivot has to push it forwards.

(f) DERIVE · U2 **[3]**

Derive, from conservation of angular momentum about the pivot and the impulse–momentum theorem, the distance x below the pivot at which the clay must strike for the pivot to exert no horizontal impulse during the collision. Show that the result does not depend on the mass of the clay.

1. Angular momentum about the pivot: $mvx = (ML^2/3 + mx^2)\omega$
2. No pivot impulse means linear momentum is conserved: $mv = (ML/2 + mx)\omega$
3. Dividing the two equations eliminates v and ω: $x(ML/2 + mx) = ML^2/3 + mx^2$, so the mx^2 terms cancel and $x = 2L/3 = 0.800$ m whatever the value of m

Candidates who write the linear-momentum condition with the rod alone, mv = (ML/2)ω, lose the cancellation and find an x that depends on m. The point 2L/3 is the rod's centre of percussion, the sweet spot of a bat.

16

CORE COURSE

DATA ANALYSIS · DEMANDING · UPI-01.09 · UPI-14.04 · UPI-14.07 **[12]**

A glider of mass 0.250 kg on a level air track is held between two springs and oscillates with a period of 0.800 s. A card sail fixed to the glider increases the air drag. A motion sensor records the amplitude of the oscillation, the greatest displacement from equilibrium, at the end of every fifth oscillation, with an uncertainty of ±0.3 mm in each reading. Two damping models are proposed. Model V (viscous drag): the resistive force is −bv, so the amplitude decays as A = A₀e^(−bt/(2m)). Model F (sliding friction): a resistive force of constant magnitude acts, which reduces the amplitude by the same amount in every oscillation.

(a) DESCRIBE · U1 **[2]**

Describe the pattern that each model predicts in amplitudes recorded at equal intervals of time.

1. Model V: each amplitude is the same fixed fraction of the one before, so successive ratios are constant and ln A falls linearly with t
2. Model F: each amplitude is smaller than the one before by the same fixed amount, so successive differences are constant and A falls linearly with t, reaching zero at a finite time

Many candidates describe both models as 'the amplitude decreases', which is true of both and distinguishes nothing. The marks are for naming the quantity that stays constant.

(b) JUSTIFY · U4 **[3]**

Justify, using calculations from the table, which model the data support.

1. Successive differences are 22.4, 16.8, 11.2, 8.7, 5.6 and 4.5 mm: they fall steadily by a factor of five, far beyond the ±0.3 mm reading uncertainty, so Model F is rejected
2. Successive ratios are 0.720, 0.708, 0.725, 0.706, 0.732 and 0.706: constant at about 0.72, with a scatter of about ±2% that the ±0.3 mm reading uncertainty can account for (combining fractional uncertainties in quadrature, it is ±2.4% in 15.3/20.9 and ±3.4% in 10.8/15.3)
3. Concludes that the data support Model V, viscous drag

An answer that says only that A against t is curved earns part of the credit: curvature rules out Model F but does not by itself show that the decay is exponential. The ratio test, set against the reading uncertainty, does both.

(c) DETERMINE · U4 **[3]**

Determine the damping constant b by a method that uses all of the readings.

1. Uses $\ln A = \ln A_0 - (b/2m)t$: tabulates ln(A/mm) = 4.382, 4.054, 3.709, 3.388, 3.040, 2.728, 2.380 and plots or fits ln A against t
2. Best-fit gradient = $-0.0833\ s^{-1}$ (accept -0.080 to $-0.086\ s^{-1}$)
3. $b = -2m \times \text{gradient} = 2 \times 0.250\ kg \times 0.0833\ s^{-1} = 0.0416\ kg/s$ (accept 0.040 kg/s to 0.043 kg/s)

A single pair of adjacent readings gives b anywhere between 0.039 kg/s and 0.044 kg/s depending on the pair chosen, which is why the question asks for a method that uses every reading. Forgetting the factor of 2 in b/(2m) halves the answer.

(d) CALCULATE · U3 **[2]**

Calculate the fraction of the glider's mechanical energy that is lost during one oscillation.

1 Mechanical energy is proportional to A^2, so over one period it falls by the factor $e^{(-bT/m)} = e^{(-0.0416 \times 0.800/0.250)} = e^{(-0.133)} = 0.875$

2 Fraction lost per oscillation = $1 - 0.875 = 0.125$, about 12.5% (accept 12% to 13%)

Candidates who apply the amplitude ratio directly, $1 - e^{(-bT/(2m))} = 6.4\%$, have forgotten that the energy of an oscillator scales as the square of its amplitude.

(e) PREDICT · U1 **[2]**

A 0.250 kg brass block is fixed to the glider, which keeps the same sail and the same springs. Predict the time for the amplitude to fall to half its initial value, before and after the block is added, and give the reason for the change.

1 Before: $t_{½} = 2m \ln 2/b = (2 \times 0.250 \times 0.693)/0.0416 = 8.33$ s

2 The sail, and so b, is unchanged while m doubles, so the decay constant $b/(2m)$ halves and $t_{½}$ doubles to 16.7 s (the period also lengthens, by $\sqrt{2}$ to 1.13 s, but that does not alter the decay time)

The usual prediction is that a heavier glider is damped faster because more mass means more friction. That reasoning belongs to Model F, which the data rejected; under viscous drag the resistive force depends on speed, not on weight, so extra inertia makes the decay slower.

17

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION: MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-11.06 · UPI-13.04 · UPI-13.05 · UPI-13.06 **[15]**

A communications satellite is raised from a circular parking orbit of radius $r_1 = 6.78 \times 10^6$ m to a circular orbit of radius $r_2 = 4.22 \times 10^7$ m by a two-burn transfer. A short engine burn at radius r_1 places it on an elliptical transfer orbit whose nearest point to Earth's centre (perigee) is at r_1 and whose farthest point (apogee) is at r_2. A second short burn at apogee makes the orbit circular. For Earth, $GM = 3.986 \times 10^{14}$ m^3/s^2. Treat both burns as instantaneous, treat Earth as a spherically symmetric mass, and ignore every other body.

(a) DERIVE · U2 **[2]**

Derive, from Newton's law of gravitation and Newton's second law, an expression for the speed v of a satellite in a circular orbit of radius r.

1 The gravitational force provides the centripetal acceleration: $GMm/r^2 = mv^2/r$

2 $v = \sqrt{(GM/r)}$, which is independent of the satellite's mass

Candidates who quote $v = \sqrt{(2GM/r)}$ have used the escape condition, which sets the kinetic energy equal to the depth of the potential well rather than balancing force against acceleration; it overstates the orbital speed by a factor of $\sqrt{2}$.

(b) EXPLAIN · U1 **[2]**

Explain why the satellite's angular momentum about Earth's centre is constant along the transfer orbit, and why its magnitude at perigee and at apogee is simply mrv.

1 The gravitational force is central, directed along the line to Earth's centre, so its torque $r \times F$ about the centre is zero and the angular momentum $L = r \times mv$ is constant between the burns

2 At perigee and apogee r is at its minimum or maximum, so $dr/dt = 0$ and the velocity is perpendicular to the radius vector; there $|r \times mv| = mrv \sin 90° = mrv$, which is not true elsewhere on the ellipse

Candidates who write $L = mrv$ at a general point of the ellipse set up equations that hold only at its two ends. The perpendicularity at perigee and apogee is what makes part (c) tractable.

(c) DERIVE · U2 **[4]**

Using conservation of angular momentum and conservation of mechanical energy between perigee and apogee, derive an expression for the speed v_p at perigee on the transfer orbit, in terms of GM, r_1 and r_2.

1. Angular momentum: $r_1 v_p = r_2 v_a$, so $v_a = v_p\, r_1/r_2$
2. Energy: $\tfrac{1}{2}mv_p^2 - GMm/r_1 = \tfrac{1}{2}mv_a^2 - GMm/r_2$
3. Substituting for v_a: $\tfrac{1}{2}v_p^2(1 - r_1^2/r_2^2) = GM(1/r_1 - 1/r_2) = GM(r_2 - r_1)/(r_1 r_2)$
4. Using $1 - r_1^2/r_2^2 = (r_2 - r_1)(r_2 + r_1)/r_2^2$ and cancelling $(r_2 - r_1)$ gives $v_p = \sqrt{(2GMr_2/[r_1(r_1 + r_2)])}$

Candidates who use the near-Earth form mgh for the potential energy cannot reach the result: g is far from constant over a sixfold change in radius, and the −GMm/r form is required.

(d) SHOW THAT · U1 **[2]**

Show that your expression reduces to the circular-orbit speed at r_1 when $r_2 = r_1$, and to the escape speed from r_1 as $r_2 \to \infty$, and state what each limit means physically.

1. $r_2 = r_1$ gives $v_p = \sqrt{(2GMr_1/(2r_1^2))} = \sqrt{(GM/r_1)}$: an ellipse whose perigee and apogee coincide is the circular orbit, and no burn is needed
2. As $r_2 \to \infty$, $r_2/(r_1 + r_2) \to 1$ and $v_p \to \sqrt{(2GM/r_1)}$: an ellipse whose apogee recedes without limit becomes the escape trajectory, with zero total mechanical energy

Limit checks are where an algebraic slip in part (c) reveals itself. A candidate whose expression fails either limit should return to part (c) rather than carry the error into the numbers.

(e) CALCULATE · U3 **[3]**

Calculate the change in speed required at each burn, and the total.

1. First burn: $v_1 = \sqrt{(GM/r_1)} = 7.67 \times 10^3$ m/s and $v_p = 1.007 \times 10^4$ m/s, so $\Delta v_1 = 2.40 \times 10^3$ m/s
2. Second burn: $v_a = v_p\, r_1/r_2 = 1.62 \times 10^3$ m/s and $v_2 = \sqrt{(GM/r_2)} = 3.07 \times 10^3$ m/s, so $\Delta v_2 = 1.46 \times 10^3$ m/s
3. Total change in speed = 3.85×10^3 m/s

The satellite speeds up at both burns yet ends at 3.07×10^3 m/s, well under half its initial 7.67×10^3 m/s. Candidates who find this contradictory have not separated the burns from the coast between them, during which the climb converts kinetic energy into gravitational potential energy.

(f) CALCULATE · U3 **[2]**

Calculate the time the satellite spends on the transfer orbit, from perigee to apogee.

1. The semi-major axis is $a = (r_1 + r_2)/2 = 2.45 \times 10^7$ m, and Kepler's third law gives the period $T = 2\pi\sqrt{(a^3/GM)} = 3.81 \times 10^4$ s
2. The transfer is half an orbit: $T/2 = 1.91 \times 10^4$ s, about 5.3 hours

Using r_2 in Kepler's third law in place of the semi-major axis gives the period of the final orbit, 8.63×10^4 s, and a transfer time more than twice too long.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION: FROM A CONSERVATION LAW TO THE EQUATION OF MOTION · DISCRIMINATING · UPI-09.07 · UPI-10.08 · UPI-14.02 · UPI-14.06 **[15]**

A seismic isolation bearing is modelled as a uniform solid cylinder of mass m and radius r that rolls without slipping on the inside of a fixed, concave cylindrical track of radius R, where R > r and both axes are horizontal and parallel. The moment of inertia of the cylinder about its own axis is $\tfrac{1}{2}mr^2$. Let φ be the angle between the vertical and the line from the track's axis of curvature to the cylinder's axis, so that $\varphi = 0$ at the lowest position. Take $g = 9.81$ m/s², and ignore rolling resistance and air resistance.

(a) EXPLAIN · U1 **[2]**

Explain why the speed of the cylinder's axis is $(R - r)|d\varphi/dt|$, and use the rolling condition to obtain the cylinder's angular speed ω about its own axis in terms of $d\varphi/dt$.

1. The cylinder's axis stays a fixed distance $R - r$ from the track's axis of curvature, so it moves on a circle of that radius with speed $v = (R - r)|d\varphi/dt|$
2. Rolling without slipping means the point of the cylinder in contact with the fixed track is instantaneously at rest, so $v = r\omega$ and $\omega = (R - r)|d\varphi/dt|/r$

Candidates who take the axis to move on a circle of radius R, or who set $\omega = d\varphi/dt$, build the wrong kinetic energy into part (b) and cannot recover. The radius of the path is $R - r$, and the spin rate is $(R - r)/r$ times the rate at which the line to the axis turns, not equal to it.

(b) DERIVE · U2 **[5]**

Derive, from conservation of mechanical energy, the equation of motion for φ.

1. Gravitational potential energy measured from the lowest position: $U = mg(R - r)(1 - \cos\varphi)$
2. Kinetic energy is translational plus rotational: $K = \tfrac{1}{2}m(R - r)^2(d\varphi/dt)^2 + \tfrac{1}{2}(\tfrac{1}{2}mr^2)\omega^2$
3. Substituting $\omega = (R - r)(d\varphi/dt)/r$ gives $K = \tfrac{3}{4}m(R - r)^2(d\varphi/dt)^2$
4. Sets the time derivative of $K + U$ to zero: $(3/2)m(R - r)^2(d\varphi/dt)(d^2\varphi/dt^2) + mg(R - r)\sin\varphi\,(d\varphi/dt) = 0$
5. Divides by $d\varphi/dt$, which is non-zero during the motion, to obtain $d^2\varphi/dt^2 = -[2g/(3(R - r))]\sin\varphi$

Leaving out the rotational kinetic energy, the error this part is set to catch, gives $d^2\varphi/dt^2 = -[g/(R - r)]\sin\varphi$, the equation for a particle sliding on a frictionless track. Static friction does no work on a cylinder that rolls without slipping, which is why energy conservation is legitimate here even though friction acts.

(c) SHOW THAT · U2 **[2]**

Show that, for small oscillations, the motion is simple harmonic with period $T = 2\pi\sqrt{(3(R - r)/(2g))}$.

1. For small φ in radians $\sin\varphi \approx \varphi$, so $d^2\varphi/dt^2 = -[2g/(3(R - r))]\varphi$, which has the form of the simple harmonic equation $d^2\varphi/dt^2 = -\omega_0^2\varphi$
2. $\omega_0 = \sqrt{(2g/(3(R - r)))}$, so $T = 2\pi/\omega_0 = 2\pi\sqrt{(3(R - r)/(2g))}$

Confusing ω_0, the angular frequency of the oscillation, with ω, the spin rate of the cylinder, is the error to avoid: the two symbols describe different motions of the same body.

(d) COMPARE · U1 **[2]**

Compare this period with that of a small block sliding without friction whose centre of mass moves on the same circle of radius $R - r$, and explain the difference.

1. The sliding block is a simple pendulum of length $R - r$ with $T = 2\pi\sqrt{((R - r)/g)}$; the rolling cylinder's period is longer by a factor of $\sqrt{(3/2)} \approx 1.22$
2. In both cases the component of gravity along the path, $mg\sin\varphi$, drives the motion, but the rolling cylinder must also gain rotational kinetic energy: its effective inertia is $3m/2$ rather than m, so it accelerates along the path at only two-thirds the rate

Candidates who say the cylinder is slower because friction dissipates energy reach the right direction for the wrong reason. Static friction does no work on a cylinder that rolls without slipping: it slows the translation only by supplying the torque that spins the cylinder up, so it diverts energy into rotation without removing any, and that is the whole of the effect.

(e) CALCULATE · U3 **[2]**

The engineers want an isolation period of 2.50 s. Calculate the value of $R - r$ required.

1. Rearranges $T = 2\pi\sqrt{(3(R - r)/(2g))}$ to $R - r = gT^2/(6\pi^2)$
2. $R - r = (9.81 \times 2.50^2)/(6\pi^2) = 1.04$ m

Candidates who use the simple-pendulum period here obtain $R - r = 1.55$ m, a bearing 50% larger than the specification requires.

(f) PREDICT · U1 **[2]**

Predict the period of the bearing if the solid cylinder is replaced by a thin-walled tube of the same mass and outer radius, whose moment of inertia about its own axis is mr^2, with $R - r$ unchanged, and give the reason.

1 For a thin-walled tube $I = mr^2$, so $K = m(R - r)^2(d\varphi/dt)^2$ and $\omega_0^2 = g/(2(R - r))$, giving $T = 2\pi\sqrt{(2(R - r)/g)}$

2 $T = 2.50\ \text{s} \times \sqrt{(4/3)} = 2.89\ \text{s}$: a larger share of the energy is rotational, so the effective inertia rises to $2m$ and the oscillation is slower

Neither the mass nor the radius of the rolling body appears in any period in this question except through $R - r$; only the shape factor $I/(mr^2)$ survives. Candidates who expect a heavier roller to change the period should notice that m has cancelled.

University Physics II · Midterm Examination

Electrostatics through magnetic forces: charge and Coulomb's law, electric fields and Gauss's law, electric potential, capacitance and dielectrics, DC and multi-loop circuits with RC transients, and the Lorentz force on moving charges. Most marks sit on setting up a field, circuit or force model before calculating, and on testing every result against symmetry, conservation and limiting cases.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**72**	**13**	**8**

SECTION	**UPII · Midterm · University Physics II — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Routine 6 · Demanding 6 · Discriminating 1** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UPII-01 · UPII-02 · UPII-03 · UPII-04 · UPII-05 · UPII-06 · UPII-07 · UPII-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Physics II · Midterm Examination

Answer all questions. 72 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-01.01 · UPII-01.02 · UPII-01.03 **[1]**

A neutral solid metal sphere stands on an insulating support. A glass rod carrying positive charge is held close to the left-hand side of the sphere without touching it. While the rod is held in place, a wire connected to earth briefly touches the right-hand side of the sphere, the side farther from the rod, and is then removed. Finally the rod is taken away. What is the final charge on the sphere?

A Positive, the same sign as the rod, spread uniformly over the surface

B Zero, because the earth connection was broken before the rod was removed

C Negative, concentrated on the left-hand side where the rod had been

D Negative, spread uniformly over the surface

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-02.02 · UPII-02.03 **[1]**

A point charge $+9Q$ is fixed at $x = 0$ and a point charge $-4Q$ is fixed at $x = L$ on the x-axis, where $Q > 0$. Apart from points infinitely far away, where is the electric field zero?

A $x = 3L$

B $x = 1.8L$

C $x = 0.6L$

D Nowhere: the field is non-zero at every point a finite distance from the charges

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-03.01 · UPII-03.02 · UPII-03.08 **[1]**

A very long, straight, thin line of charge with uniform linear charge density λ passes perpendicularly through the centres of two opposite faces of an imaginary cube of edge length a. What is the electric flux through one of the four faces that the line does not pass through?

A $\lambda a/(6\varepsilon_0)$

B $\lambda a/(4\varepsilon_0)$

C $\lambda a/(\pi\varepsilon_0)$

D $\lambda a/\varepsilon_0$

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-02.03 · UPII-04.04 **[1]**

Four point charges sit at the corners of a square of side a centred on the origin in the xy-plane: +q at (−a/2, a/2) and at (a/2, a/2), and −q at (−a/2, −a/2) and at (a/2, −a/2), with q > 0. Take the electric potential to be zero infinitely far away and k = 1/(4πε₀). Which statement describes the electric potential V and the electric field E at the origin?

A V = 0 and E = 0

B V > 0 and E = 0

C V = 0, and E has magnitude $4\sqrt{2}kq/a^2$ in the +y direction

D V = 0, and E has magnitude $4\sqrt{2}kq/a^2$ in the −y direction

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-05.02 **[1]**

Two initially uncharged capacitors, of capacitance 2.0 μF and 6.0 μF, are connected in series across a 12 V battery and allowed to charge fully. What is the potential difference across the 6.0 μF capacitor?

A 3.0 V

B 6.0 V

C 9.0 V

D 12 V

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-06.03 · UPII-06.05 **[1]**

Wire X and wire Y are made of the same metal, are at the same temperature, and are joined end to end so that they carry the same steady current. The diameter of X is twice the diameter of Y. What are the ratio of the drift speed in X to that in Y, v_X/v_Y, and the ratio of the electric field inside X to that inside Y, E_X/E_Y?

A v_X/v_Y = 1/2 and E_X/E_Y = 1/2

B v_X/v_Y = 1/4 and E_X/E_Y = 1/4

C v_X/v_Y = 1/4 and E_X/E_Y = 1

D v_X/v_Y = 1 and E_X/E_Y = 1/4

Answer

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-07.06 · UPII-07.07 **[1]**

A charged capacitor discharges through a fixed resistor. The potential difference across the capacitor falls to half its initial value in 3.0 s. How long after the discharge begins has the energy stored in the capacitor fallen to one-eighth of its initial value?

A 1.5 s

B 3.0 s

C 4.5 s

D 9.0 s

Answer

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-08.03 · UPII-08.05 **[1]**

In a mass spectrometer, ions start from rest, are accelerated through the same potential difference, and then enter a uniform magnetic field perpendicular to their velocity, in which they move on circular arcs. Ion X is $^{4}He^{+}$ (charge +e, mass 4u) and ion Y is $^{16}O^{2+}$ (charge +2e, mass 16u), where u is the atomic mass unit. What is the ratio of the radius of Y's path to the radius of X's path?

A 1/2

B 1

C √2

D 2

Answer

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9 CORE COURSE

STRUCTURED · ROUTINE · UPII-01.01 · UPII-01.03 · UPII-01.04 · UPII-01.05 · UPII-04.01 · UPII-04.03 · UPII-04.04 **[10]**

Three small charged beads are fixed on a horizontal insulating board, which defines the xy-plane. Bead 1 carries +4.0 nC at the origin, bead 2 carries −2.0 nC at (0.30 m, 0) and bead 3 carries +3.0 nC at (0, 0.40 m). Point P is at (0.30 m, 0.40 m). Bead 2 was initially neutral and gained its charge by being rubbed with a cloth that was also initially neutral, and no charge escaped to the surroundings. A fourth bead carrying +1.0 nC is available for part (d), and in part (e) bead 2 is replaced by a bead carrying +2.0 nC at the same position. Treat all beads as point charges and ignore the surroundings. Take $k = 1/(4\pi\varepsilon_0) = 8.99 \times 10^9$ N m² C⁻² and $e = 1.60 \times 10^{-19}$ C.

(a) STATE **[1]**

State the charge on the cloth after the rubbing.

(b) CALCULATE **[1]**

Calculate the number of electrons transferred during the rubbing, and state whether they moved from the cloth to the bead or from the bead to the cloth.

(c) CALCULATE **[4]**

Calculate the magnitude and direction of the net electrostatic force on bead 1.

(d) DETERMINE **[2]**

Determine the electric potential at P, and the work an external agent must do to bring the +1.0 nC bead slowly from a very large distance to rest at P.

(e) COMPARE **[2]**

Bead 2 is replaced by the +2.0 nC bead. Compare the net force on bead 1 in the original and the new arrangements, and state one measurement at P that would distinguish the two arrangements.

10 CORE COURSE

STRUCTURED · DEMANDING · UPII-03.03 · UPII-03.05 · UPII-03.07 · UPII-05.06 · UPII-05.07 · UPII-05.08 **[13]**

A length of high-voltage coaxial cable has a solid inner conductor of radius a = 1.0 mm and a thin outer conducting sheath of radius b = 4.0 mm on the same axis. The space between them is filled with polyethylene of dielectric constant $\kappa = 2.3$, which breaks down when the electric field in it reaches $E_{bd} = 2.0 \times 10^{7}$ V m^{-1}. The inner conductor carries free charge $+\lambda$ per unit length and the sheath carries $-\lambda$ per unit length. The cable is long enough for end effects to be neglected. A redesign with a thinner inner conductor, of radius 0.50 mm and the same sheath, is under consideration. Take $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}.

(a) DERIVE **[3]**

Using Gauss's law, derive an expression for the magnitude of the electric field in the polyethylene at a distance r from the axis, where a < r < b. Justify your choice of Gaussian surface.

(b) SKETCH **[2]**

Sketch a graph of the magnitude of the electric field against r from r = 0 to r = 2b, marking on the r-axis the radii at which the graph changes form.

(c) SHOW THAT **[2]**

Show that the capacitance per unit length of the cable is $C/L = 2\pi\kappa\varepsilon_0/\ln(b/a)$.

(d) DETERMINE **[3]**

Determine the largest potential difference that can be applied between the conductors of the present design without breakdown of the polyethylene.

(e) EVALUATE **[3]**

The proposed redesign thins the inner conductor to 0.50 mm on the grounds that a wider gap needs a larger potential difference to break down. Evaluate this proposal, and determine the inner radius that gives the highest voltage rating for b = 4.0 mm.

11 CORE COURSE

STRUCTURED · DEMANDING · UPII-06.06 · UPII-06.08 · UPII-07.01 · UPII-07.02 · UPII-07.03 · UPII-07.06 · UPII-07.08 **[13]**

A 12.0 V supply of internal resistance $r_1 = 0.50\ \Omega$ and a rechargeable 9.0 V cell of internal resistance $r_2 = 1.5\ \Omega$ are connected in parallel across a load of resistance $R_L = 4.0\ \Omega$. Both positive terminals are joined to node X, both negative terminals are joined to node Y, and the load connects X to Y. The connecting wires have negligible resistance, and the emfs and internal resistances stay constant. In parts (d) and (e) an initially uncharged 0.47 F capacitor, with a manufacturing tolerance of ±20%, is also connected directly between X and Y, and the design specification requires the load voltage to stay above 5.0 V for at least 1.0 s after both sources are disconnected.

(a) DETERMINE **[4]**

Using Kirchhoff's rules, determine the current in the supply, in the cell and in the load, stating the direction of each, and the potential difference V_{XY} between X and Y.

(b) SHOW THAT **[2]**

Show that the power delivered by the emf of the 12.0 V supply, 41.1 W, equals the total rate of energy transfer in the two internal resistances, the load and the cell.

(c) PREDICT **[2]**

Without the capacitor, the load resistance is gradually reduced from 4.0 Ω. Predict the load resistance at which the current in the 9.0 V cell becomes zero, and what the cell does when the load resistance is smaller still.

(d) CALCULATE **[3]**

With the load at 4.0 Ω, the 0.47 F capacitor is connected between X and Y and the circuit is left for a long time. At $t = 0$ both the supply and the cell are disconnected, leaving only the capacitor and the load. Calculate the time at which the potential difference across the load has fallen to 5.0 V.

(e) EVALUATE **[2]**

Evaluate whether every capacitor within the ±20% tolerance meets the specification that the load voltage stays above 5.0 V for at least 1.0 s after disconnection.

12 CORE COURSE

DATA ANALYSIS · DEMANDING · UPII-06.03 · UPII-08.02 · UPII-08.08 **[11]**

A thin rectangular strip of doped semiconductor lies in the xy-plane. It is 20 mm long in the x-direction, w = 5.0 mm wide in the y-direction and t = 0.50 mm thick in the z-direction. A steady current I = 4.0 mA flows through it in the +x direction. A uniform magnetic field of adjustable magnitude is applied along the z-axis, and a high-resistance voltmeter connected to contacts on the two long edges records the Hall voltage $V_H = V(y = w) - V(y = 0)$. A positive value of B means the field points in the +z direction; the final reading was taken with the field reversed. Take $e = 1.60 \times 10^{-19}$ C. For part (e), copper has density 8.96×10^{3} kg m^{-3} and molar mass 63.5 g mol^{-1}, and $N_A = 6.02 \times 10^{23}$ mol^{-1}.

Hall voltage readings at I = 4.0 mA (the last row was taken with the field reversed)

B / T	V_H / mV
0.00	0.42
0.10	2.95
0.20	5.40
0.30	7.95
0.40	10.39
0.50	12.92
0.60	15.40
−0.50	−12.08

(a) JUSTIFY **[2]**

Justify, from the sign of the readings for positive B, whether the charge carriers in the strip are positive or negative.

(b) EXPLAIN **[2]**

The reading at B = 0 is not zero. Explain how this offset can arise, and use the readings at B = +0.50 T and B = −0.50 T to obtain the Hall voltage at 0.50 T with the offset removed.

(c) DETERMINE **[3]**

Using the seven readings with $B \geq 0$, determine the gradient of a graph of V_H against B.

(d) CALCULATE **[2]**

Calculate the number density of charge carriers in the semiconductor.

(e) ESTIMATE **[2]**

Estimate the Hall voltage that the same current and a field of 0.50 T would produce in a copper strip of the same dimensions, stating your assumption about the number of conduction electrons, and say what your estimate implies for the choice of material in a Hall probe.

13

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPII-01.06 · UPII-02.05 · UPII-03.02 · UPII-04.06 · UPII-04.07 **[17]**

A thin flat annulus, shaped like a washer, of inner radius a and outer radius b lies in the xy-plane centred on the origin and carries a uniform surface charge density $\sigma > 0$. The z-axis passes through the centre of the hole, perpendicular to the annulus. Take the electric potential to be zero infinitely far away and neglect gravity. For part (e), the particle is an electron of charge magnitude $q = 1.60 \times 10^{-19}$ C and mass $m = 9.11 \times 10^{-31}$ kg, with $a = 2.0$ cm, $b = 6.0$ cm, $\sigma = 1.0 \times 10^{-9}$ C m^{-2} and $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}.

(a) DERIVE **[4]**

Starting from the potential of a point charge, derive an expression for the electric potential $V(z)$ at a point on the z-axis.

(b) DETERMINE **[2]**

Determine the z-component of the electric field on the axis from your answer to (a), and explain why the potential on the axis alone is enough to give the complete field there.

(c) SHOW THAT **[3]**

Show that for $|z| \gg b$ the field on the axis reduces to that of a point charge $Q = \pi\sigma(b^2 - a^2)$, and state the field obtained in the limit $a \to 0$, $b \to \infty$, naming the charge distribution this limit describes.

(d) DERIVE **[3]**

A particle of charge $-q$ and mass m is released from rest on the axis at $z = z_0$, where $0 < z_0 \ll a$. Using Newton's second law, derive the equation of motion for small z, show that the motion along the axis is simple harmonic, and derive its angular frequency ω.

(e) CALCULATE **[2]**

Calculate the period of the oscillation for the electron described in the stem.

(f) PREDICT **[3]**

In practice the electron cannot be released exactly on the axis. By applying Gauss's law to a small closed cylinder of radius ρ and length $2h$, centred on the origin with its axis along z and with ρ and h both much smaller than a, predict the radial component of the electric field at a small distance ρ from the axis near the centre, and hence predict whether the oscillation in (d) survives a small sideways displacement.

MARK SCHEME

University Physics II · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-01.01 · UPII-01.02 · UPII-01.03 **[1]**

A neutral solid metal sphere stands on an insulating support. A glass rod carrying positive charge is held close to the left-hand side of the sphere without touching it. While the rod is held in place, a wire connected to earth briefly touches the right-hand side of the sphere, the side farther from the rod, and is then removed. Finally the rod is taken away. What is the final charge on the sphere?

KEY **D** **Negative, spread uniformly over the surface**

A is tempting because charging by contact does leave an object with the same sign as its source, but the rod never touches the sphere. While the wire touches the sphere, the sphere and the earth form one conductor, and electrons are drawn up the wire towards the positive rod, so the sphere ends with an excess of electrons and a charge opposite in sign to the rod's. B would be right only if the last two steps were reversed: removing the rod first lets the extra electrons flow back to earth. Here the wire is removed while the rod still holds them on the sphere, so they are trapped. C has the right sign but keeps the distribution the rod induced. Once the rod has gone nothing holds the excess charge on one side, and on an isolated conducting sphere it spreads uniformly over the outer surface. The point where the earth wire touches is irrelevant: touching the far side, where induced positive charge sat, still lets electrons flow onto the sphere.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-02.02 · UPII-02.03 **[1]**

A point charge +9Q is fixed at x = 0 and a point charge −4Q is fixed at x = L on the x-axis, where Q > 0. Apart from points infinitely far away, where is the electric field zero?

KEY **A** **x = 3L**

C is the null point for two like charges, where the fields oppose each other between the charges. Here the charges have opposite signs, so between them both fields point in the +x direction and add; any cancellation must lie outside the pair, on the side of the smaller-magnitude charge, where its greater nearness can compensate for its smaller size. Setting $9Q/x^2 = 4Q/(x - L)^2$ for $x > L$ gives $3(x - L) = 2x$, so $x = 3L$. B comes from writing the same balance with $1/r$ instead of $1/r^2$, $9/x = 4/(x - L)$: the right region with the wrong law. D is tempting for anyone who pictures field lines running from the positive to the negative charge and concludes that opposite charges never cancel. Beyond $x = L$ the two fields do point in opposite directions; just beyond L the −4Q field dominates, while far away the +9Q field dominates because $9Q > 4Q$, so the two must be equal somewhere between.

U3

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-03.01 · UPII-03.02 · UPII-03.08 **[1]**

A very long, straight, thin line of charge with uniform linear charge density λ passes perpendicularly through the centres of two opposite faces of an imaginary cube of edge length a. What is the electric flux through one of the four faces that the line does not pass through?

KEY **B** **λa/(4ε₀)**

D is the flux through the whole closed cube, which by Gauss's law is the enclosed charge λa divided by ε₀; the question asks for one face. A shares that total equally among all six faces, which ignores the direction of the field: the field of a long line is radial and perpendicular to the line, so it is parallel to the two faces the line pierces and carries no flux through them. All of λa/ε₀ therefore passes through the other four faces, and the fourfold rotational symmetry about the line makes their shares equal. C is the most careful-looking wrong answer. It takes the field at the centre of a face, λ/(2πε₀(a/2)) = λ/(πε₀a), and multiplies it by the area a², as if that field were uniform and normal over the whole face. Away from the centre the field is both weaker and tilted from the normal, so this overestimates. The cube is not a surface on which Gauss's law gives the field, yet it still gives the flux exactly.

U1

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-02.03 · UPII-04.04 **[1]**

Four point charges sit at the corners of a square of side a centred on the origin in the xy-plane: +q at (−a/2, a/2) and at (a/2, a/2), and −q at (−a/2, −a/2) and at (a/2, −a/2), with q > 0. Take the electric potential to be zero infinitely far away and k = 1/(4πε₀). Which statement describes the electric potential V and the electric field E at the origin?

KEY **D** **V = 0, and E has magnitude 4√2kq/a² in the −y direction**

A treats the field as if it were the potential. Potential is a scalar, and the four contributions ±kq/(a/√2) cancel in pairs, so V = 0; the field is a vector, and here its contributions do not cancel. Each positive charge pushes a test charge at the origin away from itself and each negative charge pulls it towards itself, so all four fields have a component in the −y direction while their x-components cancel. Each field has magnitude kq/(a²/2) = 2kq/a² and a y-component of magnitude 2kq/(√2a²), and four of these give 4√2kq/a². A would be correct for the alternating arrangement with like charges on opposite corners, which is why it is tempting. C has the right magnitude but reverses the direction: the field at the centre points from the positive pair towards the negative pair. B comes from adding kq/r for every charge as if potential could not be negative, and then assuming a symmetric arrangement must cancel the field.

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-05.02 **[1]**

Two initially uncharged capacitors, of capacitance 2.0 μF and 6.0 μF, are connected in series across a 12 V battery and allowed to charge fully. What is the potential difference across the 6.0 μF capacitor?

KEY **A** **3.0 V**

Because both capacitors start uncharged and the two plates joined between them form an isolated conductor, the capacitors carry equal charge. The series combination is (2.0 × 6.0)/(2.0 + 6.0) μF = 1.5 μF, so the charge is 18 μC and the potential difference across the 6.0 μF capacitor is 18 μC/6.0 μF = 3.0 V. C is the potential difference across the 2.0 μF capacitor, and it is chosen by anyone who divides the voltage in proportion to capacitance, as series resistors divide it in proportion to resistance; at equal charge V = Q/C, so the larger capacitor takes the smaller share. B splits the 12 V equally, which is true only for equal capacitances. D is the answer for a parallel connection, in which each capacitor sits directly across the battery.

U3

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-06.03 · UPII-06.05 **[1]**

Wire X and wire Y are made of the same metal, are at the same temperature, and are joined end to end so that they carry the same steady current. The diameter of X is twice the diameter of Y. What are the ratio of the drift speed in X to that in Y, v_X/v_Y, and the ratio of the electric field inside X to that inside Y, E_X/E_Y?

KEY **B** **v_X/v_Y = 1/4 and E_X/E_Y = 1/4**

The current I = nqAv is the same in both wires and so is the carrier density n, because the metal is the same; doubling the diameter quadruples the cross-sectional area A, so the drift speed in X is a quarter of that in Y. The field follows from the microscopic form of Ohm's law, E = ρJ = ρI/A: the same resistivity and current through four times the area give a quarter of the field. A scales both quantities with the diameter rather than with the area. C gets the drift speed right but assumes the field is the same throughout a connected circuit, as if the battery alone set it; the field in each segment is whatever is needed to drive the common current through that segment's resistance per unit length. D assumes carriers in the same metal must drift at the same speed, when the drift speed is set by the current density and not by the material alone.

U1

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-07.06 · UPII-07.07 **[1]**

A charged capacitor discharges through a fixed resistor. The potential difference across the capacitor falls to half its initial value in 3.0 s. How long after the discharge begins has the energy stored in the capacitor fallen to one-eighth of its initial value?

KEY **C** **4.5 s**

The stored energy is U = ½CV², so it decays as e^(−2t/RC), twice as fast as the voltage: it halves every 1.5 s, and one-eighth is three energy halvings, 4.5 s. Equivalently, U = U₀/8 requires V = V₀/√8, and RC ln√8 = 1.5 × RC ln 2 = 1.5 × 3.0 s. D applies the voltage's halving time of 3.0 s to the energy three times over, forgetting that energy goes as the square of the voltage. B is the time at which the energy has fallen to one-quarter, because the voltage has then halved; it answers a different fraction. A is the energy's halving time itself, one of the three steps required, and is chosen by candidates who correctly see that energy decays twice as fast but stop there.

U3

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-08.03 · UPII-08.05 **[1]**

In a mass spectrometer, ions start from rest, are accelerated through the same potential difference, and then enter a uniform magnetic field perpendicular to their velocity, in which they move on circular arcs. Ion X is $^{4}He^{+}$ (charge +e, mass 4u) and ion Y is $^{16}O^{2+}$ (charge +2e, mass 16u), where u is the atomic mass unit. What is the ratio of the radius of Y's path to the radius of X's path?

KEY **C** **√2**

After acceleration through a potential difference V each ion has kinetic energy qV, so its momentum is √(2mqV) and its radius is r = mv/(qB) = √(2mV/q)/B. The radius therefore depends on √(m/q): √(16/2) against √(4/1), a ratio of √2. D is the ratio of m/q, which is the answer if both ions enter the field with the same speed, as they would after a velocity selector; it is the right formula applied to the wrong entry condition. B is the answer for equal kinetic energies, when r ∝ √m/q, but Y's double charge gives it twice the kinetic energy from the same potential difference. A is the answer for equal momenta, when r ∝ 1/q. Each wrong option is the correct result for a different real arrangement, which is why the entry condition has to be read before any formula is chosen.

U2

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9 CORE COURSE

STRUCTURED · ROUTINE · UPII-01.01 · UPII-01.03 · UPII-01.04 · UPII-01.05 · UPII-04.01 · UPII-04.03 · UPII-04.04 **[10]**

Three small charged beads are fixed on a horizontal insulating board, which defines the xy-plane. Bead 1 carries +4.0 nC at the origin, bead 2 carries −2.0 nC at (0.30 m, 0) and bead 3 carries +3.0 nC at (0, 0.40 m). Point P is at (0.30 m, 0.40 m). Bead 2 was initially neutral and gained its charge by being rubbed with a cloth that was also initially neutral, and no charge escaped to the surroundings. A fourth bead carrying +1.0 nC is available for part (d), and in part (e) bead 2 is replaced by a bead carrying +2.0 nC at the same position. Treat all beads as point charges and ignore the surroundings. Take $k = 1/(4\pi\varepsilon_0) = 8.99 \times 10^{9}$ N m² C⁻² and $e = 1.60 \times 10^{-19}$ C.

(a) STATE · U1 **[1]**

State the charge on the cloth after the rubbing.

1 +2.0 nC: the cloth loses exactly the electrons the bead gains, so the charge of cloth and bead together stays zero

An answer of zero or −2.0 nC treats rubbing as creating charge rather than transferring it.

(b) CALCULATE · U3 **[1]**

Calculate the number of electrons transferred during the rubbing, and state whether they moved from the cloth to the bead or from the bead to the cloth.

1 $N = (2.0 \times 10^{-9}\ \text{C})/(1.60 \times 10^{-19}\ \text{C}) = 1.25 \times 10^{10}$ electrons, moving from the cloth to the bead

Candidates who say positive charge moved to the cloth have missed the model used at this level: charging by friction transfers electrons, while the positive charge stays bound in the nuclei.

(c) CALCULATE · U3 **[4]**

Calculate the magnitude and direction of the net electrostatic force on bead 1.

1 Force from bead 2: $(8.99 \times 10^{9})(4.0 \times 10^{-9})(2.0 \times 10^{-9})/(0.30)^2 = 7.99 \times 10^{-7}$ N, attractive, so along +x towards bead 2

2 Force from bead 3: $(8.99 \times 10^{9})(4.0 \times 10^{-9})(3.0 \times 10^{-9})/(0.40)^2 = 6.74 \times 10^{-7}$ N, repulsive, so along −y away from bead 3

3 Combines the perpendicular components: $|F| = \sqrt{[(7.99 \times 10^{-7})^2 + (6.74 \times 10^{-7})^2]} = 1.05 \times 10^{-6}$ N

4 Direction: $\tan^{-1}(6.74/7.99) = 40°$ from the +x axis, rotated towards the −y axis (components $+7.99 \times 10^{-7}$ N along x and -6.74×10^{-7} N along y)

Adding the two magnitudes to get 1.47×10^{-6} N, or fixing directions from the algebraic sign of q_1q_2 instead of from attraction and repulsion, are the errors this part is set to catch.

(d) DETERMINE · U3 **[2]**

Determine the electric potential at P, and the work an external agent must do to bring the +1.0 nC bead slowly from a very large distance to rest at P.

1 Distances from P are 0.50 m, 0.40 m and 0.30 m, so $V_P = (8.99 \times 10^{9})[(4.0/0.50) - (2.0/0.40) + (3.0/0.30)] \times 10^{-9}\ \text{V} = 117$ V

2 $W = qV_P = (1.0 \times 10^{-9}\ \text{C})(117\ \text{V}) = 1.17 \times 10^{-7}$ J, positive work done by the agent

Potential is a signed scalar sum, so the −2.0 nC bead subtracts. Adding magnitudes gives 207 V; resolving potentials into components as if they were vectors earns nothing.

(e) COMPARE · U1 **[2]**

Bead 2 is replaced by the +2.0 nC bead. Compare the net force on bead 1 in the original and the new arrangements, and state one measurement at P that would distinguish the two arrangements.

1 The force from bead 2 reverses to −x with the same size, so the net force on bead 1 keeps its magnitude of 1.05×10^{-6} N but now points at 40° from the −x axis towards −y: same magnitude, different direction

2 The potential at P distinguishes them: 117 V originally against 207 V in the new arrangement (equivalently 1.17×10^{-7} J against 2.07×10^{-7} J of work to bring the +1.0 nC bead to P); accept instead the electric field at P, about 386 N C^{-1} almost exactly along +x originally against 448 N C^{-1} at 31° above the +x axis in the new arrangement

The part checks that candidates see a force magnitude at one point does not fix a charge configuration; a second, independent measurement is needed to separate them.

10 CORE COURSE

STRUCTURED · DEMANDING · UPII-03.03 · UPII-03.05 · UPII-03.07 · UPII-05.06 · UPII-05.07 · UPII-05.08 **[13]**

A length of high-voltage coaxial cable has a solid inner conductor of radius a = 1.0 mm and a thin outer conducting sheath of radius b = 4.0 mm on the same axis. The space between them is filled with polyethylene of dielectric constant κ = 2.3, which breaks down when the electric field in it reaches $E_{bd} = 2.0 \times 10^{7}$ V m^{-1}. The inner conductor carries free charge +λ per unit length and the sheath carries −λ per unit length. The cable is long enough for end effects to be neglected. A redesign with a thinner inner conductor, of radius 0.50 mm and the same sheath, is under consideration. Take $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}.

(a) DERIVE · U2 **[3]**

Using Gauss's law, derive an expression for the magnitude of the electric field in the polyethylene at a distance r from the axis, where a < r < b. Justify your choice of Gaussian surface.

1. The charge distribution is unchanged by translation along the axis and by rotation about it, so E is radial and depends only on r; a coaxial cylinder of radius r and length ℓ therefore has E normal to and constant over its curved surface, and zero flux through its flat ends
2. Gauss's law in the dielectric in terms of free charge, $\oint D \cdot dA = q_{free}$ with $D = \kappa\varepsilon_0 E$ (or with the bound charge included, which reduces the enclosed charge by the factor κ): $\kappa\varepsilon_0 E(2\pi r\ell) = \lambda\ell$
3. $E = \lambda/(2\pi\kappa\varepsilon_0 r)$, directed radially outward

   Two errors are targeted: choosing a spherical surface because Gauss's law 'uses spheres', and dropping κ by applying the vacuum form of the law to the free charge alone.

(b) SKETCH · U1 **[2]**

Sketch a graph of the magnitude of the electric field against r from r = 0 to r = 2b, marking on the r-axis the radii at which the graph changes form.

1. E = 0 for r < a (inside the conducting inner conductor) and E = 0 for r > b (a Gaussian cylinder there encloses zero net charge), with a and b marked on the r-axis
2. At r = a the field jumps from zero to its maximum, then falls along a 1/r curve (concave up, not a straight line) to one-quarter of that maximum at r = b, where it drops to zero

   Curves falling as $1/r^2$, or falling continuously to zero instead of jumping at the conductor surfaces, describe a point charge or a smeared distribution rather than this cable.

(c) SHOW THAT · U2 **[2]**

Show that the capacitance per unit length of the cable is $C/L = 2\pi\kappa\varepsilon_0/\ln(b/a)$.

1. Potential difference between the conductors: $V = \int_a^b E\,dr = (\lambda/(2\pi\kappa\varepsilon_0)) \int_a^b dr/r = (\lambda/(2\pi\kappa\varepsilon_0)) \ln(b/a)$
2. Capacitance per unit length is charge per unit length over potential difference: $C/L = \lambda/V = 2\pi\kappa\varepsilon_0/\ln(b/a)$

   Writing $V = E(b - a)$, the parallel-plate relation, is the error caught here; it does not hold for a field that varies as 1/r.

(d) DETERMINE · U3 **[3]**

Determine the largest potential difference that can be applied between the conductors of the present design without breakdown of the polyethylene.

1. The field varies as 1/r, so it is greatest at r = a, the surface of the inner conductor, and breakdown begins there
2. Eliminating λ between the results of (a) and (c): $E(a) = V/(a \ln(b/a))$, so $V_{max} = E_{bd}\, a \ln(b/a) = (2.0 \times 10^{7}\ \text{V m}^{-1})(1.0 \times 10^{-3}\ \text{m}) \ln 4$
3. $V_{max} = 2.8 \times 10^{4}$ V (27.7 kV)

   Multiplying the breakdown field by the gap, $(2.0 \times 10^{7})(3.0 \times 10^{-3}) = 6.0 \times 10^{4}$ V, assumes a uniform field and more than doubles the true rating.

(e) EVALUATE · U3 **[3]**

The proposed redesign thins the inner conductor to 0.50 mm on the grounds that a wider gap needs a larger potential difference to break down. Evaluate this proposal, and determine the inner radius that gives the highest voltage rating for b = 4.0 mm.

1. With a = 0.50 mm, $V_{max} = (2.0 \times 10^{7})(0.50 \times 10^{-3}) \ln 8 = 2.1 \times 10^{4}$ V, lower than 2.8×10^{4} V: thinning the conductor raises the field at its surface faster than ln(b/a) grows, so the proposal lowers the rating
2. Maximising $V_{max} = E_{bd}\, a \ln(b/a)$ with respect to a: $dV_{max}/da = E_{bd}[\ln(b/a) - 1] = 0$, so $a = b/e = 1.47$ mm
3. The optimum rating is $E_{bd}\, b/e = 2.9 \times 10^{4}$ V, about 6% above the present design; the judgement is to reject the proposal and make the inner conductor slightly thicker, not thinner

The intuition that a wider gap withstands more is right for parallel plates, where the field is uniform, and wrong here, where the field concentrates on the thin conductor. The top answers find the optimum rather than only rejecting the proposal.

11

CORE COURSE

STRUCTURED · DEMANDING · UPII-06.06 · UPII-06.08 · UPII-07.01 · UPII-07.02 · UPII-07.03 · UPII-07.06 · UPII-07.08 **[13]**

A 12.0 V supply of internal resistance $r_1 = 0.50\ \Omega$ and a rechargeable 9.0 V cell of internal resistance $r_2 = 1.5\ \Omega$ are connected in parallel across a load of resistance $R_L = 4.0\ \Omega$. Both positive terminals are joined to node X, both negative terminals are joined to node Y, and the load connects X to Y. The connecting wires have negligible resistance, and the emfs and internal resistances stay constant. In parts (d) and (e) an initially uncharged 0.47 F capacitor, with a manufacturing tolerance of ±20%, is also connected directly between X and Y, and the design specification requires the load voltage to stay above 5.0 V for at least 1.0 s after both sources are disconnected.

(a) DETERMINE · U3 **[4]**

Using Kirchhoff's rules, determine the current in the supply, in the cell and in the load, stating the direction of each, and the potential difference V_{XY} between X and Y.

1. Defines I_1 and I_2 as the currents leaving the positive terminals of the supply and the cell and I_L as the load current from X to Y, and applies the junction rule at X: $I_1 + I_2 = I_L$
2. Loop rule for each branch: $12.0 - 0.50I_1 = V_{XY}$, $9.0 - 1.5I_2 = V_{XY}$ and $V_{XY} = 4.0I_L$ (or an equivalent pair of independent loop equations)
3. Solving $(12.0 - V_{XY})/0.50 + (9.0 - V_{XY})/1.5 = V_{XY}/4.0$ gives $V_{XY} = 10.3$ V (10.29 V)
4. $I_1 = 3.43$ A out of the supply's positive terminal; $I_2 = -0.857$ A, that is 0.857 A into the cell's positive terminal; $I_L = 2.57$ A from X to Y through the load

Candidates who assume every source drives current out of its positive terminal, and then quietly drop the minus sign on I_2, discard the physics of the question: the negative sign is the result.

(b) SHOW THAT · U3 **[2]**

Show that the power delivered by the emf of the 12.0 V supply, 41.1 W, equals the total rate of energy transfer in the two internal resistances, the load and the cell.

1. Dissipation: $I_1^2 r_1 = (3.43)^2(0.50) = 5.88$ W, $I_2^2 r_2 = (0.857)^2(1.5) = 1.10$ W and $I_L^2 R_L = (2.57)^2(4.0) = 26.4$ W, totalling 33.4 W
2. The cell stores $\varepsilon_2 |I_2| = (9.0\ \text{V})(0.857\ \text{A}) = 7.71$ W as chemical energy because current is driven into its positive terminal, and 33.4 W + 7.71 W = 41.1 W = (12.0 V)(3.429 A)

Counting only the I^2R terms leaves 7.7 W unaccounted for; the audit balances only when the cell is recognised as acting as a load.

(c) PREDICT · U1 **[2]**

Without the capacitor, the load resistance is gradually reduced from 4.0 Ω. Predict the load resistance at which the current in the 9.0 V cell becomes zero, and what the cell does when the load resistance is smaller still.

1. Zero current in the cell means no potential drop across r_2, so V_XY = 9.0 V; then I_1 = (12.0 − 9.0)/0.50 = 6.0 A, all of which passes through the load, so R_L = 9.0 V/6.0 A = 1.5 Ω
2. For R_L < 1.5 Ω, V_XY falls below 9.0 V, so current leaves the cell's positive terminal: the cell discharges and shares the load current with the supply (at R_L = 1.0 Ω it delivers 0.55 A)

The part rewards reasoning from the condition V_XY = ε_2 rather than re-solving the whole network for trial resistances.

(d) CALCULATE · U3 **[3]**

With the load at 4.0 Ω, the 0.47 F capacitor is connected between X and Y and the circuit is left for a long time. At t = 0 both the supply and the cell are disconnected, leaving only the capacitor and the load. Calculate the time at which the potential difference across the load has fallen to 5.0 V.

1. In the steady state no current flows into the capacitor, so the currents found in (a) are unchanged and the capacitor is charged to V_0 = V_XY = 10.3 V
2. After disconnection it discharges through the load alone: V = V_0e^(−t/RC) with RC = (4.0 Ω)(0.47 F) = 1.88 s
3. t = RC ln(V_0/V) = (1.88 s) ln(10.29/5.0) = 1.36 s

Two errors are targeted: starting the discharge from the 12.0 V emf instead of the terminal voltage, and including internal resistances in the time constant after the sources have been removed.

(e) EVALUATE · U4 **[2]**

Evaluate whether every capacitor within the ±20% tolerance meets the specification that the load voltage stays above 5.0 V for at least 1.0 s after disconnection.

1. The worst case is the smallest capacitance, 0.80 × 0.47 F = 0.376 F, giving RC = (4.0 Ω)(0.376 F) = 1.504 s; the starting voltage of 10.3 V does not depend on C
2. t = (1.504 s) ln(10.29/5.0) = 1.08 s, which exceeds 1.0 s, so every capacitor in tolerance meets the specification, but with a margin of only about 0.08 s (about 8%), so a lower load resistance or a lower starting voltage could cause failure

Checking the nominal value (1.36 s) and declaring a comfortable pass is the error; a design judgement has to be made at the edge of the tolerance band.

12

CORE COURSE

DATA ANALYSIS · DEMANDING · UPII-06.03 · UPII-08.02 · UPII-08.08 **[11]**

A thin rectangular strip of doped semiconductor lies in the xy-plane. It is 20 mm long in the x-direction, w = 5.0 mm wide in the y-direction and t = 0.50 mm thick in the z-direction. A steady current I = 4.0 mA flows through it in the +x direction. A uniform magnetic field of adjustable magnitude is applied along the z-axis, and a high-resistance voltmeter connected to contacts on the two long edges records the Hall voltage V_H = V(y = w) − V(y = 0). A positive value of B means the field points in the +z direction; the final reading was taken with the field reversed. Take e = 1.60 × 10^{-19} C. For part (e), copper has density 8.96 × 10^3 kg m^{-3} and molar mass 63.5 g mol^{-1}, and N_A = 6.02 × 10^{23} mol^{-1}.

(a) JUSTIFY · U1 **[2]**

Justify, from the sign of the readings for positive B, whether the charge carriers in the strip are positive or negative.

1. For current in +x, positive carriers move in +x and negative carriers in −x; in both cases F = qv × B with B along +z points in −y ($q(v\,\hat{x}) \times (B\,\hat{z}) = -qvB\,\hat{y}$ for positive carriers, $(-e)(-v\,\hat{x}) \times (B\,\hat{z}) = -evB\,\hat{y}$ for electrons), so carriers of either sign accumulate on the y = 0 edge
2. $V_H = V(y = w) - V(y = 0)$ is positive, so the y = 0 edge is at the lower potential and the charge accumulated there is negative: the carriers are electrons and the material is n-type

Applying the right-hand rule to the conventional current alone gives the same deflection for both signs and never reaches the answer; the polarity is informative precisely because the deflection direction does not depend on the carrier sign.

(b) EXPLAIN · U4 **[2]**

The reading at B = 0 is not zero. Explain how this offset can arise, and use the readings at B = +0.50 T and B = −0.50 T to obtain the Hall voltage at 0.50 T with the offset removed.

1. The two edge contacts are not exactly opposite each other, so they also pick up part of the resistive potential drop along the strip; this offset keeps its sign when B is reversed, whereas the Hall voltage reverses sign
2. Half the difference of the two readings removes the offset: $V_H = [12.92 - (-12.08)]/2 = 12.50$ mV (half their sum, 0.42 mV, recovers the offset and agrees with the zero-field reading)

Subtracting the zero-field reading from the +0.50 T reading gives the same 12.50 mV here only because the offset stayed constant through the run; reversing the field measures the offset at the same field magnitude, so it also cancels any part of the contact voltage that changes with the size of B but not with its direction. The part checks that the physical origin of the offset is understood.

(c) DETERMINE · U4 **[3]**

Using the seven readings with B ≥ 0, determine the gradient of a graph of V_H against B.

1. Uses all seven readings through a single best-fit straight line of V_H against B, drawn on a graph or found by least squares, rather than one reading or two adjacent readings
2. Reads the gradient from two widely separated points on that line, for example about (0.00 T, 0.43 mV) and (0.60 T, 15.40 mV), giving (15.40 − 0.43) mV/(0.60 − 0.00) T, or takes the least-squares slope directly
3. Gradient = 25.0 mV T^{-1} = 2.50×10^{-2} V T^{-1} (accept 24.6 to 25.4 mV T^{-1})

Dividing a single reading by its field, 12.92/0.50 = 25.8 mV T^{-1}, builds the offset into the result; the gradient of a line is immune to a constant offset, which is why it is used.

(d) CALCULATE · U3 **[2]**

Calculate the number density of charge carriers in the semiconductor.

1. Balancing the magnetic and Hall-field forces on the carriers gives $V_H = IB/(n\,e\,t)$, so the gradient equals $I/(n\,e\,t)$ and $n = I/(e\,t \times \text{gradient})$
2. $n = (4.0 \times 10^{-3})/[(1.60 \times 10^{-19})(0.50 \times 10^{-3})(2.50 \times 10^{-2})] = 2.0 \times 10^{21}$ m^{-3}

Using the width w instead of the thickness t, the dimension along B, gives a value ten times too small, 2.0×10^{20} m^{-3}.

(e) ESTIMATE · U4 **[2]**

Estimate the Hall voltage that the same current and a field of 0.50 T would produce in a copper strip of the same dimensions, stating your assumption about the number of conduction electrons, and say what your estimate implies for the choice of material in a Hall probe.

1. Assuming one conduction electron per copper atom: $n \approx (8.96 \times 10^{3}\ \text{kg m}^{-3}/0.0635\ \text{kg mol}^{-1})(6.02 \times 10^{23}\ \text{mol}^{-1}) \approx 8.5 \times 10^{28}\ \text{m}^{-3}$
2. $V_H = IB/(n e t) \approx (4.0 \times 10^{-3})(0.50)/[(8.5 \times 10^{28})(1.60 \times 10^{-19})(0.50 \times 10^{-3})] \approx 3 \times 10^{-10}$ V, about 4×10^{7} times smaller than the semiconductor's 12.5 mV and far below its 0.42 mV offset, so Hall probes are made from low-carrier-density semiconductors

   Candidates who expect the better conductor to give the larger signal have the dependence upside down: $V_H \propto 1/n$.

13

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPII-01.06 · UPII-02.05 · UPII-03.02 · UPII-04.06 · UPII-04.07 **[17]**

A thin flat annulus, shaped like a washer, of inner radius a and outer radius b lies in the xy-plane centred on the origin and carries a uniform surface charge density $\sigma > 0$. The z-axis passes through the centre of the hole, perpendicular to the annulus. Take the electric potential to be zero infinitely far away and neglect gravity. For part (e), the particle is an electron of charge magnitude $q = 1.60 \times 10^{-19}$ C and mass $m = 9.11 \times 10^{-31}$ kg, with a = 2.0 cm, b = 6.0 cm, $\sigma = 1.0 \times 10^{-9}$ C m^{-2} and $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}.

(a) DERIVE · U2 **[4]**

Starting from the potential of a point charge, derive an expression for the electric potential V(z) at a point on the z-axis.

1. Divides the annulus into thin rings of radius r and width dr, each carrying $dq = \sigma(2\pi r\,dr)$, all of it at the same distance $\sqrt{(r^2 + z^2)}$ from the field point
2. $dV = dq/(4\pi\varepsilon_0\sqrt{(r^2 + z^2)}) = (\sigma/(2\varepsilon_0))\, r\, dr/\sqrt{(r^2 + z^2)}$, added as scalars with no components to resolve
3. Integrates from r = a to r = b: $\int_a^b r\,dr/\sqrt{(r^2 + z^2)} = \sqrt{(b^2 + z^2)} - \sqrt{(a^2 + z^2)}$
4. $V(z) = (\sigma/(2\varepsilon_0))[\sqrt{(b^2 + z^2)} - \sqrt{(a^2 + z^2)}]$

   Integrating the field instead takes on a vector integral the problem does not need; using r instead of $\sqrt{(r^2 + z^2)}$ for the distance produces a potential that does not depend on z.

(b) DETERMINE · U2 **[2]**

Determine the z-component of the electric field on the axis from your answer to (a), and explain why the potential on the axis alone is enough to give the complete field there.

1. $E_z = -dV/dz = (\sigma z/(2\varepsilon_0))[1/\sqrt{(a^2 + z^2)} - 1/\sqrt{(b^2 + z^2)}]$, positive for $z > 0$, so the field on the axis points away from the annulus
2. V known only along the axis gives only $\partial V/\partial z$, but rotational symmetry about the z-axis forbids any transverse component on the axis, so $E_x = E_y = 0$ there and E_z is the whole field

   Dropping the minus sign in $E = -dV/dz$ gives a field pointing towards a positive distribution; the symmetry argument is needed because a potential known on a line cannot by itself give derivatives across that line.

(c) SHOW THAT · U2 **[3]**

Show that for $|z| \gg b$ the field on the axis reduces to that of a point charge $Q = \pi\sigma(b^2 - a^2)$, and state the field obtained in the limit $a \to 0$, $b \to \infty$, naming the charge distribution this limit describes.

1. For $|z| \gg b$, $1/\sqrt{(a^2 + z^2)} \approx (1/|z|)(1 - a^2/(2z^2))$ and likewise for b, so the bracket in E_z is approximately $(b^2 - a^2)/(2|z|^3)$
2. $E_z \approx \sigma(b^2 - a^2)/(4\varepsilon_0 z^2)$ for $z > 0$, which equals $Q/(4\pi\varepsilon_0 z^2)$ with $Q = \pi\sigma(b^2 - a^2)$, the total charge of the annulus
3. With $a \to 0$ and $b \to \infty$ the bracket tends to $1/|z|$, so $E_z = \sigma/(2\varepsilon_0)$ directed away from the plane on each side and independent of distance: the field of an infinite uniformly charged plane

   The expansion must be carried to second order; keeping only $1/|z|$ from each term makes the two cancel and suggests, wrongly, that the far field vanishes.

(d) DERIVE · U2 **[3]**

A particle of charge $-q$ and mass m is released from rest on the axis at $z = z_0$, where $0 < z_0 \ll a$. Using Newton's second law, derive the equation of motion for small z, show that the motion along the axis is simple harmonic, and derive its angular frequency ω.

1. For $|z| \ll a$, $1/\sqrt{(a^2 + z^2)} \approx 1/a$ and $1/\sqrt{(b^2 + z^2)} \approx 1/b$, so $E_z \approx (\sigma z/(2\varepsilon_0))(1/a - 1/b) = \sigma(b - a)z/(2\varepsilon_0 ab)$
2. Newton's second law with force $-qE_z$: $m\,d^2z/dt^2 = -[q\sigma(b - a)/(2\varepsilon_0 ab)]z$, a restoring force proportional to displacement, so the motion is simple harmonic about the centre
3. $\omega = \sqrt{[q\sigma(b - a)/(2\varepsilon_0 mab)]}$

   Setting $z = 0$ in the prefactor as well as inside the bracket removes the force entirely; the approximation must keep the leading power of z.

(e) CALCULATE · U3 **[2]**

Calculate the period of the oscillation for the electron described in the stem.

1. $\omega^2 = (1.60 \times 10^{-19})(1.0 \times 10^{-9})(0.040)/[2(8.85 \times 10^{-12})(9.11 \times 10^{-31})(0.020)(0.060)] = 3.3 \times 10^{14}\ \mathrm{s^{-2}}$, so $\omega = 1.82 \times 10^{7}\ \mathrm{rad\ s^{-1}}$
2. $T = 2\pi/\omega = 3.5 \times 10^{-7}\ \mathrm{s}$ ($3.45 \times 10^{-7}\ \mathrm{s}$)

   Leaving a and b in centimetres makes ω^2 a hundred times too small and the period ten times too long, since ω^2 scales as 1/length; a dimensional check catches it.

(f) PREDICT · U1 **[3]**

In practice the electron cannot be released exactly on the axis. By applying Gauss's law to a small closed cylinder of radius ρ and length 2h, centred on the origin with its axis along z and with ρ and h both much smaller than a, predict the radial component of the electric field at a small distance ρ from the axis near the centre, and hence predict whether the oscillation in (d) survives a small sideways displacement.

1. The cylinder lies inside the hole and encloses no charge; with $E_z \approx Kz$, where $K = \sigma(b - a)/(2\varepsilon_0 ab)$, the outward flux through the two end caps is $2(Kh)(\pi\rho^2)$
2. Zero total flux requires an equal inward flux through the curved surface: $E_\rho(2\pi\rho)(2h) = -2\pi Kh\rho^2$, so $E_\rho = -K\rho/2$, directed towards the axis
3. The force on the electron, $-qE_\rho$, points away from the axis and grows with ρ, so the centre is unstable sideways: a small transverse displacement grows, the electron drifts off the axis towards the annulus and the axial oscillation is not sustained, as Earnshaw's theorem requires of any electrostatic equilibrium

   This part separates the top grade. The one-dimensional analysis in (d) looks complete, and only a candidate who uses Gauss's law as a constraint on the field, rather than as a way to compute it, sees that stability along the axis forces instability across it.

University Physics II · Final Examination

A cumulative final on calculus-based electricity, magnetism, optics and modern physics, weighted towards sources of magnetic fields, electromagnetic induction, inductance and AC circuits, Maxwell's equations and electromagnetic waves, geometric and physical optics, and the photon and matter-wave evidence of modern physics. The earlier units return through superposed Coulomb forces and stable equilibrium, electric potential, capacitor design, internal resistance, RC transients and charged-particle motion.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**117**	**19**	**10**

SECTION — **UPII · Final · University Physics II — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Routine 6 · Demanding 11 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPII-01 · UPII-02 · UPII-03 · UPII-04 · UPII-05 · UPII-06 · UPII-07 · UPII-08 · UPII-09 · UPII-10 · UPII-11 · UPII-12 · UPII-13 · UPII-14 · UPII-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Physics II · Final Examination

Answer all questions. 117 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-01.04 · UPII-01.05 · UPII-02.03 **[1]**

A thin, smooth insulating ring of radius R is held fixed, and gravity can be neglected. Point charges +Q are fixed at the two ends of one diameter of the ring, and a small bead carrying charge +q, with q > 0, is threaded on the ring so that it can slide freely round it. The bead is placed at a point on the ring a quarter of a turn from each fixed charge. Which statement is correct?

A The electric field of the fixed charges is zero at that point, so the bead would stay there even if the ring were removed.

B The field there is not zero but points radially outwards, at right angles to the ring, and a bead nudged a short way along the ring is pushed back towards that point.

C The field there is not zero but points radially outwards, and a bead nudged a short way along the ring is pushed on towards the nearer fixed charge.

D The bead cannot be in equilibrium at that point, because both fixed charges repel it; only a negatively charged bead could rest there.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-04.06 **[1]**

In a region of space the electric potential is $V(x, y) = \alpha x^2 y - \beta y$, with $\alpha = 20$ V/m³ and $\beta = 30$ V/m. What is the electric field at the point $(x, y) = (1.0\text{ m}, 2.0\text{ m})$?

A $(-80\,\hat{\imath} + 10\,\hat{\jmath})$ V/m

B $(+80\,\hat{\imath} - 10\,\hat{\jmath})$ V/m

C $(-40\,\hat{\imath} + 10\,\hat{\jmath})$ V/m

D $(+10\,\hat{\imath} - 80\,\hat{\jmath})$ V/m

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-06.06 · UPII-06.08 **[1]**

A cordless drill runs from a battery pack. A high-resistance voltmeter across the pack's terminals reads 18.0 V while the drill is switched off and 16.8 V while the drill draws a steady current of 8.0 A. Model the pack as a constant emf in series with a constant internal resistance. At what rate is energy dissipated as heat inside the pack while the drill runs?

A 144 W

B 134 W

C 9.6 W

D 2.16×10^3 W

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-08.02 · UPII-08.03 **[1]**

Electrons leave a small source inside a long solenoid, each moving at 30° to the solenoid's axis. Inside the solenoid the magnetic field is uniform and parallel to the axis, and there is no electric field, so each electron follows a helix whose axis is parallel to the field. What is the ratio of the pitch of the helix, the distance an electron moves parallel to the axis during one revolution, to the radius of the helix?

A $2\pi/\sqrt{3}$

B $\pi\sqrt{3}$

C 2π

D $2\pi\sqrt{3}$

Answer ☐

---

5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-09.05 **[1]**

A long straight hollow cylindrical conductor has inner radius 2.0 cm and outer radius 4.0 cm, and carries a steady current of 12 A distributed uniformly over the metal's cross-section. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A. What is the magnitude of the magnetic field at a distance of 3.0 cm from the axis?

A zero

B 8.0×10^{-5} T

C 4.5×10^{-5} T

D 3.3×10^{-5} T

Answer ☐

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-10.03 · UPII-10.06 **[1]**

A strong cylindrical magnet is dropped down the centre of a long vertical copper pipe and soon falls at a steady speed far below that of free fall. Copper is not ferromagnetic. Which statement correctly explains the motion?

A The magnet magnetises the copper, which then attracts it upwards in the way an iron pipe would.

B Eddy currents flow only in the pipe just below the magnet, where the flux is increasing; above the magnet the flux is decreasing, so no current is induced there.

C Eddy currents circulate in opposite senses in the pipe just below and just above the magnet; each exerts an upward force on the magnet, and the gravitational energy the magnet loses is dissipated in the pipe.

D Eddy currents circulate in the same sense above and below the magnet, and the steady speed is reached once the pipe has warmed enough for its resistance to stop them growing.

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-11.06 · UPII-11.07 **[1]**

A series RLC circuit is driven by a sinusoidal source at its resonant frequency f_0. The source frequency is then raised to $2f_0$ with the source amplitude unchanged. Which statement describes the circuit at $2f_0$? (X_L and X_C are the inductive and capacitive reactances.)

A X_L = 2X_C, and the current leads the source emf

B X_L = 4X_C, and the current lags the source emf

C X_L = 4X_C, and the current leads the source emf

D X_L = X_C still, because L and C have not changed, so the impedance is still R

Answer

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-12.06 · UPII-12.07 **[1]**

The electric field of a plane electromagnetic wave in vacuum is $E = E_0 \sin(kz + \omega t)\,\hat{\jmath}$, where E_0, k and ω are positive constants and c is the speed of light. Which expression gives the magnetic field of the wave?

A $B = (E_0/c)\sin(kz + \omega t)\,\hat{\imath}$

B $B = -(E_0/c)\sin(kz + \omega t)\,\hat{\imath}$

C $B = cE_0 \sin(kz + \omega t)\,\hat{\imath}$

D $B = (E_0/c)\sin(kz + \omega t)\,\hat{k}$

Answer

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-13.02 · UPII-13.03 **[1]**

Light travelling in glass of refractive index 1.52 meets a flat boundary with water of refractive index 1.33. Above what angle of incidence, measured from the normal, is the light totally internally reflected?

A 41.1°

B 48.8°

C 61.0°

D No angle: total internal reflection requires the second medium to be air or vacuum

Answer

10
CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPII-15.05 **[1]**

An electron is accelerated from rest through a potential difference of 150 V. Take $h = 6.63 \times 10^{-34}$ J·s, $m_e = 9.11 \times 10^{-31}$ kg, $e = 1.60 \times 10^{-19}$ C and $c = 3.00 \times 10^{8}$ m/s. What is the electron's de Broglie wavelength?

A 8.29 nm

B 0.142 nm

C 0.00243 nm

D 0.100 nm

Answer

11 CORE COURSE

STRUCTURED · DEMANDING · UPII-03.04 · UPII-04.08 · UPII-05.05 · UPII-05.07 · UPII-05.08 **[12]**

The storage capacitor of a laboratory pulse generator is a hollow metal globe of radius b = 0.200 m with a metal ball of radius a = 0.120 m supported by an insulating rod at its centre. Transformer oil, of dielectric constant $\kappa = 2.2$ and dielectric strength 1.2×10^{7} V/m, fills the globe exactly up to the horizontal plane through its centre, so that the oil occupies the lower half of the gap between ball and globe; the upper half of the gap contains air, of dielectric constant 1.00 and dielectric strength 3.0×10^{6} V/m. The ball is charged positive, to a potential difference of 100 kV relative to the globe, and the capacitor is then disconnected from the supply. Treat the wall of the globe as thin and neglect the supporting rod. Take $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m.

(a) SHOW THAT **[4]**

Starting from Gauss's law, show that if air alone filled the whole gap the capacitance would be $C_0 = 4\pi\varepsilon_0 ab/(b - a)$. Hence, by explaining why the field in the half-filled gap is radial and has the same magnitude in the oil and in the air at the same distance from the centre, show that the half-filled capacitor has $C = 2\pi(1 + \kappa)\varepsilon_0 ab/(b - a)$.

(b) CALCULATE **[3]**

Calculate the capacitance, the charge on the ball, and the magnitude of the electric field in the gap next to the surface of the ball, when the potential difference is 100 kV.

(c) DETERMINE **[3]**

Determine the free charge per unit area on the half of the ball's surface in contact with the oil and on the half in contact with the air, and check that together they carry the charge found in (b).

(d) PREDICT **[2]**

With the capacitor still disconnected, the oil is drained out through a valve in the bottom of the globe and air takes its place. Predict the potential difference that the charge on the ball would produce with air alone in the gap, and whether the air could withstand it.

12 CORE COURSE

STRUCTURED · DEMANDING · UPII-07.02 · UPII-07.03 · UPII-07.05 · UPII-07.07 **[11]**

An ideal battery of emf ε = 9.0 V, a switch and a resistor R_1 = 30 kΩ are connected in series from the battery's positive terminal to a node P. From node P back to the battery's negative terminal run two branches in parallel: one contains a resistor R_2 = 60 kΩ, the other an initially uncharged capacitor C = 220 μF. The switch is closed at t = 0.

(a) DETERMINE **[2]**

Determine the current in R_1 immediately after the switch is closed, and a long time afterwards.

(b) DERIVE **[4]**

Using Kirchhoff's loop and junction rules, derive the differential equation obeyed by the capacitor voltage V(t), and hence obtain V(t) = V∞(1 − e^(−t/τ)), giving V∞ and τ in terms of ε, R_1, R_2 and C.

(c) CALCULATE **[2]**

Calculate τ, and the time at which the capacitor voltage reaches 4.5 V.

(d) EVALUATE **[2]**

A student argues that because the capacitor charges through R_1, the time constant must be R_1C = 6.6 s. Evaluate this argument.

(e) CALCULATE **[1]**

Calculate the energy stored in the capacitor a long time after the switch is closed.

13 CORE COURSE

STRUCTURED · DEMANDING · UPII-09.02 · UPII-09.05 · UPII-10.01 · UPII-10.02 · UPII-10.03 · UPII-11.02 **[11]**

A long straight insulated wire lies in the plane of a flat rectangular coil of N = 200 turns. The coil's two long sides, each of length ℓ = 0.200 m, are parallel to the wire at distances a = 0.020 m and b = 0.080 m from it. The coil's ends are connected to an oscilloscope of very high input resistance. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) DERIVE **[3]**

Starting from Ampère's law and the definition of magnetic flux, derive an expression for the magnetic flux through one turn of the coil when the wire carries a steady current I.

(b) CALCULATE **[2]**

Calculate the mutual inductance of the wire and the coil.

(c) CALCULATE **[2]**

The wire carries $I(t) = I_0 \sin(2\pi f t)$ with I_0 = 15 A and f = 50 Hz. Calculate the amplitude of the emf induced in the coil.

(d) EXPLAIN **[2]**

The coil is now closed through a resistor. At an instant when the wire current is in its positive direction and increasing, explain the direction of the induced current in the long side nearer the wire, and explain whether the wire attracts or repels the coil.

(e) PREDICT **[2]**

The coil is slid within its plane until the wire lies along the coil's centre line, parallel to and midway between its long sides. Predict the emf induced by the same alternating current, giving the reason.

14 CORE COURSE

STRUCTURED · DEMANDING · UPII-11.05 · UPII-11.06 · UPII-11.07 · UPII-11.08 **[11]**

A series circuit contains a resistor R = 25 Ω, an inductor L = 80 mH of negligible resistance and a capacitor C = 50 μF. It is driven by a sinusoidal source of rms emf 120 V whose frequency can be varied.

(a) CALCULATE **[2]**

Calculate the resonant frequency, and the rms current and average power delivered at resonance.

(b) DETERMINE **[4]**

The source frequency is set to 50 Hz. Determine the impedance, the rms current, the phase angle between the current and the source emf, and the average power delivered, stating whether the current leads or lags the emf.

(c) SKETCH **[3]**

Sketch the phasor diagram for the circuit at 50 Hz, with the current phasor drawn horizontally, showing the rms voltage phasors across R, L and C and the source emf, each labelled with its magnitude.

(d) EXPLAIN **[2]**

At resonance the rms voltage across the capacitor is 192 V, well above the 120 V rms of the source. Explain how this is consistent with Kirchhoff's loop rule and with conservation of energy.

15 CORE COURSE

STRUCTURED · DEMANDING · UPII-13.05 · UPII-13.06 · UPII-13.07 **[10]**

A compact telephoto camera lens is modelled as two thin lenses on a common axis: a converging lens of focal length $f_1 = +10.0$ cm, followed 6.0 cm behind it by a diverging lens of focal length $f_2 = -12.0$ cm. The image sensor sits behind the second lens. Use the real-is-positive convention, $1/u + 1/v = 1/f$ with magnification $m = -v/u$, in which a virtual object has a negative object distance u.

(a) DETERMINE **[3]**

Light from a very distant object enters the first lens. Determine where the final image forms, and state whether it is real or virtual.

(b) SHOW THAT **[3]**

The distant object subtends an angle of 0.50° at the lens. Show that the final image is 0.131 cm tall, and hence that the combination behaves as a single lens of focal length 15.0 cm.

(c) COMPARE **[2]**

Compare this combination with a single converging lens that forms an image of the same size of the same distant object, in terms of the distance from the front of the lens to the sensor.

(d) PREDICT **[2]**

The camera is refocused on an object 2.00 m in front of the first lens, with the lens separation unchanged. Predict whether the sensor must move towards or away from the second lens, and by how much.

16 CORE COURSE

STRUCTURED · DEMANDING · UPII-14.03 · UPII-14.05 · UPII-14.08 **[10]**

A student uses a transmission grating ruled with 450 lines per millimetre, illuminated at normal incidence, to look for the deuterium line in the light of a discharge lamp that contains hydrogen and deuterium in comparable amounts. The red Balmer line of ordinary hydrogen has wavelength 656.28 nm; the corresponding line of deuterium has wavelength 656.10 nm. The lamp also emits the blue-green Balmer line of hydrogen, of wavelength 486.13 nm.

(a) CALCULATE **[2]**

Calculate the grating spacing, and the highest order in which light of wavelength 656 nm can be observed.

(b) CALCULATE **[1]**

Calculate the angle at which the hydrogen line appears in second order.

(c) SHOW THAT **[2]**

By differentiating the grating equation, show that two lines separated in wavelength by a small amount $\Delta\lambda$ appear in order m with an angular separation $\Delta\theta = m\Delta\lambda/(d\cos\theta)$.

(d) CALCULATE **[1]**

Calculate the angular separation of the hydrogen and deuterium lines in second order.

(e) DETERMINE **[2]**

Determine the minimum number of grating lines that must be illuminated to resolve the two lines in second order, and the corresponding minimum width of the illuminated region of the grating.

(f) COMPARE **[2]**

Compare working in third order with working in second order for this measurement.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPII-15.04 **[12]**

In a photoelectric experiment, monochromatic light falls on a clean metal cathode in an evacuated tube. For each wavelength the reverse potential difference between anode and cathode is increased until the photocurrent just falls to zero, and this stopping potential V_s is recorded. Each stopping potential has an uncertainty of ± 0.03 V, and each wavelength is known to ± 1 nm. Take $e = 1.60 \times 10^{-19}$ C and $c = 3.00 \times 10^{8}$ m/s.

Stopping potential against wavelength for one cathode

Reading	Wavelength λ (nm)	Stopping potential V_s (V)
1	300	1.95
2	340	1.44
3	380	1.08
4	420	0.66
5	460	0.52
6	500	0.28

(a) STATE **[1]**

State the photoelectric equation relating the stopping potential V_s to the frequency f of the light.

(b) EVALUATE **[2]**

Calculate the frequency of the light for each reading, and evaluate whether all six readings are consistent with a single straight line.

(c) DETERMINE **[3]**

Determine the Planck constant from the gradient of a graph of V_s against f.

(d) DETERMINE **[2]**

Determine the work function of the cathode, in electronvolts, and the threshold wavelength beyond which no photoelectrons are emitted.

(e) ESTIMATE **[2]**

Estimate the uncertainty in your value of the Planck constant, stating the assumptions you make.

(f) JUSTIFY **[2]**

Justify the claim that these data cannot be explained by a classical wave model, in which the energy an electron absorbs grows with the intensity of the light and the time it is exposed.

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18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPII-12.04 · UPII-12.05 · UPII-12.07 **[15]**

A parallel-plate capacitor has circular plates of radius R = 5.0 cm separated by d = 2.0 mm in vacuum. It is being charged by a steady current I = 0.40 A carried by long straight leads that meet the centres of the plates along their common axis. Neglect fringing, so that the electric field is uniform between the plates and zero outside the gap. Take $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) EXPLAIN **[2]**

Ampère's law without Maxwell's correction gives contradictory results for the magnetic field at a point beside the gap, depending on which surface is chosen to span a circular path around the axis. Explain the contradiction, and how the displacement current resolves it.

(b) DERIVE **[4]**

Using the Ampère–Maxwell law, derive the magnitude of the magnetic field between the plates at a distance r < R from the axis.

(c) DETERMINE **[2]**

Determine the magnitude of the magnetic field in the midplane of the gap at a distance r > R from the axis, and state how it compares with the field at the same distance from one of the leads.

(d) SKETCH **[2]**

Sketch a graph of the magnitude of B against r in the midplane of the gap, from r = 0 to r = 3R, marking the value at r = R.

(e) SHOW THAT **[4]**

At the edge of the gap, r = R, show that the Poynting vector $S = E \times B/\mu_0$ points radially inwards, and that the total power flowing in through the cylindrical surface of radius R and height d equals the rate of increase of the energy stored in the capacitor, $d/dt(q^2/2C)$.

(f) PREDICT **[1]**

The capacitor is later discharged through the same leads. Predict the direction of the Poynting vector at the edge of the gap during the discharge.

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19 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPII-09.04 · UPII-10.07 · UPII-12.03 **[15]**

A long solenoid of radius a = 4.0 cm has n = 8000 turns per metre and carries a current I_0 = 20 A. Coaxial with it, outside the winding, a thin insulating ring of radius b = 6.0 cm and mass m = 0.50 g carries a charge Q = 0.50 μC spread uniformly around it. The ring can turn freely about the common axis, and the apparatus is in vacuum. The solenoid current is reduced steadily to zero over T = 0.50 s. Treat the solenoid as ideal, with no magnetic field outside its winding, and neglect the magnetic field of the ring itself. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) DESCRIBE **[2]**

Describe the electric field induced while the current is falling: its direction, the shape of its field lines, and why it cannot be described by an electric potential.

(b) DERIVE **[4]**

Using Faraday's law in the form $\oint E \cdot dl = -d\Phi_B/dt$, derive the magnitude of the induced electric field at distance r from the axis, for r < a and for r > a, in terms of the rate of change dB/dt of the field inside the solenoid.

(c) CALCULATE **[2]**

Calculate the magnetic field inside the solenoid before the current starts to fall, and the magnitude of the induced electric field at the ring while the current is falling.

(d) DERIVE **[3]**

Using the rotational form of Newton's second law, $\tau = dL/dt$, derive the angular momentum the ring has acquired once the current has fallen to zero, and show that it depends neither on T nor on b, provided $b > a$.

(e) CALCULATE **[2]**

Calculate the angular speed of the ring once the current has fallen to zero.

(f) EVALUATE **[2]**

A student claims that the ring starts to turn because the solenoid's magnetic field pushes on its charges. Evaluate this claim.

MARK SCHEME

University Physics II · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-01.04 · UPII-01.05 · UPII-02.03 **[1]**

A thin, smooth insulating ring of radius R is held fixed, and gravity can be neglected. Point charges +Q are fixed at the two ends of one diameter of the ring, and a small bead carrying charge +q, with q > 0, is threaded on the ring so that it can slide freely round it. The bead is placed at a point on the ring a quarter of a turn from each fixed charge. Which statement is correct?

KEY **B** **The field there is not zero but points radially outwards, at right angles to the ring, and a bead nudged a short way along the ring is pushed back towards that point.**

Each fixed charge is a distance R√2 from the quarter point and produces a field of magnitude kQ/(2R²) there, where k = 1/(4πε₀), directed away from itself. By symmetry the components along the ring cancel and the components along the radius add, giving √2kQ/(2R²) radially outwards; the only point where the field of the two fixed charges vanishes is the centre of the ring. A mistakes an equilibrium for a null point: the bead is at rest only because the ring pushes inwards on it, balancing the outward electric force, and without the ring it would be driven away from the centre. C forgets what a nudge along the ring does to the two distances: the bead moves closer to one fixed charge and further from the other, so the push along the ring from the nearer charge grows while the opposing push from the further charge shrinks, and the bead is driven back; the quarter point is a minimum of potential energy along the ring. D confuses equilibrium with the absence of force: both repulsions act, but their components along the ring are equal and opposite and the ring supplies the rest. A negative bead would also be in equilibrium at that point, but a nudge would then be amplified rather than undone, so its equilibrium would be unstable.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-04.06 **[1]**

In a region of space the electric potential is V(x, y) = αx²y − βy, with α = 20 V/m³ and β = 30 V/m. What is the electric field at the point (x, y) = (1.0 m, 2.0 m)?

KEY **A** **(−80 î + 10 ĵ) V/m**

The field is minus the gradient of the potential: $E_x = -\partial V/\partial x = -2\alpha xy = -80$ V/m and $E_y = -\partial V/\partial y = -(\alpha x^2 - \beta) = +10$ V/m. B gives both components the sign of the gradient itself, the field of a candidate who forgets that E points down the potential slope. C drops the factor of 2 from differentiating x², a slip that leaves E_y untouched and so looks half right. D attaches each derivative to the wrong component, taking E_x from ∂V/∂y and E_y from ∂V/∂x; the numbers are the right ones, which is exactly what makes it convincing.

U2

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-06.06 · UPII-06.08 **[1]**

A cordless drill runs from a battery pack. A high-resistance voltmeter across the pack's terminals reads 18.0 V while the drill is switched off and 16.8 V while the drill draws a steady current of 8.0 A. Model the pack as a constant emf in series with a constant internal resistance. At what rate is energy dissipated as heat inside the pack while the drill runs?

KEY **C** **9.6 W**

The terminal voltage falls by 1.2 V when 8.0 A flows, and that fall is the potential difference across the internal resistance, so $r = (1.2\ V)/(8.0\ A) = 0.15\ \Omega$ and the heating rate inside the pack is $I^2r = (8.0\ A)^2(0.15\ \Omega) = 9.6\ W$, the same as $(1.2\ V)(8.0\ A)$. A is $\varepsilon I = (18.0\ V)(8.0\ A)$, the whole rate at which the emf does work, which counts the energy delivered to the drill as well as the heat left in the pack. B is the terminal voltage times the current, $(16.8\ V)(8.0\ A) = 134\ W$: this is the power delivered to the drill, the energy that leaves the pack rather than the energy that stays in it. D is ε^2/r, the heating rate if the terminals were short-circuited; it puts the whole emf across the internal resistance, which happens only when the external resistance is zero, not while the drill is running.

U3

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-08.02 · UPII-08.03 **[1]**

Electrons leave a small source inside a long solenoid, each moving at 30° to the solenoid's axis. Inside the solenoid the magnetic field is uniform and parallel to the axis, and there is no electric field, so each electron follows a helix whose axis is parallel to the field. What is the ratio of the pitch of the helix, the distance an electron moves parallel to the axis during one revolution, to the radius of the helix?

KEY **D** **$2\pi\sqrt{3}$**

Resolve the velocity along and across the field. The perpendicular component $v \sin 30°$ drives the circular motion, with radius $r = mv \sin 30°/(eB)$ and period $T = 2\pi m/(eB)$, which does not depend on the speed. The magnetic force has no component along the field, so the parallel component $v \cos 30°$ stays constant and carries the electron a pitch $p = (v \cos 30°)T = 2\pi mv \cos 30°/(eB)$ in one revolution. The ratio is $p/r = 2\pi \cot 30° = 2\pi\sqrt{3} \approx 10.9$, whatever the speed, the field strength or the electron's charge and mass. A comes from interchanging the components, putting sin 30° into the pitch and cos 30° into the radius. B compares the pitch with the diameter rather than the radius, or counts half a period as one revolution. C assumes the electron advances one circumference per revolution, which is true only when the two components are equal, at 45° to the field.

U2

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-09.05 **[1]**

A long straight hollow cylindrical conductor has inner radius 2.0 cm and outer radius 4.0 cm, and carries a steady current of 12 A distributed uniformly over the metal's cross-section. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A. What is the magnitude of the magnetic field at a distance of 3.0 cm from the axis?

KEY **D** **3.3×10^{-5} T**

By symmetry B is azimuthal and uniform on a circle of radius 3.0 cm about the axis, so Ampère's law gives $B(2\pi r) = \mu_0 I_{enc}$. The enclosed current is the fraction of the metal inside that circle, $I_{enc} = 12\ A \times (3.0^2 - 2.0^2)/(4.0^2 - 2.0^2) = 5.0\ A$, so $B = \mu_0(5.0\ A)/(2\pi \times 0.030\ m) = 3.3 \times 10^{-5}$ T. B uses the whole 12 A, as if the point lay outside the conductor. C takes the enclosed fraction as r^2/b^2, the rule for a solid wire, and so counts current in the hollow core where there is none. A carries over the electrostatic result that the field inside a conductor vanishes; that is true of E in equilibrium, not of the magnetic field of a current flowing through the metal.

U3

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-10.03 · UPII-10.06 **[1]**

A strong cylindrical magnet is dropped down the centre of a long vertical copper pipe and soon falls at a steady speed far below that of free fall. Copper is not ferromagnetic. Which statement correctly explains the motion?

KEY **C** **Eddy currents circulate in opposite senses in the pipe just below and just above the magnet; each exerts an upward force on the magnet, and the gravitational energy the magnet loses is dissipated in the pipe.**

Below the magnet the flux through each ring of the pipe is increasing, so the induced current opposes the increase and repels the magnet; above it the flux is decreasing, so the induced current acts to maintain the flux and attracts the magnet back. The two rings carry currents in opposite senses, yet both forces on the magnet point up, and at the steady speed their sum equals the weight while the lost gravitational energy becomes internal energy of the copper. A fails for copper, whose magnetisation is negligible, and would fail even for iron: a long uniform ferromagnetic pipe pulls the magnet sideways towards its wall, not upwards along the axis. A copper pipe with a slot cut along its length also slows the magnet far less, because the slot interrupts the circulating currents, which a magnetisation mechanism cannot explain. B applies Lenz's law only to increasing flux, but a decreasing flux induces a current just as surely. D gets the sense of the currents wrong and invents a thermal cause for the steady speed, which is really set by a drag force that grows in proportion to the speed.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-11.06 · UPII-11.07 **[1]**

A series RLC circuit is driven by a sinusoidal source at its resonant frequency f_0. The source frequency is then raised to $2f_0$ with the source amplitude unchanged. Which statement describes the circuit at $2f_0$? (X_L and X_C are the inductive and capacitive reactances.)

KEY **B** **$X_L = 4X_C$, and the current lags the source emf**

At f_0 the two reactances are equal. Doubling the frequency doubles $X_L = 2\pi fL$ and halves $X_C = 1/(2\pi fC)$, so $X_L = 4X_C$; the circuit is inductive, and in an inductive circuit the current lags the emf. A doubles X_L but forgets that X_C falls at the same time. C has the ratio right and the phase backwards: a leading current belongs to a capacitive circuit, below resonance. D treats resonance as a property of the components alone, when it is a property of those components at one particular frequency.

U1

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPII-12.06 · UPII-12.07 **[1]**

The electric field of a plane electromagnetic wave in vacuum is $E = E_0 \sin(kz + \omega t)\,\hat{j}$, where E_0, k and ω are positive constants and c is the speed of light. Which expression gives the magnetic field of the wave?

KEY **A** **$B = (E_0/c)\sin(kz + \omega t)\,\hat{i}$**

The phase $kz + \omega t$ stays constant if z decreases as t increases, so this wave travels towards $-z$. The energy flux $S = E \times B/\mu_0$ must then point along $-\hat{k}$; with E along $\hat{j}$ that needs B along $+\hat{i}$, because $\hat{j} \times \hat{i} = -\hat{k}$, and the amplitudes are related by $B_0 = E_0/c$. Faraday's law confirms it: $-\partial E_y/\partial z = -kE_0\cos(kz + \omega t)$ equals $-\partial B_x/\partial t$ only for the field in A. B gives the magnetic field the orientation, $-\hat{i}$, that belongs to a wave travelling towards $+z$; with this E it would make S point along $+\hat{k}$, against the direction the wave actually travels, and it is chosen by a candidate who assumes a wave always moves forwards without reading the sign in the phase. C inverts the amplitude relation, giving a magnetic field about 10^{17} times too large. D puts B along the direction of propagation, but a field that varies only with z can have no varying z-component without violating Gauss's law for magnetism: electromagnetic waves in vacuum are transverse.

U2

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-13.02 · UPII-13.03 **[1]**

Light travelling in glass of refractive index 1.52 meets a flat boundary with water of refractive index 1.33. Above what angle of incidence, measured from the normal, is the light totally internally reflected?

KEY **C** **61.0°**

Total internal reflection needs only that the second medium have the lower refractive index. At the critical angle the refracted ray grazes the boundary, so 1.52 sin θc = 1.33 sin 90° and θc = sin⁻¹(1.33/1.52) = 61.0°. A is sin⁻¹(1/1.52), the critical angle for glass to air, the value a candidate remembers rather than works out. B is sin⁻¹(1/1.33), the critical angle for water to air, reached by dropping the glass out of the problem. D mistakes the everyday example for the condition: every boundary from a higher to a lower index has a critical angle, and it is larger the closer the two indices are.

U3

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPII-15.05 **[1]**

An electron is accelerated from rest through a potential difference of 150 V. Take h = 6.63 × 10⁻³⁴ J·s, m_e = 9.11 × 10⁻³¹ kg, e = 1.60 × 10⁻¹⁹ C and c = 3.00 × 10⁸ m/s. What is the electron's de Broglie wavelength?

KEY **D** **0.100 nm**

The kinetic energy is eV = 2.40 × 10⁻¹⁷ J, tiny beside $m_e c^2$, so the classical momentum $p = \sqrt{(2m_e eV)}$ = 6.61 × 10⁻²⁴ kg·m/s is adequate and λ = h/p = 1.00 × 10⁻¹⁰ m = 0.100 nm. A is hc/(eV), the wavelength of a 150 eV photon, which applies E = pc to a particle with mass. B drops the factor of 2 inside the square root, which is what happens when the kinetic energy is written as mv² instead of ½mv². C is h/(m_ec), the Compton wavelength, which replaces the electron's actual speed with the speed of light.

U3

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11 CORE COURSE

STRUCTURED · DEMANDING · UPII-03.04 · UPII-04.08 · UPII-05.05 · UPII-05.07 · UPII-05.08 **[12]**

The storage capacitor of a laboratory pulse generator is a hollow metal globe of radius b = 0.200 m with a metal ball of radius a = 0.120 m supported by an insulating rod at its centre. Transformer oil, of dielectric constant κ = 2.2 and dielectric strength 1.2 × 10⁷ V/m, fills the globe exactly up to the horizontal plane through its centre, so that the oil occupies the lower half of the gap between ball and globe; the upper half of the gap contains air, of dielectric constant 1.00 and dielectric strength 3.0 × 10⁶ V/m. The ball is charged positive, to a potential difference of 100 kV relative to the globe, and the capacitor is then disconnected from the supply. Treat the wall of the globe as thin and neglect the supporting rod. Take ε_0 = 8.85 × 10⁻¹² F/m.

(a) SHOW THAT · U2 **[4]**

Starting from Gauss's law, show that if air alone filled the whole gap the capacitance would be $C_0 = 4\pi\varepsilon_0 ab/(b - a)$. Hence, by explaining why the field in the half-filled gap is radial and has the same magnitude in the oil and in the air at the same distance from the centre, show that the half-filled capacitor has $C = 2\pi(1 + \kappa)\varepsilon_0 ab/(b - a)$.

1. Gaussian sphere of radius r, a < r < b, concentric with the ball: by symmetry E is radial and uniform over it, so $4\pi r^2 E = Q/\varepsilon_0$ and $E = Q/(4\pi\varepsilon_0 r^2)$
2. $V = \int E\,dr$ from r = a to r = b = $(Q/(4\pi\varepsilon_0))(1/a - 1/b)$, so $C_0 = Q/V = 4\pi\varepsilon_0 ab/(b - a)$
3. A radial field $E = A/r^2$ in each half lies along the flat oil–air boundary with no component across it; because the ball and the globe are equipotentials, $\int E\,dr$ from a to b is the same V along every radius, which forces the same A in the oil and in the air and keeps the tangential field continuous at the boundary, so the two halves share one potential difference and act as capacitors in parallel
4. The air half has capacitance $C_0/2$ and the oil half $\kappa C_0/2$, so $C = (1 + \kappa)C_0/2 = 2\pi(1 + \kappa)\varepsilon_0 ab/(b - a)$

Candidates who stack the oil and the air as capacitors in series have put the boundary in the wrong place: layers in series need a boundary across the field lines, whereas this boundary runs along them, so the halves share the potential difference and their capacitances add. The step from C_0 to C earns nothing without the reason the field is the same in both media.

(b) CALCULATE · U3 **[3]**

Calculate the capacitance, the charge on the ball, and the magnitude of the electric field in the gap next to the surface of the ball, when the potential difference is 100 kV.

1. $C = 2\pi(3.2)(8.85 \times 10^{-12}\ \text{F/m})(0.120\ \text{m})(0.200\ \text{m})/(0.080\ \text{m}) = 5.34 \times 10^{-11}\ \text{F}$ (53.4 pF)
2. $Q = CV = (5.34 \times 10^{-11}\ \text{F})(1.00 \times 10^{5}\ \text{V}) = 5.34 \times 10^{-6}\ \text{C}$
3. $E(a) = Vb/(a(b - a)) = (1.00 \times 10^{5}\ \text{V})(0.200\ \text{m})/((0.120\ \text{m})(0.080\ \text{m})) = 2.08 \times 10^{6}$ V/m, the same in the oil and in the air, and below the 3.0×10^{6} V/m that air withstands

   Dividing the field in the oil by κ, to 9.5×10^{5} V/m, treats the oil as if it faced a fixed charge of its own; here the conductors fix one potential difference along every radius, so the field is the same in both media and it is the charge on the ball that differs between the halves. Using $V/(b - a) = 1.25 \times 10^{6}$ V/m, as if the field were uniform, understates the field at the ball by 40%.

(c) DETERMINE · U2 **[3]**

Determine the free charge per unit area on the half of the ball's surface in contact with the oil and on the half in contact with the air, and check that together they carry the charge found in (b).

1. Just outside a conductor the free surface charge density is $\sigma = \kappa\varepsilon_0 E$, with $\kappa = 1.00$ for the air, and $E(a)$ is the same on both halves
2. $\sigma_air = (8.85 \times 10^{-12}\ \text{F/m})(2.08 \times 10^{6}\ \text{V/m}) = 1.84 \times 10^{-5}\ \text{C/m}^2$, and $\sigma_oil = \kappa\sigma_air = 4.06 \times 10^{-5}\ \text{C/m}^2$
3. Each half has area $2\pi a^2 = 9.05 \times 10^{-2}\ \text{m}^2$, so $Q = 2\pi a^2(\sigma_oil + \sigma_air) = 5.34 \times 10^{-6}$ C, agreeing with (b); the oil-covered half carries $\kappa/(1 + \kappa) = 69\%$ of it

   Candidates who assume a conductor always spreads its charge evenly put 2.67×10^{-6} C on each half. A conductor must be an equipotential, not uniformly charged: the bound charge on the oil next to the ball cancels part of the free charge there, so the oil-covered half draws κ times as much free charge per unit area to keep the field the same on both sides.

(d) PREDICT · U1 **[2]**

With the capacitor still disconnected, the oil is drained out through a valve in the bottom of the globe and air takes its place. Predict the potential difference that the charge on the ball would produce with air alone in the gap, and whether the air could withstand it.

1. The charge stays at 5.34×10^{-6} C while the capacitance falls to $C_0 = 2C/(1 + \kappa) =$ 33.4 pF, so the potential difference would rise by the factor $(1 + \kappa)/2 = 1.6$, to 1.60×10^{5} V
2. The field at the surface of the ball would rise by the same factor, to 3.33×10^{6} V/m, above the 3.0×10^{6} V/m that air withstands, so the air cannot withstand it: breakdown starts next to the ball while the last of the oil is still draining, before the potential difference reaches 1.60×10^{5} V

   Candidates who keep the potential difference at 100 kV, as if the supply were still connected, predict no change in the field and a safe gap. With the charge fixed, removing the dielectric lowers the capacitance and raises both the potential difference and the field; and the oil's high dielectric strength never protected the capacitor, because the air half always carried the same field.

12

CORE COURSE

STRUCTURED · DEMANDING · UPII-07.02 · UPII-07.03 · UPII-07.05 · UPII-07.07 **[11]**

An ideal battery of emf $\varepsilon = 9.0$ V, a switch and a resistor $R_1 = 30$ kΩ are connected in series from the battery's positive terminal to a node P. From node P back to the battery's negative terminal run two branches in parallel: one contains a resistor $R_2 = 60$ kΩ, the other an initially uncharged capacitor $C = 220$ μF. The switch is closed at $t = 0$.

(a) DETERMINE · U1 **[2]**

Determine the current in R_1 immediately after the switch is closed, and a long time afterwards.

1. At $t = 0$ the uncharged capacitor has no potential difference across it, so P is at the potential of the negative terminal and $I_1 = 9.0\ \text{V}/30\ \text{k}\Omega = 3.0 \times 10^{-4}$ A (0.30 mA)
2. A long time later the capacitor carries no current, so R_1 and R_2 are in series and $I_1 = 9.0\ \text{V}/90\ \text{k}\Omega = 1.0 \times 10^{-4}$ A (0.10 mA)

Candidates who treat the capacitor as an open circuit at $t = 0$ give 0.10 mA for both times. The uncharged capacitor behaves as a wire at the first instant and as a break only once charged.

(b) DERIVE · U2 **[4]**

Using Kirchhoff's loop and junction rules, derive the differential equation obeyed by the capacitor voltage V(t), and hence obtain V(t) = V∞(1 − e^(−t/τ)), giving V∞ and τ in terms of ε, R_1, R_2 and C.

1. Loop rule round the battery, R_1 and the capacitor: $\varepsilon - I_1R_1 - V = 0$, so $I_1 = (\varepsilon - V)/R_1$
2. Junction rule at P: $I_1 = V/R_2 + C\,dV/dt$
3. Combining: $C\,dV/dt = \varepsilon/R_1 - V(1/R_1 + 1/R_2)$, which rearranges to $\tau\,dV/dt = V_\infty - V$ with $V_\infty = \varepsilon R_2/(R_1 + R_2)$ and $\tau = CR_1R_2/(R_1 + R_2)$
4. Separates the variables and integrates with $V(0) = 0$: $\ln((V_\infty - V)/V_\infty) = -t/\tau$, so $V(t) = V_\infty(1 - e^{-t/\tau})$

The junction rule is the step most often missing: a single loop equation with $I_1 = C\,dV/dt$ ignores the current R_2 draws and reproduces the student's argument in part (d).

(c) CALCULATE · U3 **[2]**

Calculate τ, and the time at which the capacitor voltage reaches 4.5 V.

1. $\tau = (220 \times 10^{-6}\ \text{F})(20\ \text{k}\Omega) = 4.4$ s, with $V_\infty = 6.0$ V
2. $4.5\ \text{V} = (6.0\ \text{V})(1 - e^{-t/\tau})$ gives $e^{-t/\tau} = 0.25$, so $t = \tau \ln 4 = 6.1$ s

Using 9.0 V as the final voltage gives $t = \tau \ln 2 = 3.0$ s; the threshold is three-quarters of the way to 6.0 V, not half of the way to 9.0 V.

(d) EVALUATE · U1 **[2]**

A student argues that because the capacitor charges through R_1, the time constant must be $R_1C = 6.6$ s. Evaluate this argument.

1. The argument fails because R_2 diverts part of the current arriving at P: seen from the capacitor, the network is equivalent to a source $\varepsilon R_2/(R_1 + R_2) = 6.0$ V behind R_1 and R_2 in parallel, 20 kΩ
2. So $\tau = 4.4$ s rather than 6.6 s, and the student's value would place the 4.5 V crossing at 9.1 s instead of 6.1 s, an error of 50% in a timing design

The part rewards a judgement backed by the equivalent resistance, not a restatement of part (b). An answer that says only 'R_2 also matters' earns the first mark at most.

(e) CALCULATE · U3 **[1]**

Calculate the energy stored in the capacitor a long time after the switch is closed.

1. $U = \tfrac{1}{2}CV_\infty^2 = \tfrac{1}{2}(220 \times 10^{-6}\ \text{F})(6.0\ \text{V})^2 = 3.96 \times 10^{-3}$ J (4.0 mJ)

Using the battery emf gives 8.9 mJ; the capacitor ends at the divider voltage across R_2.

13 CORE COURSE

STRUCTURED · DEMANDING · UPII-09.02 · UPII-09.05 · UPII-10.01 · UPII-10.02 · UPII-10.03 · UPII-11.02 **[11]**

A long straight insulated wire lies in the plane of a flat rectangular coil of N = 200 turns. The coil's two long sides, each of length ℓ = 0.200 m, are parallel to the wire at distances a = 0.020 m and b = 0.080 m from it. The coil's ends are connected to an oscilloscope of very high input resistance. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) DERIVE · U2 **[3]**

Starting from Ampère's law and the definition of magnetic flux, derive an expression for the magnetic flux through one turn of the coil when the wire carries a steady current I.

1. Ampère's law round a circle of radius r centred on the wire gives $B(2\pi r) = \mu_0 I$, so $B = \mu_0 I/(2\pi r)$, directed perpendicular to the plane of the coil
2. Because B varies across the coil, takes a strip of width dr and length ℓ parallel to the wire, so $d\Phi = B\ell\, dr$
3. $\Phi = (\mu_0 I\ell/2\pi)\int dr/r$ from r = a to r = b = $(\mu_0 I\ell/2\pi)\ln(b/a)$

Multiplying the field at the coil's centre line by the coil's area gives a plausible answer that is about 13% too small. The integral is required because the field falls off across the coil.

(b) CALCULATE · U3 **[2]**

Calculate the mutual inductance of the wire and the coil.

1. $M = N\Phi/I = N\mu_0\ell \ln(b/a)/(2\pi)$
2. $M = 200 \times (2.0 \times 10^{-7}\ \text{T·m/A})(0.200\ \text{m}) \ln 4 = 1.11 \times 10^{-5}$ H (11.1 μH)

Forgetting the factor N gives 5.5×10^{-8} H. The flux linkage, not the flux through one turn, defines the mutual inductance.

(c) CALCULATE · U3 **[2]**

The wire carries $I(t) = I_0 \sin(2\pi ft)$ with I_0 = 15 A and f = 50 Hz. Calculate the amplitude of the emf induced in the coil.

1. $\varepsilon = -M\, dI/dt = -2\pi f M I_0 \cos(2\pi ft)$, so the amplitude is $2\pi f M I_0$
2. $\varepsilon_0 = 2\pi(50\ \text{Hz})(1.11 \times 10^{-5}\ \text{H})(15\ \text{A}) = 5.2 \times 10^{-2}$ V (52 mV)

Quoting the rms value, 37 mV, answers a different question; the amplitude is asked for. Using f in place of $\omega = 2\pi f$ gives 8.3 mV.

(d) EXPLAIN · U1 **[2]**

The coil is now closed through a resistor. At an instant when the wire current is in its positive direction and increasing, explain the direction of the induced current in the long side nearer the wire, and explain whether the wire attracts or repels the coil.

1. The flux of the wire's field through the coil is increasing, so by Lenz's law the induced current makes a field opposing it inside the coil, which requires the near side to carry current antiparallel to the wire's current
2. Antiparallel currents repel and parallel currents attract; the near side, antiparallel and closer, feels the stronger force, so the net force pushes the coil away from the wire

Candidates often get the direction of the current right and stop. The far side carries current parallel to the wire and is attracted, so the net force has to be argued from the 1/r dependence of the field.

(e) PREDICT · U1 **[2]**

The coil is slid within its plane until the wire lies along the coil's centre line, parallel to and midway between its long sides. Predict the emf induced by the same alternating current, giving the reason.

1. The wire's field points into the plane on one side of the wire and out of it on the other, with equal magnitudes at equal distances
2. The two halves of the coil are mirror images about the wire, so the net flux is zero for any current: M = 0 and no emf is induced

The coil is now threaded by the strongest fields in the arrangement, and candidates who reason from field strength near the wire predict a larger emf. Induction depends on net flux linkage, which the symmetry makes zero.

14 CORE COURSE

STRUCTURED · DEMANDING · UPII-11.05 · UPII-11.06 · UPII-11.07 · UPII-11.08 **[11]**

A series circuit contains a resistor R = 25 Ω, an inductor L = 80 mH of negligible resistance and a capacitor C = 50 μF. It is driven by a sinusoidal source of rms emf 120 V whose frequency can be varied.

(a) CALCULATE · U3 **[2]**

Calculate the resonant frequency, and the rms current and average power delivered at resonance.

1. $f_0 = 1/(2\pi\sqrt{(LC)}) = 1/(2\pi\sqrt{((0.080\ \text{H})(50 \times 10^{-6}\ \text{F}))}) = 79.6$ Hz
2. At resonance Z = R, so $I_{rms} = 120\ \text{V}/25\ \Omega = 4.80$ A and $P = I_{rms}^2 R = 576$ W

Quoting $\omega_0 = 500$ rad/s as the frequency in hertz is the slip this part catches; the factor 2π separates angular frequency from frequency.

(b) DETERMINE · U3 **[4]**

The source frequency is set to 50 Hz. Determine the impedance, the rms current, the phase angle between the current and the source emf, and the average power delivered, stating whether the current leads or lags the emf.

1. $X_L = 2\pi(50\ \text{Hz})(0.080\ \text{H}) = 25.1\ \Omega$ and $X_C = 1/(2\pi(50\ \text{Hz})(50 \times 10^{-6}\ \text{F})) = 63.7\ \Omega$
2. $Z = \sqrt{((25\ \Omega)^2 + (25.1\ \Omega - 63.7\ \Omega)^2)} = 45.9\ \Omega$, so $I_{rms} = 120\ \text{V}/45.9\ \Omega = 2.61$ A
3. $\tan\varphi = (X_L - X_C)/R = -1.54$, so the phase angle is 57.0°, with the current leading the emf because below resonance the circuit is capacitive
4. $P = I_{rms}^2 R = (2.61\ \text{A})^2(25\ \Omega) = 171$ W, equivalently $V_{rms} I_{rms} \cos\varphi$ with power factor 0.544

Computing the power as $V_{rms} \times I_{rms} = 314$ W ignores the power factor; only the resistor dissipates energy on average, and the check $I_{rms}^2 R$ exposes the error at once.

(c) SKETCH · U1 **[3]**

Sketch the phasor diagram for the circuit at 50 Hz, with the current phasor drawn horizontally, showing the rms voltage phasors across R, L and C and the source emf, each labelled with its magnitude.

1. $V_R = 65.3$ V drawn along the current phasor
2. $V_L = 65.7$ V drawn 90° ahead of the current and $V_C = 166$ V drawn 90° behind it, visibly longer than V_L, for phasors rotating anticlockwise
3. The source emf, 120 V, drawn as the vector sum of the three, lying 57° behind the current phasor so that the current leads

A diagram in which V_L and V_C are drawn in the same direction, or in which the emf lies ahead of the current, contradicts part (b); the sketch is marked for consistency with the candidate's own phase angle.

(d) EXPLAIN · U1 **[2]**

At resonance the rms voltage across the capacitor is 192 V, well above the 120 V rms of the source. Explain how this is consistent with Kirchhoff's loop rule and with conservation of energy.

1. The loop rule holds for instantaneous voltages, or for phasors added with their phases: at resonance V_L and V_C are equal in size and 180° apart, so they cancel and the whole source voltage appears across R
2. No energy is created: the inductor and capacitor exchange stored energy back and forth each cycle with zero average power, and the only average power drawn from the source is I^2R dissipated in the resistor

Adding the rms voltages as plain numbers at resonance, 120 V + 192 V + 192 V = 504 V, appears to exceed the 120 V source more than fourfold; this part is set to break that habit.

15 CORE COURSE

STRUCTURED · DEMANDING · UPII-13.05 · UPII-13.06 · UPII-13.07 **[10]**

A compact telephoto camera lens is modelled as two thin lenses on a common axis: a converging lens of focal length $f_1 = +10.0$ cm, followed 6.0 cm behind it by a diverging lens of focal length $f_2 = -12.0$ cm. The image sensor sits behind the second lens. Use the real-is-positive convention, $1/u + 1/v = 1/f$ with magnification $m = -v/u$, in which a virtual object has a negative object distance u.

(a) DETERMINE · U3 **[3]**

Light from a very distant object enters the first lens. Determine where the final image forms, and state whether it is real or virtual.

1. The first lens alone would form its image at its focal point, 10.0 cm behind it, which is 4.0 cm behind the second lens
2. The second lens intercepts rays that are still converging, so that point is a virtual object for it, with $u_2 = -4.0$ cm
3. $1/v_2 = 1/f_2 - 1/u_2 = -1/12.0 + 1/4.0$, so $v_2 = +6.0$ cm: a real image 6.0 cm behind the second lens, 12.0 cm behind the first

   Taking $u_2 = +4.0$ cm, as if the first image were a real object in front of the second lens, gives $v_2 = -3.0$ cm, a virtual image that no sensor could record. Declaring the sign of the virtual object is the step the part is set on.

(b) SHOW THAT · U3 **[3]**

The distant object subtends an angle of 0.50° at the lens. Show that the final image is 0.131 cm tall, and hence that the combination behaves as a single lens of focal length 15.0 cm.

1. The first image has height $h_1 = f_1\theta = (10.0\text{ cm})(8.73 \times 10^{-3}\text{ rad}) = 0.0873$ cm
2. The second lens magnifies it by $m_2 = -v_2/u_2 = -(6.0\text{ cm})/(-4.0\text{ cm}) = +1.50$, so the final image is 1.50×0.0873 cm $= 0.131$ cm tall
3. A single lens images a distant object at height $f\theta$, so $f_{eff} = (0.131\text{ cm})/(8.73 \times 10^{-3}\text{ rad}) = 15.0$ cm

   Candidates who leave θ in degrees obtain a final image 7.5 cm tall, larger than any camera sensor, and should reject it on sight; the small-angle relation $h = f\theta$ needs radians.

(c) COMPARE · U1 **[2]**

Compare this combination with a single converging lens that forms an image of the same size of the same distant object, in terms of the distance from the front of the lens to the sensor.

1. A single lens giving the same image size must have $f = 15.0$ cm, so its sensor sits 15.0 cm behind the lens
2. The combination places the sensor only 12.0 cm behind its front lens, 3.0 cm (20%) shorter for the same image size, because the diverging lens shifts the effective lens position 3.0 cm in front of the first lens

   The telephoto idea is that effective focal length and physical length are different quantities. Candidates who add the focal lengths or the separations have not used the image-size result of part (b).

(d) PREDICT · U3 **[2]**

The camera is refocused on an object 2.00 m in front of the first lens, with the lens separation unchanged. Predict whether the sensor must move towards or away from the second lens, and by how much.

1. First lens: $1/v_1 = 1/10.0 - 1/200$, so $v_1 = 10.53$ cm and the virtual object for the second lens is at $u_2 = -(10.53 - 6.0)$ cm $= -4.53$ cm
2. $1/v_2 = -1/12.0 + 1/4.53$ gives $v_2 = 7.27$ cm, so the sensor must move 1.27 cm away from the second lens

   The shift is larger than the 1.22 cm a single 15.0 cm lens would need. Candidates who treat 2.00 m as effectively infinite should notice that it is only about 13 focal lengths.

16 CORE COURSE

STRUCTURED · DEMANDING · UPII-14.03 · UPII-14.05 · UPII-14.08 **[10]**

A student uses a transmission grating ruled with 450 lines per millimetre, illuminated at normal incidence, to look for the deuterium line in the light of a discharge lamp that contains hydrogen and deuterium in comparable amounts. The red Balmer line of ordinary hydrogen has wavelength 656.28 nm; the corresponding line of deuterium has wavelength 656.10 nm. The lamp also emits the blue-green Balmer line of hydrogen, of wavelength 486.13 nm.

(a) CALCULATE · U3 **[2]**

Calculate the grating spacing, and the highest order in which light of wavelength 656 nm can be observed.

1 $d = (1.00 \times 10^{-3}\ \text{m})/450 = 2.22 \times 10^{-6}\ \text{m}$

2 $d \sin\theta = m\lambda$ with $\sin\theta < 1$ requires $m < d/\lambda = 3.39$, so the highest order is $m = 3$

Rounding $d/\lambda = 3.39$ to the nearest integer rather than down is harmless here but not in general; the order must be the largest integer with $\sin\theta$ below 1.

(b) CALCULATE · U3 **[1]**

Calculate the angle at which the hydrogen line appears in second order.

1 $\sin\theta = 2(656.28 \times 10^{-9}\ \text{m})/(2.222 \times 10^{-6}\ \text{m}) = 0.591$, so $\theta = 36.2°$

Using the small-angle form $\theta \approx m\lambda/d$ gives 33.8°; at an angle this large the approximation fails.

(c) SHOW THAT · U2 **[2]**

By differentiating the grating equation, show that two lines separated in wavelength by a small amount $\Delta\lambda$ appear in order m with an angular separation $\Delta\theta = m\Delta\lambda/(d\cos\theta)$.

1 Differentiates $d\sin\theta = m\lambda$ with d and m fixed: $d\cos\theta\, d\theta = m\, d\lambda$

2 For small separations replaces the differentials by differences, $\Delta\theta = m\Delta\lambda/(d\cos\theta)$, with $\Delta\theta$ in radians

Candidates who differentiate $\sin\theta$ as $\cos\theta$ but forget that the result is in radians later quote a separation in degrees that is 57 times too small.

(d) CALCULATE · U3 **[1]**

Calculate the angular separation of the hydrogen and deuterium lines in second order.

1 $\Delta\theta = 2(0.18 \times 10^{-9}\ \text{m})/((2.222 \times 10^{-6}\ \text{m})\cos 36.2°) = 2.0 \times 10^{-4}\ \text{rad}$ (0.0115°)

Subtracting two angles each computed to 0.01° loses the answer entirely; the differential form is the reliable route for a separation this small.

(e) DETERMINE · U3 **[2]**

Determine the minimum number of grating lines that must be illuminated to resolve the two lines in second order, and the corresponding minimum width of the illuminated region of the grating.

1 Resolving power $\lambda/\Delta\lambda = mN$, so $N \geq (656.19\ \text{nm})/(2 \times 0.18\ \text{nm}) = 1.82 \times 10^{3}$ lines (1823)

2 Width $\geq 1823/(450\ \text{lines per mm}) = 4.1\ \text{mm}$

Resolution is set by the number of lines illuminated, not by the angular separation alone. A grating lit by a 2 mm beam separates the line centres in angle but cannot resolve them, however carefully it is read.

(f) COMPARE · U1 **[2]**

Compare working in third order with working in second order for this measurement.

1. Third order spreads the lines further: m/cos θ rises from 2.48 at 36.2° to 6.47 at 62.4°, so Δθ = 5.2 × 10^{-4} rad, about 2.6 times larger, and only 1216 lines (2.7 mm of grating) are needed to resolve the pair
2. Against that, gives a supported cost of third order: the pair lies at 62.4°, near the limit of usable angles, and higher orders are usually fainter; or the third-order spectrum overlaps the fourth, so the 486.13 nm line appears in fourth order at 61.0°, only 1.3° from the red pair, and each line must be identified with care

A comparison that lists only the advantage of higher order earns one mark. The fourth-order image of the 486.13 nm line, 1.3° from the red pair, is the practical trap, and a candidate who checks where other lamp lines fall has done what a spectroscopist would.

17

CORE COURSE

DATA ANALYSIS · DEMANDING · UPII-15.04 **[12]**

In a photoelectric experiment, monochromatic light falls on a clean metal cathode in an evacuated tube. For each wavelength the reverse potential difference between anode and cathode is increased until the photocurrent just falls to zero, and this stopping potential V_s is recorded. Each stopping potential has an uncertainty of ±0.03 V, and each wavelength is known to ±1 nm. Take e = 1.60 × 10^{-19} C and c = 3.00 × 10^{8} m/s.

(a) STATE · U2 **[1]**

State the photoelectric equation relating the stopping potential V_s to the frequency f of the light.

1. $eV_s = hf - \varphi$, where φ is the work function of the cathode, so V_s against f is a straight line of gradient h/e and intercept −φ/e

Writing $V_s = hf - \varphi$ without the charge e mixes volts with joules; the equation must balance in units before a gradient can be interpreted.

(b) EVALUATE · U4 **[2]**

Calculate the frequency of the light for each reading, and evaluate whether all six readings are consistent with a single straight line.

1. f = c/λ gives 10.00, 8.82, 7.89, 7.14, 6.52 and 6.00 × 10^{14} Hz for readings 1 to 6
2. Readings 1, 2, 3, 5 and 6 lie within ±0.03 V of a straight line, but reading 4 lies about 0.10 V below it, more than three times its uncertainty, so it is anomalous and is excluded from the fit

A line forced through all six points gives h ≈ 6.70 × 10^{-34} J·s and a shifted intercept. The anomalous reading is only found by comparing each residual with the stated uncertainty, which is the skill being tested.

(c) DETERMINE · U4 **[3]**

Determine the Planck constant from the gradient of a graph of V_s against f.

1. Plots V_s against f, or fits a line, using readings 1, 2, 3, 5 and 6
2. Gradient = (1.95 V − 0.28 V)/((10.00 − 6.00) × 10^{14} Hz) = 4.2 × 10^{-15} V·s, or 4.13 × 10^{-15} V·s by least squares (accept 4.0 to 4.3 × 10^{-15} V·s)
3. h = e × gradient = 6.6 × 10^{-34} J·s from the least-squares gradient, or 6.7 × 10^{-34} J·s from the two end points (accept 6.4 to 6.9 × 10^{-34} J·s)

Plotting against wavelength rather than frequency produces a curve, and a gradient read from it has no meaning. Candidates must linearise first.

(d) DETERMINE · U4 **[2]**

Determine the work function of the cathode, in electronvolts, and the threshold wavelength beyond which no photoelectrons are emitted.

1. Extrapolating to f = 0 gives an intercept of about −2.19 V, so φ = 2.2 eV (accept 2.1 to 2.3 eV)
2. $f_0 = \varphi/h = 5.3 \times 10^{14}$ Hz, so $\lambda_0 = c/f_0 = 5.7 \times 10^{2}$ nm (accept 540 to 590 nm, or a value consistent with the candidate's own gradient and intercept)

   Reading the intercept from a graph whose frequency axis starts at 6×10^{14} Hz rather than at zero gives a wrong work function; the intercept must be calculated from the line's equation or read from a true origin.

(e) ESTIMATE · U4 **[2]**

Estimate the uncertainty in your value of the Planck constant, stating the assumptions you make.

1. Takes the ±0.03 V uncertainty as the dominant one and draws the steepest and shallowest lines through the error bars of the extreme readings at 300 nm and 500 nm, with gradients of about 4.33 and 4.03×10^{-15} V·s
2. Half the spread gives $h = (6.6 \pm 0.2) \times 10^{-34}$ J·s, an uncertainty of about 4%; the ±1 nm wavelength uncertainty shifts each frequency by at most 0.33% and the gradient by about 1%, small beside the 4% from the voltages

   An estimate without its assumption, which uncertainty dominates, earns nothing for the second mark. Candidates who quote the least-squares standard error alone, about 0.07×10^{-34} J·s, should say that it reflects the scatter and not the ±0.03 V on each reading.

(f) JUSTIFY · U1 **[2]**

Justify the claim that these data cannot be explained by a classical wave model, in which the energy an electron absorbs grows with the intensity of the light and the time it is exposed.

1. In the wave model the maximum energy of an ejected electron should depend on intensity and exposure time, not on frequency, and light of any frequency should eventually eject electrons
2. The data show the maximum kinetic energy eV_s rising linearly with frequency, and the line extrapolates to a threshold near 5.3×10^{14} Hz (about 570 nm) below which no electrons would be emitted: a frequency dependence and a threshold the wave model does not predict, but which follow if each electron absorbs a single quantum of energy hf

   Answers that describe photons without pointing to a feature of this table, the linear dependence on f or the threshold it extrapolates to, have asserted the claim rather than justified it.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPII-12.04 · UPII-12.05 · UPII-12.07 **[15]**

A parallel-plate capacitor has circular plates of radius R = 5.0 cm separated by d = 2.0 mm in vacuum. It is being charged by a steady current I = 0.40 A carried by long straight leads that meet the centres of the plates along their common axis. Neglect fringing, so that the electric field is uniform between the plates and zero outside the gap. Take $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) EXPLAIN · U1 **[2]**

Ampère's law without Maxwell's correction gives contradictory results for the magnetic field at a point beside the gap, depending on which surface is chosen to span a circular path around the axis. Explain the contradiction, and how the displacement current resolves it.

1. A flat disc spanning a circle round the axis in the midplane of the gap is crossed by no conduction current, but a surface with the same boundary that bulges out to cut one of the leads is pierced by the current I, so $\oint B \cdot dl$ would have to be both zero and $\mu_0 I$
2. Maxwell added the displacement current $\varepsilon_0 \, d\Phi_E/dt$; the electric flux through the gap is q/ε_0, so $\varepsilon_0 \, d\Phi_E/dt = dq/dt = I$, and $\oint B \cdot dl = \mu_0(I_c + \varepsilon_0 \, d\Phi_E/dt) = \mu_0 I$ for both surfaces

   Candidates who say only that 'a changing electric field makes a magnetic field' have named the correction without showing that it removes the contradiction; the second mark needs $\varepsilon_0 \, d\Phi_E/dt$ equal to I.

(b) DERIVE · U2 **[4]**

Using the Ampère–Maxwell law, derive the magnitude of the magnetic field between the plates at a distance $r < R$ from the axis.

1. By symmetry B is azimuthal and uniform in magnitude on a circle of radius r centred on the axis in a plane between the plates, so $\oint B \cdot dl = 2\pi r B$
2. No conduction current crosses the disc bounded by that circle; the field $E = q/(\varepsilon_0 \pi R^2)$ is uniform, so $\Phi_E = \pi r^2 E = q r^2/(\varepsilon_0 R^2)$
3. The displacement current through the disc is $\varepsilon_0 \, d\Phi_E/dt = I r^2/R^2$
4. $2\pi r B = \mu_0 I r^2/R^2$, so $B = \mu_0 I r/(2\pi R^2)$

   The commonest error is to take the whole displacement current I through a circle of radius $r < R$, which gives $\mu_0 I/(2\pi r)$ and a field that diverges on the axis. Only the fraction r^2/R^2 of the changing flux is enclosed.

(c) DETERMINE · U2 **[2]**

Determine the magnitude of the magnetic field in the midplane of the gap at a distance $r > R$ from the axis, and state how it compares with the field at the same distance from one of the leads.

1. A disc of radius $r > R$ encloses all the changing electric flux, so the displacement current through it is I and $2\pi r B = \mu_0 I$, giving $B = \mu_0 I/(2\pi r)$
2. This is identical to the field at distance r from the long lead: outside the gap the magnetic field cannot distinguish conduction current from displacement current

   Candidates who expect the field beside the gap to be weaker, because no charge flows there, are reasoning from conduction current alone; the result is the physical content of Maxwell's correction.

(d) SKETCH · U1 **[2]**

Sketch a graph of the magnitude of B against r in the midplane of the gap, from $r = 0$ to $r = 3R$, marking the value at $r = R$.

1. B rises linearly from zero on the axis to a maximum at $r = R$, marked as $\mu_0 I/(2\pi R) = 1.6 \times 10^{-6}$ T
2. Beyond $r = R$, B falls as $1/r$, continuous at $r = R$, reaching one-third of the maximum (5.3×10^{-7} T) at $r = 3R$

   A sketch with a jump at $r = R$, or with the peak anywhere but at the edge, is inconsistent with parts (b) and (c) and is not credited for shape.

(e) SHOW THAT · U2 **[4]**

At the edge of the gap, r = R, show that the Poynting vector $S = E \times B/\mu_0$ points radially inwards, and that the total power flowing in through the cylindrical surface of radius R and height d equals the rate of increase of the energy stored in the capacitor, $d/dt(q^2/2C)$.

1. E is directed along the axis from the positive plate, the same sense as the current in the leads, and B at r = R circulates in the sense given by the right-hand rule about that current; with the axis as $\hat{z}$, $E \times B$ lies along $\hat{z} \times \hat{\phi} = -\hat{r}$, radially inwards
2. $|S| = EB/\mu_0 = (q/(\varepsilon_0 \pi R^2))(I/(2\pi R)) = qI/(2\pi^2 \varepsilon_0 R^3)$
3. Power entering through the side of the gap $= |S| \times 2\pi Rd = qId/(\pi \varepsilon_0 R^2)$
4. With $C = \varepsilon_0 \pi R^2/d$, $d/dt(q^2/2C) = (q/C)(dq/dt) = qId/(\pi \varepsilon_0 R^2)$, equal to the power entering: the energy reaches the capacitor through the sides of the gap, not along the wires

The inward direction is the discriminating step. A candidate who takes B in the wrong sense finds energy leaving a capacitor that is being charged, and should treat that contradiction as a signal to recheck, not as a result.

(f) PREDICT · U1 **[1]**

The capacitor is later discharged through the same leads. Predict the direction of the Poynting vector at the edge of the gap during the discharge.

1. E keeps its direction while the current, and so B, reverses, so S points radially outwards: energy leaves the gap through its sides

Candidates who reverse both E and B predict inward flow again; the field between the plates depends on the charge, which keeps its sign while it decreases.

19

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPII-09.04 · UPII-10.07 · UPII-12.03 **[15]**

A long solenoid of radius a = 4.0 cm has n = 8000 turns per metre and carries a current I_0 = 20 A. Coaxial with it, outside the winding, a thin insulating ring of radius b = 6.0 cm and mass m = 0.50 g carries a charge Q = 0.50 μC spread uniformly around it. The ring can turn freely about the common axis, and the apparatus is in vacuum. The solenoid current is reduced steadily to zero over T = 0.50 s. Treat the solenoid as ideal, with no magnetic field outside its winding, and neglect the magnetic field of the ring itself. Take $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.

(a) DESCRIBE · U1 **[2]**

Describe the electric field induced while the current is falling: its direction, the shape of its field lines, and why it cannot be described by an electric potential.

1. By axial symmetry the induced field is azimuthal, with closed circular field lines centred on the axis and a magnitude that depends only on r; its sense is the one that would drive a current opposing the fall in flux, the same sense as the solenoid current
2. Its circulation round a closed field line, $\oint E \cdot dl = -d\Phi_B/dt$, is not zero, whereas any field derived from a potential has zero circulation round every closed path, so no potential function exists

Describing the field as radial, by analogy with the field of a charge, is the misconception this part is set to catch; an induced field has no sources or sinks, only circulation.

(b) DERIVE · U2 **[4]**

Using Faraday's law in the form $\oint E \cdot dl = -d\Phi_B/dt$, derive the magnitude of the induced electric field at distance r from the axis, for r < a and for r > a, in terms of the rate of change dB/dt of the field inside the solenoid.

1. Chooses a circular path of radius r centred on the axis, on which E is tangential and uniform, so $\oint E \cdot dl = 2\pi rE$
2. For r < a the flux enclosed is $\pi r^2 B$, so $2\pi rE = \pi r^2 |dB/dt|$ and $E = (r/2)|dB/dt|$
3. For r > a only the solenoid's cross-section carries flux, $\pi a^2 B$, so $2\pi rE = \pi a^2 |dB/dt|$
4. $E = (a^2/(2r))|dB/dt|$, which agrees with the inside result at r = a

Using the area of the path instead of the area that carries flux for r > a gives a field that grows without limit outside the solenoid. The continuity check at r = a catches it immediately, which is why the prompt asks for both regions.

(c) CALCULATE · U3 **[2]**

Calculate the magnetic field inside the solenoid before the current starts to fall, and the magnitude of the induced electric field at the ring while the current is falling.

1. $B_0 = \mu_0 n I_0 = (4\pi \times 10^{-7}\ \text{T·m/A})(8000\ \text{m}^{-1})(20\ \text{A}) = 0.201\ \text{T}$, so $|dB/dt| = 0.201\ \text{T}/0.50\ \text{s} = 0.402\ \text{T/s}$
2. At r = b: $E = (a^2/(2b))|dB/dt| = ((0.040\ \text{m})^2/(2 \times 0.060\ \text{m}))(0.402\ \text{T/s}) = 5.4 \times 10^{-3}\ \text{V/m}$

   Substituting r = b into the inside expression (r/2)|dB/dt| gives 1.2 × 10⁻² V/m, more than twice too large, and is the error of a candidate who has not decided which region the ring is in.

(d) DERIVE · U2 **[3]**

Using the rotational form of Newton's second law, τ = dL/dt, derive the angular momentum the ring has acquired once the current has fallen to zero, and show that it depends neither on T nor on b, provided b > a.

1. Every element of the ring's charge feels a tangential force at radius b, so the torque about the axis is $\tau = QEb = (Qa^2/2)|dB/dt|$
2. Integrating over the whole fall in field, $L = \int \tau\, dt = (Qa^2/2)\int |dB/dt|\, dt = Qa^2 B_0/2$
3. The ring radius b has cancelled and only the total change in B appears, so neither b nor the duration T of the decrease affects L

   The cancellation of b is the result to notice: a larger ring feels a weaker field but with a longer lever arm. Candidates who stop at τ and multiply by T without integrating obtain the right number here only because the fall is uniform, and have not shown the independence of T.

(e) CALCULATE · U3 **[2]**

Calculate the angular speed of the ring once the current has fallen to zero.

1. $L = (0.50 \times 10^{-6}\ \text{C})(0.040\ \text{m})^2(0.201\ \text{T})/2 = 8.0 \times 10^{-11}\ \text{kg·m}^2\text{/s}$
2. Moment of inertia $mb^2 = (0.50 \times 10^{-3}\ \text{kg})(0.060\ \text{m})^2 = 1.8 \times 10^{-6}\ \text{kg·m}^2$, so $\omega = L/(mb^2) = 4.5 \times 10^{-5}\ \text{rad/s}$

   Using $Qb^2B_0/2$, the ring's own area rather than the solenoid's, gives an angular speed 2.25 times too large; the flux lives only inside the winding.

(f) EVALUATE · U1 **[2]**

A student claims that the ring starts to turn because the solenoid's magnetic field pushes on its charges. Evaluate this claim.

1. The claim fails: a magnetic force qv × B needs moving charges and the ring starts from rest, and in any case the ideal solenoid's field is zero at r = b, so no magnetic force acts on the ring at any time
2. The torque comes from the induced, non-conservative electric field, which exists at the ring even though B is zero there: what matters is the changing flux enclosed by the ring's path, not the field at the ring's own position

   This is the discriminating idea of the question. Candidates who hold that a field must be present where a force acts are obliged by the ideal solenoid to accept that a changing magnetic flux elsewhere produces an electric field here.

University Physics III · Midterm Examination

Oscillations, mechanical waves and wave phenomena, geometric and physical optics, and temperature and kinetic theory: derive the equation of motion, the total phase or the molecular flux before quoting a formula, then test each result against the approximation that made it valid.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**74**	**14**	**8**

SECTION — **UPIII · Midterm · University Physics III — Midterm examination**

IN THE FORMAT OF — A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation

DEMAND — **Routine 5 · Demanding 7 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPIII-01 · UPIII-02 · UPIII-03 · UPIII-04 · UPIII-05 · UPIII-06

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Physics III · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-01.03 · UPIII-01.04 **[1]**

A block on a light horizontal spring moves in simple harmonic motion on a frictionless surface. At one instant its speed is 60% of its maximum speed. What fraction of the oscillator's total mechanical energy is stored as elastic potential energy at that instant?

A 0.36

B 0.40

C 0.80

D 0.64

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-01.09 **[1]**

A lightly damped oscillator of mass m and natural angular frequency ω_0 is driven by a force $F_0\cos \omega t$ with $\omega = 3\omega_0$. Which statement correctly describes its steady-state displacement?

A Its amplitude is about $F_0/(8m\omega_0^2)$, and it lags the driving force by nearly π.

B Its amplitude is about $F_0/(8m\omega_0^2)$, and it lags the driving force by $\pi/2$.

C Its amplitude is about $F_0/(9m\omega_0^2)$, and it lags the driving force by nearly π.

D Its amplitude is about $F_0/(8m\omega_0^2)$, and it is nearly in phase with the driving force.

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.05 · UPIII-02.08 **[1]**

A narrow tube is either open at both ends or closed at one end; you are not told which. Two of its successive resonant frequencies are 497 Hz and 639 Hz, with no resonance between them. Ignoring end corrections, what is the lowest resonant frequency of the air column?

A 35.5 Hz

B 71.0 Hz

C 142 Hz

D 355 Hz

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.07 · UPIII-03.03 **[1]**

Two small loudspeakers are driven in phase by the same signal generator at a single frequency. A point P is the same distance from both, and reflections are negligible. With either loudspeaker operating alone, the sound-intensity level at P is 70.0 dB. What is the level at P when both operate together?

A 70.0 dB

B 73.0 dB

C 76.0 dB

D 140 dB

Answer ☐

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-03.05 **[1]**

Two loudspeakers emit steady tones of equal amplitude at 396.0 Hz and 401.0 Hz, and a microphone records their superposition. Which statement describes the recorded signal?

A A 398.5 Hz tone whose loudness peaks 2.5 times per second

B A 797 Hz tone whose loudness peaks 5 times per second

C A 398.5 Hz tone whose loudness peaks 5 times per second

D A steady 5.0 Hz tone, below the range of hearing

Answer ☐

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-04.03 · UPIII-04.04 **[1]**

A step-index optical fibre has a core of refractive index 1.52 and a cladding of refractive index 1.46. Its flat entry face is perpendicular to its axis and is in air of refractive index 1.00. What is the largest angle to the fibre axis at which a ray in air can enter the core and still be guided by total internal reflection at the core–cladding boundary?

A 16.2°

B 41.1°

C 73.8°

D 25.0°

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-05.09 **[1]**

Unpolarized light of intensity I_0 passes in turn through three ideal linear polarizers whose transmission axes are at 30°, 90° and 135° to the vertical, all measured in the same rotational sense. What intensity emerges from the third polarizer?

A $I_0/16$

B $I_0/8$

C $I_0/(4\sqrt{2})$

D 0

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-06.02 **[1]**

An aluminium bar 1.50 m long fits snugly, and without stress, between two rigid supports at 12 °C. It is heated uniformly to 47 °C, and the supports prevent any change in its length. For aluminium, Young's modulus is 7.0×10^{10} Pa and the coefficient of linear expansion is $2.3 \times 10^{-5}\ \text{K}^{-1}$. What is the compressive stress in the bar?

A 3.8×10^{7} Pa

B 5.6×10^{7} Pa

C 8.5×10^{7} Pa

D 5.2×10^{8} Pa

Answer

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9 CORE COURSE

STRUCTURED · ROUTINE · UPIII-02.02 · UPIII-02.03 · UPIII-02.05 **[8]**

A vibrating-wire strain gauge bolted to a concrete beam contains a steel wire of length L = 0.150 m and linear mass density $\mu = 3.85 \times 10^{-4}$ kg/m, clamped at both ends. An electromagnet plucks the wire and a pick-up coil measures its fundamental frequency, which is 800 Hz when the gauge is installed.

(a) STATE **[1]**

State the linear wave equation obeyed by small transverse displacements y(x, t) of a wire under tension T with linear mass density μ, and the wave speed it implies.

(b) CALCULATE **[2]**

Calculate the speed of transverse waves on the wire and the tension in the wire when the gauge is installed.

(c) PREDICT **[2]**

Loading the beam raises the tension in the wire by 3.0%. Treating L and μ as unchanged, predict the new fundamental frequency.

(d) CALCULATE **[3]**

In a separate test, a long wire with the same μ and the same tension as in (b) carries a sinusoidal transverse travelling wave of frequency 800 Hz and amplitude 0.20 mm, and its far end is terminated so that no wave is reflected. Calculate the average power the wave transmits along the wire.

10 CORE COURSE

STRUCTURED · DEMANDING · UPIII-01.01 · UPIII-01.02 · UPIII-01.08 · UPIII-01.09 **[12]**

A cylindrical instrument buoy floats upright in calm sea water. Its mass is m = 850 kg, its horizontal cross-sectional area at the waterline is A = 0.500 m², and its sides are vertical from 0.40 m below to 0.40 m above the calm waterline. Take the density of sea water as ρ = 1025 kg/m³ and g = 9.81 m/s². Model the water as exerting a hydrostatic buoyant force together with a linear damping force of magnitude b|ż| that opposes the buoy's vertical velocity, ignore any water carried along with the buoy, and write γ = b/m and Q = ω₀/γ, where ω₀ is the undamped angular frequency.

(a) DERIVE **[3]**

Ignoring damping, derive from Newton's second law and Archimedes' principle the equation of motion for a small vertical displacement z of the buoy from its floating equilibrium, and hence identify ω₀.

(b) CALCULATE **[2]**

Calculate the undamped period of the buoy's vertical oscillation.

(c) DETERMINE **[3]**

The buoy is pushed down 0.120 m in calm water and released. After 5 complete oscillations its amplitude has fallen to 0.030 m. Determine γ and Q.

(d) PREDICT **[2]**

A regular train of surface waves of amplitude $\eta_0 = 0.050$ m, with period equal to your answer to (b), now passes the buoy. Taking the driving force as $\rho g A \eta$, where $\eta = \eta_0 \cos \omega t$ is the vertical displacement of the water surface, predict the amplitude of the buoy's steady-state oscillation.

(e) EVALUATE **[2]**

Evaluate whether the model used in (d) can be trusted for these waves.

11 CORE COURSE

STRUCTURED · DEMANDING · UPIII-05.01 · UPIII-05.04 · UPIII-05.07 **[10]**

A discharge lamp contains a mixture of hydrogen and deuterium. Its blue-green Balmer emission consists of two lines, at $\lambda_1 = 486.13$ nm from hydrogen and $\lambda_2 = 486.00$ nm from deuterium. To show that both lines are present, a student may use either a transmission grating ruled with 1200 lines per millimetre over a width of 25.0 mm, used at normal incidence and able to be illuminated over any chosen width, or a Michelson interferometer whose movable mirror can be moved through up to 4.0 mm from the position at which the two arms are equal in length.

(a) CALCULATE **[2]**

Calculate the chromatic resolving power needed to separate the two lines.

(b) DETERMINE **[3]**

Determine the highest diffraction order the grating produces at this wavelength, and hence the minimum width of grating that must be illuminated to resolve the lines.

(c) SHOW THAT **[3]**

As the mirror moves a distance x from the position of equal arm lengths, the detector records fringes whose visibility falls periodically to a minimum and recovers. Show that the mirror displacement between successive visibility minima is $\lambda_1\lambda_2/[2(\lambda_1 - \lambda_2)] \approx \lambda^2/(2\Delta\lambda)$.

(d) SKETCH **[2]**

Assuming the two lines are of equal intensity, sketch a graph of fringe visibility against mirror displacement from 0 to 4.0 mm, marking the displacements at which the visibility is a minimum.

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12 CORE COURSE

STRUCTURED · DISCRIMINATING · UPIII-04.03 · UPIII-04.07 **[11]**

Reflective road markings contain transparent glass beads, each a sphere of radius R = 0.300 mm, half-embedded in white paint. Light from a car's headlamps enters the exposed front of a bead and is focused towards the back surface, where the paint scatters it; light scattered from a small bright patch there is sent back towards the car by the bead. Treat the incident light as a parallel beam travelling along an axis through the bead's centre, in air of refractive index 1.00. For refraction at a single spherical surface use $n_1/s + n_2/s' = (n_2 - n_1)/R$, where s is positive for a real object in front of the surface, s' is positive for a real image behind it, and R is positive when the centre of curvature lies behind the surface.

(a) SHOW THAT **[2]**

Show that, in the paraxial limit, refraction at the front surface alone would bring the beam to a focus a distance $nR/(n-1)$ behind that surface, where n is the refractive index of the glass.

(b) DETERMINE **[2]**

Determine the refractive index a bead must have for the paraxial focus to lie exactly on the paint at its back surface, and state where the paraxial focus lies for a bead of ordinary glass with $n = 1.50$.

(c) SHOW THAT **[3]**

Take the centre of the bead as origin, with the incident light travelling in the $+x$ direction, so that the axis meets the back surface at $(R, 0)$. A ray entering parallel to the axis at height h above it has angle of incidence i, where $\sin i = h/R$, and angle of refraction r. Without making the paraxial approximation, show that this ray reaches the back surface at the point whose radius makes an angle $i - 2r$ with the $+x$ axis, on the opposite side of the axis from the point of entry.

(d) CALCULATE **[2]**

For a bead with $n = 2.00$, calculate the perpendicular distance from the axis of the point at which the back surface is reached by the ray entering at $h = 0.50R$, and by the ray entering at $h = 0.90R$.

(e) EVALUATE **[2]**

High-index road-marking beads are typically made from glass with n close to 1.9 rather than exactly 2.00. Evaluate this choice in the light of your results.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIII-03.06 · UPIII-03.07 · UPIII-03.09 **[10]**

To measure the speed of sound outdoors, a group mounts a buzzer of steady but unknown frequency f_0 on the roof of a car. On a calm day at 21 °C the car is driven at a series of steady speeds v_s along a straight runway directly towards a microphone at the same height, and a spectrum analyser records the frequency f received while the car is still far away. Each speed is logged by satellite navigation. In still air the moving-source Doppler relation is $f = f_0 v/(v - v_s)$, where v is the speed of sound.

Received frequency at each approach speed

Run	Approach speed v_s (m/s)	Received frequency f (Hz)	1/f (10^{-4} s)
1	4.0	2175.6	4.5964
2	8.0	2201.0	4.5434
3	12.0	2227.8	4.4887
4	16.0	2254.6	4.4354
5	20.0	2282.9	4.3804
6	24.0	2311.2	4.3268

(a) SHOW THAT **[2]**

Show that the relation predicts a straight line when 1/f is plotted against v_s, and identify its gradient and its intercept on the 1/f axis.

(b) DETERMINE **[4]**

Use the readings to determine f_0 and the speed of sound v.

(c) JUSTIFY **[2]**

Justify, using the readings, the use of the exact Doppler relation rather than the low-speed approximation $f \approx f_0(1 + v_s/v)$.

(d) PREDICT **[2]**

On a second day a steady wind of 5.0 m/s blows along the runway from the microphone towards the approaching car. Predict how the intercept and the gradient of the plot of 1/f against v_s would change, and the value of v that the analysis in (b) would then return.

14 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPIII-06.03 · UPIII-06.04 · UPIII-06.05 · UPIII-06.06 · UPIII-06.07 **[15]**

A sealed glass bulb of volume $V = 1.50 \times 10^{-3}\ \text{m}^3$ contains argon at a temperature of 293 K and an initial pressure of 3.0 Pa. A circular hole 60 μm in diameter is then opened in a thin foil in the bulb's wall, leading directly to a high vacuum, while the bulb is held at 293 K. Model the escape as effusion, in which atoms leave through the hole independently of one another, and treat the argon as an ideal gas with a Maxwell–Boltzmann distribution of speeds, whose mean speed is $\langle v\rangle = \sqrt{(8kT/(\pi m))}$. Take the Boltzmann constant $k = 1.381 \times 10^{-23}$ J/K, the mass of an argon atom as 39.95 u with $1\ \text{u} = 1.661 \times 10^{-27}$ kg, and the effective diameter of an argon atom as about 0.36 nm.

(a) SHOW THAT **[4]**

Consider a gas with n atoms per unit volume and an isotropic distribution of velocities. Show that the number of atoms striking unit area of the wall per unit time is $\Phi = \tfrac{1}{4}n\langle v\rangle$.

(b) DERIVE **[3]**

Derive, from conservation of the number of atoms and the ideal-gas equation of state, an expression for the pressure in the bulb as a function of time t, identifying the time constant τ in terms of V, the area A of the hole and $\langle v\rangle$.

(c) CALCULATE **[3]**

Calculate $\langle v\rangle$ for the argon, the time constant τ, and the time taken for the pressure to fall from 3.0 Pa to 0.30 Pa.

(d) ESTIMATE **[3]**

Estimate the mean free path of the argon atoms at the initial pressure, stating the model you use, and hence decide whether treating the escape as effusion is justified.

(e) EXPLAIN **[2]**

If the bulb were thermally insulated instead of held at 293 K, the argon remaining in it would cool as it leaked. Explain why, given that the gas does no work on any piston.

MARK SCHEME

University Physics III · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-01.03 · UPIII-01.04 **[1]**

A block on a light horizontal spring moves in simple harmonic motion on a frictionless surface. At one instant its speed is 60% of its maximum speed. What fraction of the oscillator's total mechanical energy is stored as elastic potential energy at that instant?

KEY **D** **0.64**

Kinetic energy goes as the square of the speed, so at 0.60 of the maximum speed the block holds $0.60^2 = 0.36$ of the total energy as kinetic energy, and the remaining 0.64 is elastic potential energy. A is that kinetic fraction, chosen by a candidate who answers for the wrong store. B shares the energy in proportion to speed, $1 - 0.60 = 0.40$, which would only be right if kinetic energy were linear in v. C is the displacement at that instant as a fraction of the amplitude, $x/A = \sqrt{(1 - 0.36)} = 0.80$: a correct number for a different quantity, left unconverted to energy, which goes as x^2.

U1

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-01.09 **[1]**

A lightly damped oscillator of mass m and natural angular frequency ω_0 is driven by a force $F_0\cos\omega t$ with $\omega = 3\omega_0$. Which statement correctly describes its steady-state displacement?

KEY **A** **Its amplitude is about $F_0/(8m\omega_0^2)$, and it lags the driving force by nearly π.**

With light damping the steady-state amplitude is $F_0/(m\sqrt{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}) \approx F_0/(m|\omega_0^2 - 9\omega_0^2|) = F_0/(8m\omega_0^2)$, and well above resonance the response is controlled by inertia, so the displacement is almost in antiphase with the force. B keeps the right amplitude but quotes the phase lag at resonance, where the damping alone limits the motion; above resonance the lag moves on towards π. C has the right phase but uses $F_0/(m\omega^2)$, the response of a free mass with no spring: it looks plausible because inertia dominates, but the spring term ω_0^2 still subtracts from ω^2 and turns 9 into 8. D describes driving well below resonance, where the spring controls the motion and the displacement follows the force; it attracts the candidate who reasons that a mass must move the way it is pushed.

U1

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.05 · UPIII-02.08 **[1]**

A narrow tube is either open at both ends or closed at one end; you are not told which. Two of its successive resonant frequencies are 497 Hz and 639 Hz, with no resonance between them. Ignoring end corrections, what is the lowest resonant frequency of the air column?

KEY **B** **71.0 Hz**

If the tube were open at both ends, every integer harmonic would be present, successive resonances would differ by $f_1 = 639 - 497 = 142$ Hz, and 497 Hz would have to be a whole-number multiple of 142 Hz; it is not, since $497/142 = 3.5$. The tube must therefore be closed at one end, where only odd harmonics occur and successive resonances differ by $2f_1$, so $f_1 = 142/2 = 71.0$ Hz, and the given frequencies are the 7th and 9th harmonics ($497/71 = 7$, $639/71 = 9$). C is the spacing itself, the answer for the open tube that the whole-number test rules out. D is $497 - 142 = 355$ Hz, the 5th harmonic and the resonance just below 497 Hz, chosen by a candidate who steps down once and takes that as the lowest. A halves the spacing twice: once correctly, because successive odd harmonics are $2f_1$ apart, and again from recalling that closing one end halves an open tube's fundamental, a correction the first step already contains.

U3

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.07 · UPIII-03.03 **[1]**

Two small loudspeakers are driven in phase by the same signal generator at a single frequency. A point P is the same distance from both, and reflections are negligible. With either loudspeaker operating alone, the sound-intensity level at P is 70.0 dB. What is the level at P when both operate together?

KEY **C** **76.0 dB**

The sources are coherent and the path difference to P is zero, so the pressure amplitudes add in phase: the amplitude doubles, the intensity quadruples, and the level rises by 10 $\log_{10} 4$ = 6.0 dB to 76.0 dB. B is the correct answer for incoherent sources, such as two independent noise generators, whose intensities add because the interference term averages to zero; it is the most tempting option because intensity addition is the rule candidates meet first. D adds the two decibel levels, which are logarithms of intensity ratios and cannot be added. A treats the level as fixed by either source alone, as if the second contributed nothing at a point where it arrives exactly in step with the first.

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-03.05 **[1]**

Two loudspeakers emit steady tones of equal amplitude at 396.0 Hz and 401.0 Hz, and a microphone records their superposition. Which statement describes the recorded signal?

KEY **C** **A 398.5 Hz tone whose loudness peaks 5 times per second**

Adding the two waves gives $2A\cos(2\pi \times 2.5t)\cos(2\pi \times 398.5t)$: an oscillation at the mean frequency whose amplitude is modulated by an envelope of frequency 2.5 Hz. Loudness depends on the magnitude of that envelope, which reaches a maximum twice in each of its cycles, so the loudness peaks 401.0 − 396.0 = 5.0 times per second. A is the trap: it reads the envelope's own frequency as the rate of loudness peaks, missing that the envelope's positive and negative extremes are equally loud. B adds the two frequencies instead of averaging them. D mistakes the beat for a sound of its own: a linear superposition contains no component at the difference frequency, which appears only as a slow variation in loudness.

U1

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-04.03 · UPIII-04.04 **[1]**

A step-index optical fibre has a core of refractive index 1.52 and a cladding of refractive index 1.46. Its flat entry face is perpendicular to its axis and is in air of refractive index 1.00. What is the largest angle to the fibre axis at which a ray in air can enter the core and still be guided by total internal reflection at the core–cladding boundary?

KEY **D** **25.0°**

The critical angle at the core–cladding boundary is $\theta_c = \sin^{-1}(1.46/1.52) = 73.8°$, measured from the normal to that boundary, so a guided ray inside the core makes at most $90° - \theta_c = 16.2°$ with the axis. Snell's law at the entry face then gives $\sin\theta = 1.52\sin(90° - \theta_c) = \sqrt{(1.52^2 - 1.46^2)} = 0.423$, so $\theta = 25.0°$. C is the critical angle itself, measured from the normal to the core–cladding boundary rather than from the fibre axis. A is the correct angle inside the core, chosen by a candidate who forgets to apply Snell's law at the entry face: the ray bends towards the normal as it passes from air into the glass, so its angle to the axis is larger in air than inside. B is the critical angle for a glass–air boundary, $\sin^{-1}(1/1.52)$, which would decide total internal reflection at the wall of an unclad fibre but plays no part at a core–cladding boundary.

U3

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-05.09 **[1]**

Unpolarized light of intensity I_0 passes in turn through three ideal linear polarizers whose transmission axes are at 30°, 90° and 135° to the vertical, all measured in the same rotational sense. What intensity emerges from the third polarizer?

KEY **A** **$I_0/16$**

The first polarizer transmits half of the unpolarized light, $I_0/2$, now polarized at 30° to the vertical. Malus's law depends only on the angle between the light's polarization and the next transmission axis: 60° at the second polarizer, giving a factor $\cos^2 60° = 1/4$, and 45° at the third, giving $\cos^2 45° = 1/2$, so the result is $(I_0/2)(1/4)(1/2) = I_0/16$. B applies Malus's law correctly twice but omits the factor 1/2 at the first polarizer, treating unpolarized light as though it were already polarized along the first axis. C uses $\cos\theta$, the factor by which the field amplitude falls, in place of $\cos^2\theta$, the factor for intensity: $(I_0/2)(1/2)(1/\sqrt{2}) = I_0/(4\sqrt{2})$. D measures each angle from the vertical rather than from the previous transmission axis, so the horizontal second polarizer appears crossed with the light and to transmit nothing; the light reaching it is polarized at 30° to the vertical, only 60° from that axis.

U3

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-06.02 **[1]**

An aluminium bar 1.50 m long fits snugly, and without stress, between two rigid supports at 12 °C. It is heated uniformly to 47 °C, and the supports prevent any change in its length. For aluminium, Young's modulus is 7.0×10^{10} Pa and the coefficient of linear expansion is $2.3 \times 10^{-5}\ \mathrm{K^{-1}}$. What is the compressive stress in the bar?

KEY **B** **5.6×10^7 Pa**

The supports prevent the free thermal strain $\alpha\Delta T$, so the stress is $\sigma = Y\alpha\Delta T = 7.0 \times 10^{10} \times 2.3 \times 10^{-5} \times 35 = 5.6 \times 10^7$ Pa, whatever the bar's length or cross-section. C is Y multiplied by the free expansion $\Delta L = \alpha L \Delta T = 1.21 \times 10^{-3}$ m, the extension used as though it were the strain; A divides the strain by the length a second time. Both come from believing the length of the bar must matter, when strain is a fractional change and the length cancels. D uses the final absolute temperature, 320 K, in place of the 35 K rise: expansion responds to the change in temperature, and a change of 35 °C is a change of 35 K.

U3

9

CORE COURSE

STRUCTURED · ROUTINE · UPIII-02.02 · UPIII-02.03 · UPIII-02.05 **[8]**

A vibrating-wire strain gauge bolted to a concrete beam contains a steel wire of length L = 0.150 m and linear mass density μ = 3.85 × 10⁻⁴ kg/m, clamped at both ends. An electromagnet plucks the wire and a pick-up coil measures its fundamental frequency, which is 800 Hz when the gauge is installed.

(a) STATE · U1 **[1]**

State the linear wave equation obeyed by small transverse displacements y(x, t) of a wire under tension T with linear mass density μ, and the wave speed it implies.

1 $\partial^2 y/\partial t^2 = (T/\mu)\,\partial^2 y/\partial x^2$, so the wave speed is $v = \sqrt{(T/\mu)}$

(b) CALCULATE · U3 **[2]**

Calculate the speed of transverse waves on the wire and the tension in the wire when the gauge is installed.

1 both clamped ends are nodes, so the fundamental has $\lambda_1 = 2L = 0.300$ m and $v = f_1\lambda_1 = 800 \times 0.300 = 240$ m/s

2 $T = \mu v^2 = 3.85 \times 10^{-4} \times 240^2 = 22.2$ N

(c) PREDICT · U3 **[2]**

Loading the beam raises the tension in the wire by 3.0%. Treating L and μ as unchanged, predict the new fundamental frequency.

1 with L and μ fixed, $f_1 = (1/2L)\sqrt{(T/\mu)}$, so f_1 is proportional to $\sqrt{T}$

2 $f_1 = 800 \times \sqrt{1.030} = 812$ Hz (811.9 Hz), a rise of about 1.5%

Candidates who scale the frequency in proportion to the tension predict 824 Hz; the square root is the whole of this part.

(d) CALCULATE · U3 **[3]**

In a separate test, a long wire with the same μ and the same tension as in (b) carries a sinusoidal transverse travelling wave of frequency 800 Hz and amplitude 0.20 mm, and its far end is terminated so that no wave is reflected. Calculate the average power the wave transmits along the wire.

1 $\omega = 2\pi \times 800 = 5.03 \times 10^3$ rad/s

2 uses the average transmitted power $P = \tfrac{1}{2}\mu\omega^2 A^2 v$

3 $P = \tfrac{1}{2} \times 3.85 \times 10^{-4} \times (5.03 \times 10^3)^2 \times (2.0 \times 10^{-4})^2 \times 240 = 4.7 \times 10^{-2}$ W (46.7 mW)

Omitting the ½, which comes from averaging $\sin^2$ over a cycle, doubles the answer; using f in place of ω makes it about 39 times too small.

10

CORE COURSE

STRUCTURED · DEMANDING · UPIII-01.01 · UPIII-01.02 · UPIII-01.08 · UPIII-01.09 **[12]**

A cylindrical instrument buoy floats upright in calm sea water. Its mass is m = 850 kg, its horizontal cross-sectional area at the waterline is A = 0.500 m², and its sides are vertical from 0.40 m below to 0.40 m above the calm waterline. Take the density of sea water as ρ = 1025 kg/m³ and g = 9.81 m/s². Model the water as exerting a hydrostatic buoyant force together with a linear damping force of magnitude $b|\dot{z}|$ that opposes the buoy's vertical velocity, ignore any water carried along with the buoy, and write γ = b/m and $Q = \omega_0/\gamma$, where ω_0 is the undamped angular frequency.

(a) DERIVE · U2 **[3]**

Ignoring damping, derive from Newton's second law and Archimedes' principle the equation of motion for a small vertical displacement z of the buoy from its floating equilibrium, and hence identify ω_0.

1. at equilibrium the buoyant force $\rho g V_0$ on the submerged volume V_0 balances the weight mg
2. a further downward displacement z submerges an extra volume Az because the sides are vertical, so the net force is $-\rho g A z$: a restoring force proportional to the displacement
3. Newton's second law gives $m\,d^2z/dt^2 = -\rho g A z$, the simple-harmonic equation $d^2z/dt^2 = -\omega_0^2 z$ with $\omega_0 = \sqrt{(\rho g A/m)}$

Candidates who take $\rho g A z$ as the whole buoyant force, without first cancelling the weight against the equilibrium buoyancy, are left with a constant term in the equation of motion; the equilibrium step is what makes the restoring force linear.

(b) CALCULATE · U3 **[2]**

Calculate the undamped period of the buoy's vertical oscillation.

1. $\rho g A = 1025 \times 9.81 \times 0.500 = 5.03 \times 10^3$ N/m, so $\omega_0 = \sqrt{(5.03 \times 10^3/850)} = 2.43$ rad/s
2. $T = 2\pi/\omega_0 = 2.58$ s

Only the waterline area enters. Candidates who bring in the buoy's height or total volume have lost track of which quantity sets how the buoyant force changes with z.

(c) DETERMINE · U4 **[3]**

The buoy is pushed down 0.120 m in calm water and released. After 5 complete oscillations its amplitude has fallen to 0.030 m. Determine γ and Q.

1. for light damping the amplitude decays as $z_0 e^{(-\gamma t/2)}$, and 5 oscillations take $5T = 12.9$ s (the damped period differs from T by about 0.1%)
2. $\gamma = 2\ln(0.120/0.030)/(5T) = 2 \times 1.386/12.92 = 0.215\ s^{-1}$
3. $Q = \omega_0/\gamma = 2.43/0.215 = 11.3$

The factor of 2 is what this part is set to catch: the amplitude decays at $\gamma/2$ and the energy at γ. Candidates who write the amplitude as $z_0 e^{(-\gamma t)}$ obtain $Q \approx 22.7$.

(d) PREDICT · U3 **[2]**

A regular train of surface waves of amplitude $\eta_0 = 0.050$ m, with period equal to your answer to (b), now passes the buoy. Taking the driving force as $\rho g A \eta$, where $\eta = \eta_0 \cos \omega t$ is the vertical displacement of the water surface, predict the amplitude of the buoy's steady-state oscillation.

1. at resonance the steady-state amplitude is $F_0/(b\omega_0) = F_0/(m\gamma\omega_0)$, with $F_0 = \rho g A \eta_0 = 5.03 \times 10^3 \times 0.050 = 251$ N
2. since $\rho g A = m\omega_0^2$, this is $Q\eta_0 = 11.3 \times 0.050 = 0.57$ m

Candidates who use the undamped response $F_0/(m|\omega_0^2 - \omega^2|)$ find an infinite amplitude and stop; at resonance only the damping limits the motion.

(e) EVALUATE · U1 **[2]**

Evaluate whether the model used in (d) can be trusted for these waves.

1. the predicted 0.57 m amplitude exceeds the 0.40 m over which the sides are vertical (relative to the moving surface the excursion is also 0.57 m, since z lags η by $\pi/2$ at resonance and $\sqrt{(0.57^2 + 0.050^2)} = 0.57$ m), so the waterline area, and with it the restoring force per unit displacement, would change during each cycle: the linear restoring force the model rests on breaks down
2. judgement: the 0.57 m prediction is not reliable; at resonance the linear model holds only for wave amplitudes below about $0.40\ \text{m}/Q \approx 0.035$ m

The discriminating step is to test the answer against the range of validity given in the stem rather than accept it: a Q of 11 turns a modest wave into a displacement the model was never built for.

11 CORE COURSE

STRUCTURED · DEMANDING · UPIII-05.01 · UPIII-05.04 · UPIII-05.07 **[10]**

A discharge lamp contains a mixture of hydrogen and deuterium. Its blue-green Balmer emission consists of two lines, at $\lambda_1 = 486.13$ nm from hydrogen and $\lambda_2 = 486.00$ nm from deuterium. To show that both lines are present, a student may use either a transmission grating ruled with 1200 lines per millimetre over a width of 25.0 mm, used at normal incidence and able to be illuminated over any chosen width, or a Michelson interferometer whose movable mirror can be moved through up to 4.0 mm from the position at which the two arms are equal in length.

(a) CALCULATE · U3 **[2]**

Calculate the chromatic resolving power needed to separate the two lines.

1 $R = \lambda/\Delta\lambda$, with $\lambda \approx 486.07$ nm and $\Delta\lambda = 486.13 - 486.00 = 0.13$ nm

2 $R = 486.07/0.13 = 3.7 \times 10^3$ (3739)

Candidates who invert the definition and quote $\Delta\lambda/\lambda \approx 2.7 \times 10^{-4}$ have written the fractional separation of the lines, not the resolving power, which must be a large number.

(b) DETERMINE · U3 **[3]**

Determine the highest diffraction order the grating produces at this wavelength, and hence the minimum width of grating that must be illuminated to resolve the lines.

1 $d = 1/(1200\ \text{mm}^{-1}) = 833$ nm, and at normal incidence $\sin\theta = m\lambda/d \le 1$ requires $m \le d/\lambda = 1.71$, so only the first order exists

2 $R = mN$ with $m = 1$ requires at least $N = 3739$ illuminated lines

3 minimum width $= Nd = 3739 \times 833$ nm $= 3.12$ mm, well within the 25.0 mm available

The trap is to halve the width by working in second order. At 1200 lines/mm that order would need $\sin\theta = 1.17$, so it does not exist at this wavelength.

(c) SHOW THAT · U2 **[3]**

As the mirror moves a distance x from the position of equal arm lengths, the detector records fringes whose visibility falls periodically to a minimum and recovers. Show that the mirror displacement between successive visibility minima is $\lambda_1\lambda_2/[2(\lambda_1 - \lambda_2)] \approx \lambda^2/(2\Delta\lambda)$.

1 moving the mirror by x changes the path difference by 2x, so the interference orders of the two lines at the detector are $2x/\lambda_1$ and $2x/\lambda_2$

2 the two fringe systems are in step (maximum visibility) when the orders differ by a whole number and exactly out of step (minimum visibility) when they differ by a half-integer; the difference is $2x/\lambda_2 - 2x/\lambda_1 = 2x(\lambda_1 - \lambda_2)/(\lambda_1\lambda_2)$

3 successive minima are reached each time this difference grows by 1, so their spacing is $\Delta x = \lambda_1\lambda_2/[2(\lambda_1 - \lambda_2)] \approx \lambda^2/(2\Delta\lambda)$

Candidates who forget that the light crosses the extra arm length twice take the path difference as x and obtain $\lambda^2/\Delta\lambda$, twice the correct spacing.

(d) SKETCH · U1 **[2]**

Assuming the two lines are of equal intensity, sketch a graph of fringe visibility against mirror displacement from 0 to 4.0 mm, marking the displacements at which the visibility is a minimum.

1 visibility is 1 at zero displacement and follows $|\cos(2\pi x\Delta\lambda/\lambda^2)|$: rounded maxima, falling to a sharp zero at 0.45 mm and recovering to 1 at 0.91 mm

2 further zeros marked at 1.36, 2.27 and 3.18 mm (spacing 0.91 mm), with maxima of 1 at 1.82, 2.73 and 3.63 mm: four minima within the 4.0 mm travel

The first minimum lies at half the spacing, not at the spacing: at equal arm lengths the two lines are in step, so a sketch with its first zero at 0.91 mm is half a period out everywhere.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UPIII-04.03 · UPIII-04.07 **[11]**

Reflective road markings contain transparent glass beads, each a sphere of radius R = 0.300 mm, half-embedded in white paint. Light from a car's headlamps enters the exposed front of a bead and is focused towards the back surface, where the paint scatters it; light scattered from a small bright patch there is sent back towards the car by the bead. Treat the incident light as a parallel beam travelling along an axis through the bead's centre, in air of refractive index 1.00. For refraction at a single spherical surface use $n_1/s + n_2/s' = (n_2 - n_1)/R$, where s is positive for a real object in front of the surface, s′ is positive for a real image behind it, and R is positive when the centre of curvature lies behind the surface.

(a) SHOW THAT · U2 **[2]**

Show that, in the paraxial limit, refraction at the front surface alone would bring the beam to a focus a distance $nR/(n - 1)$ behind that surface, where n is the refractive index of the glass.

1. for a parallel beam $s \to \infty$, so $n_1/s = 0$, with $n_1 = 1.00$, $n_2 = n$, and R positive because the centre of curvature of the front surface lies inside the bead, behind that surface
2. $n/s' = (n - 1)/R$, so $s' = nR/(n - 1)$

The sign of R is what this part checks. A candidate who takes R as negative for a surface bulging towards the light finds a virtual focus in front of the bead.

(b) DETERMINE · U3 **[2]**

Determine the refractive index a bead must have for the paraxial focus to lie exactly on the paint at its back surface, and state where the paraxial focus lies for a bead of ordinary glass with n = 1.50.

1. the back surface is a full diameter, 2R, behind the front surface, so $n/(n - 1) = 2$, giving $n = 2.00$
2. for $n = 1.50$, $s' = 3R = 0.900$ mm, which is 0.300 mm beyond the back surface, inside the paint, so the illuminated patch on the paint is broad

Candidates who place the focus at the centre of the bead, a distance R behind the front surface, are led to $n/(n - 1) = 1$, which no glass satisfies; that impossibility should send them back to the geometry.

(c) SHOW THAT · U2 **[3]**

Take the centre of the bead as origin, with the incident light travelling in the +x direction, so that the axis meets the back surface at (R, 0). A ray entering parallel to the axis at height h above it has angle of incidence i, where $\sin i = h/R$, and angle of refraction r. Without making the paraxial approximation, show that this ray reaches the back surface at the point whose radius makes an angle $i - 2r$ with the +x axis, on the opposite side of the axis from the point of entry.

1. the normal at the entry point lies along the radius, so that radius makes an angle i with the −x axis, i.e. an angle $180° - i$ measured from the +x axis on the side of the entry point
2. the path inside the bead is a chord, which forms an isosceles triangle with the radii to its two ends; both base angles equal r, so the angle at the centre between the entry and exit radii is $180° - 2r$
3. the ray is bent towards the axis, so the exit radius lies $180° - 2r$ further round, at $(180° - i) - (180° - 2r) = -(i - 2r)$ from the +x axis: an angle $i - 2r$ on the opposite side of the axis from the entry point

With no figure, the geometry must be built from the words. The isosceles triangle formed by the chord and two radii is the step almost every candidate who fails this part has missed.

(d) CALCULATE · U3 **[2]**

For a bead with n = 2.00, calculate the perpendicular distance from the axis of the point at which the back surface is reached by the ray entering at h = 0.50R, and by the ray entering at h = 0.90R.

1. h = 0.50R: $i = 30.0°$, $r = \sin^{-1}(0.25) = 14.48°$, $i - 2r = 1.04°$, so the distance is $R \sin(i - 2r) = 0.300 \times 0.0182 = 5.5 \times 10^{-3}$ mm (5.5 μm) beyond the axis
2. h = 0.90R: $i = 64.16°$, $r = \sin^{-1}(0.45) = 26.74°$, $i - 2r = 10.67°$, so the distance is $0.300 \times \sin 10.67° = 5.6 \times 10^{-2}$ mm (56 μm) beyond the axis

The paraxial model says both rays land on the axis. The exact trace puts the outer ray ten times further off than the inner one: spherical aberration made concrete.

(e) EVALUATE · U1 **[2]**

High-index road-marking beads are typically made from glass with n close to 1.9 rather than exactly 2.00. Evaluate this choice in the light of your results.

1 at n = 2.00 only near-axis rays reach the axis point; rays entering further out cross the axis before the back surface and land up to 56 μm from it at h = 0.90R, so satisfying the paraxial condition exactly does not give the smallest patch on the paint

2 lowering n to 1.90 moves the paraxial focus just beyond the back surface (s′ = 2.11R), so near-axis rays land slightly on the entry side of the axis while outer rays land less far on the opposite side: at n = 1.90 the ray entering at h = 0.90R has r = 28.27° and i − 2r = 7.61°, landing 0.300 × sin 7.61° = 40 μm beyond the axis against 56 μm at n = 2.00, so the patch is narrower (about 44 μm across for h ≤ 0.90R), returning a tighter, brighter cone to the driver, and the choice is sound

The judgement rewarded is that a paraxial design condition is a starting point, not an optimum. Calling n = 1.9 a manufacturing compromise, without reference to the aberration just calculated, earns nothing.

13

CORE COURSE

DATA ANALYSIS · DEMANDING · UPIII-03.06 · UPIII-03.07 · UPIII-03.09 **[10]**

To measure the speed of sound outdoors, a group mounts a buzzer of steady but unknown frequency f_0 on the roof of a car. On a calm day at 21 °C the car is driven at a series of steady speeds v_s along a straight runway directly towards a microphone at the same height, and a spectrum analyser records the frequency f received while the car is still far away. Each speed is logged by satellite navigation. In still air the moving-source Doppler relation is $f = f_0 v/(v - v_s)$, where v is the speed of sound.

(a) SHOW THAT · U2 **[2]**

Show that the relation predicts a straight line when 1/f is plotted against v_s, and identify its gradient and its intercept on the 1/f axis.

1 inverting, $1/f = (v - v_s)/(f_0 v) = 1/f_0 - v_s/(f_0 v)$, which is linear in v_s

2 gradient $= -1/(f_0 v)$; intercept $= 1/f_0$

Candidates who plot f against v_s see a line that is almost straight and stop there; its slight curvature is exactly what the reciprocal plot removes.

(b) DETERMINE · U4 **[4]**

Use the readings to determine f_0 and the speed of sound v.

1 a best-fit straight line of 1/f against v_s, with its gradient read from widely separated points on the line, gives gradient $= -1.35 \times 10^{-6}$ s² m⁻¹ (accept -1.34×10^{-6} to -1.36×10^{-6} s² m⁻¹)

2 the intercept at $v_s = 0$ is $1/f_0 = 4.651 \times 10^{-4}$ s, so $f_0 = 2150$ Hz (accept 2148 Hz to 2152 Hz)

3 $v = -\text{intercept/gradient} = 1/(f_0 \times |\text{gradient}|)$

4 $v = 4.651 \times 10^{-4}/1.35 \times 10^{-6} = 344$ m/s (accept 341 m/s to 348 m/s), consistent with the 344 m/s expected in air at 21 °C

The intercept lies outside the range of the readings and must be found by extrapolation. Candidates who take the first reading, 2175.6 Hz, as f_0 carry a 1% error into v and find about 340 m/s.

(c) JUSTIFY · U4 **[2]**

Justify, using the readings, the use of the exact Doppler relation rather than the low-speed approximation $f \approx f_0(1 + v_s/v)$.

1. the approximation makes f linear in v_s, rising by the same $f_0 \times 4.0/v \approx 25.0$ Hz for each 4.0 m/s step and reaching 2300.0 Hz at 24.0 m/s; the readings rise by 25.4, 26.8, 26.8, 28.3 and 28.3 Hz and reach 2311.2 Hz
2. the steadily growing steps and the 11 Hz excess at 24.0 m/s are many times the scatter of the readings about the reciprocal straight line (about 0.3 Hz), so the data reject the approximation, which drops terms in $(v_s/v)^2$ and higher, and the exact relation is needed

The marks are for setting the size of a discrepancy against the size of the noise. Saying the exact relation is more accurate earns nothing without both numbers.

(d) PREDICT · U1 **[2]**

On a second day a steady wind of 5.0 m/s blows along the runway from the microphone towards the approaching car. Predict how the intercept and the gradient of the plot of 1/f against v_s would change, and the value of v that the analysis in (b) would then return.

1. in the frame of the air the microphone moves away from the car at 5.0 m/s and the car approaches at $v_s + 5.0$ m/s, so $f = f_0(v - 5.0)/(v - 5.0 - v_s)$; at $v_s = 0$ the buzzer and microphone share the same motion through the air, $f = f_0$, and the intercept is unchanged
2. $1/f = 1/f_0 - v_s/[f_0(v - 5.0)]$, so the gradient is steeper and the analysis returns $v - 5.0 = 339$ m/s: the speed of sound relative to the ground along the path, not the speed of sound in the air

The Doppler relation holds in the frame of the medium. Candidates who let the wind change only the source speed predict a shifted intercept, which contradicts the fact that a buzzer and a microphone at rest relative to each other must agree on the frequency.

14

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPIII-06.03 · UPIII-06.04 · UPIII-06.05 · UPIII-06.06 · UPIII-06.07 **[15]**

A sealed glass bulb of volume $V = 1.50 \times 10^{-3}$ m³ contains argon at a temperature of 293 K and an initial pressure of 3.0 Pa. A circular hole 60 µm in diameter is then opened in a thin foil in the bulb's wall, leading directly to a high vacuum, while the bulb is held at 293 K. Model the escape as effusion, in which atoms leave through the hole independently of one another, and treat the argon as an ideal gas with a Maxwell–Boltzmann distribution of speeds, whose mean speed is $\langle v \rangle = \sqrt{(8kT/(\pi m))}$. Take the Boltzmann constant $k = 1.381 \times 10^{-23}$ J/K, the mass of an argon atom as 39.95 u with $1\ u = 1.661 \times 10^{-27}$ kg, and the effective diameter of an argon atom as about 0.36 nm.

(a) SHOW THAT · U2 **[4]**

Consider a gas with n atoms per unit volume and an isotropic distribution of velocities. Show that the number of atoms striking unit area of the wall per unit time is $\Phi = \tfrac{1}{4} n \langle v \rangle$.

1. isotropy: the fraction of atoms moving at between θ and θ + dθ to the inward normal of the wall is $2\pi \sin\theta\, d\theta/4\pi = \tfrac{1}{2} \sin\theta\, d\theta$
2. atoms with speeds between v and v + dv moving in that direction reach an area A of wall within a time dt only if they lie in a slanted cylinder of volume $Av \cos\theta\, dt$, so they make $n\, g(v)dv \times \tfrac{1}{2} \sin\theta\, d\theta \times Av \cos\theta\, dt$ impacts, where g(v) is the normalised speed distribution
3. integrating over the directions heading towards the wall, $\int_0^{(\pi/2)} \tfrac{1}{2} \sin\theta \cos\theta\, d\theta = \tfrac{1}{4}$
4. integrating over speeds, $\int_0^{\infty} v\, g(v)\, dv = \langle v \rangle$, so $\Phi = \tfrac{1}{4} n \langle v \rangle$

The cos θ is where the derivation is usually lost: atoms arriving obliquely come from a thinner slab. Omitting it gives $\tfrac{1}{2} n \langle v \rangle$, and sending one sixth of the atoms along each axis direction gives $n\langle v \rangle/6$.

(b) DERIVE · U2 **[3]**

Derive, from conservation of the number of atoms and the ideal-gas equation of state, an expression for the pressure in the bulb as a function of time t, identifying the time constant τ in terms of V, the area A of the hole and ⟨v⟩.

1. atoms leave through the hole at the rate ΦA and none return from the vacuum, so dN/dt = −¼(N/V)⟨v⟩A
2. at constant T, ⟨v⟩ is constant, so integrating gives N = N₀e^(−t/τ) with τ = 4V/(A⟨v⟩)
3. pV = NkT at constant V and T makes p proportional to N, so p = p₀e^(−t/τ)

Candidates who hold the rate of escape constant predict a linear fall in pressure. The rate is proportional to the number density left in the bulb, which is what makes the decay exponential and τ independent of the starting pressure.

(c) CALCULATE · U3 **[3]**

Calculate ⟨v⟩ for the argon, the time constant τ, and the time taken for the pressure to fall from 3.0 Pa to 0.30 Pa.

1. m = 39.95 × 1.661 × 10⁻²⁷ = 6.636 × 10⁻²⁶ kg, so ⟨v⟩ = √(8 × 1.381 × 10⁻²³ × 293/(π × 6.636 × 10⁻²⁶)) = 394 m/s
2. A = π(30 × 10⁻⁶)² = 2.83 × 10⁻⁹ m², so τ = 4 × 1.50 × 10⁻³/(2.83 × 10⁻⁹ × 394) = 5.39 × 10³ s (about 1.5 h)
3. t = τ ln(3.0/0.30) = 5.39 × 10³ × 2.303 = 1.24 × 10⁴ s (about 3.4 h)

Candidates who use the rms speed, 428 m/s, obtain a time constant about 8% too short. The flux is set by the mean speed, which is why the derivation in (a) produced ⟨v⟩ and not a mean of squares.

(d) ESTIMATE · U4 **[3]**

Estimate the mean free path of the argon atoms at the initial pressure, stating the model you use, and hence decide whether treating the escape as effusion is justified.

1. model: hard spheres of diameter d ≈ 0.36 nm with Maxwell–Boltzmann relative speeds, so λ = kT/(√2πd²p), with number density p/kT = 7.4 × 10²⁰ m⁻³
2. λ = 1.381 × 10⁻²³ × 293/(√2 × π × (0.36 × 10⁻⁹)² × 3.0) ≈ 2.3 × 10⁻³ m (about 2 mm; accept about 3 mm if the √2 is omitted)
3. λ is about 40 times the 60 μm diameter of the hole, so atoms pass through it without colliding with one another; the effusion model is justified, and becomes more so as the pressure falls

An estimate earns its marks only with its model stated. The decisive comparison is the mean free path against the hole, not against the bulb: that is what separates effusion from bulk flow through the opening.

(e) EXPLAIN · U1 **[2]**

If the bulb were thermally insulated instead of held at 293 K, the argon remaining in it would cool as it leaked. Explain why, given that the gas does no work on any piston.

1. the flux weights atoms by their speed, since the rate at which atoms of speed v arrive at the hole is proportional to v g(v); faster atoms escape preferentially, and each escaping atom carries a mean kinetic energy of 2kT, more than the average of 3kT/2 per atom in the bulb
2. the gas left behind therefore loses more than the average energy per atom removed, so its mean translational kinetic energy per atom, (3/2)kT, falls and its temperature falls with it

Candidates who say the gas cools because it expands have borrowed the adiabatic-expansion argument. There is no piston; the cooling is a velocity-selection effect of the flux alone.

University Physics III · Final Examination

The whole of University Physics III, weighted towards its second half: oscillations, waves, optics and kinetic theory, then thermodynamic cycles and entropy, special relativity, the Schrödinger equation, the hydrogen atom and nuclear energetics. Most marks are for building a model from a named law, auditing it against conservation laws and the second law, and checking where its approximations break.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**118**	**19**	**10**

SECTION — **UPIII · Final · University Physics III — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Routine 7 · Demanding 10 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPIII-01 · UPIII-02 · UPIII-03 · UPIII-04 · UPIII-05 · UPIII-06 · UPIII-07 · UPIII-08 · UPIII-09 · UPIII-10 · UPIII-11

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Physics III · Final Examination

Answer all questions. 118 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-01.07 **[1]**

A particle of mass 0.150 kg moves along the positive x-axis in the potential $U(x) = A/x^2 - B/x$, where $A = 0.360$ J·m² and $B = 1.20$ J·m. What is the angular frequency of small oscillations about its stable equilibrium position?

A 0.969 rad/s

B 6.09 rad/s

C 10.5 rad/s

D 13.6 rad/s

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.02 **[1]**

The one-dimensional wave equation for waves of speed v is $\partial^2 y/\partial x^2 = (1/v^2)\,\partial^2 y/\partial t^2$. In each function below, A, k and b are positive constants and $\omega = kv$. Which function is NOT a solution of this equation?

A $y = A\cos(kx)\sin(\omega t)$

B $y = A/[1 + (x + vt)^2/b^2]$

C $y = A\sin(kx - \omega t) + A\sin(kx + \omega t)$

D $y = A\exp[-(x^2 + v^2t^2)/b^2]$

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-04.09 **[1]**

A Galilean beam expander widens a collimated laser beam of diameter 2.00 mm. The beam passes first through a thin diverging lens of focal length −25.0 mm and then through a thin converging lens of focal length +100 mm on the same axis. Which lens separation makes the beam emerge collimated, and what is its diameter then?

A Separation 125 mm; diameter 8.00 mm

B Separation 75.0 mm; diameter 0.500 mm

C Separation 75.0 mm; diameter 8.00 mm

D Separation 75.0 mm; diameter 2.00 mm

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-05.09 **[1]**

Sunlight is reflected from the calm surface of a lake, of refractive index 1.33, towards an angler whose sunglasses have polarising lenses with vertical transmission axes. At what angle below the horizontal must the angler look at the water for the reflected glare to be completely polarised, so that the sunglasses block it entirely?

A 53.1°

B 36.9°

C 41.2°

D 48.8°

Answer ☐

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-06.05 **[1]**

A sample of argon (molar mass 39.9 g/mol) is at 320 K. At what temperature would a sample of helium (molar mass 4.00 g/mol) have the same root-mean-square atomic speed? Treat both as ideal monatomic gases.

A 32.1 K

B 101 K

C 320 K

D 3190 K

Answer ☐

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-07.08 **[1]**

A rigid, thermally insulated container is divided by a partition into two equal halves. One half holds 1.00 mol of an ideal gas and the other is evacuated. The partition is punctured and the gas fills the whole container. Take R = 8.314 J/(mol·K). Which statement is correct?

A No heat enters the gas, so its entropy change is zero.

B The gas does work as it expands, so its temperature falls.

C The internal energy of the gas is unchanged, and its entropy rises by 5.76 J/K.

D The entropy of the gas rises by 5.76 J/K, but the entropy of the universe is unchanged because the container is insulated.

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-08.03 **[1]**

A bunch of protons travels at constant speed 0.800c along a straight section of a linear accelerator, between two beam-position monitors that are fixed in the laboratory 36.0 m apart. Take $c = 3.00 \times 10^8$ m/s. Which pair gives the distance between the monitors and the time the bunch takes to travel from one to the other, both as measured in the rest frame of the bunch?

A 21.6 m and 90.0 ns

B 36.0 m and 90.0 ns

C 21.6 m and 150 ns

D 60.0 m and 250 ns

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-09.02 **[1]**

Light of wavelength 400 nm ejects photoelectrons from a metal surface whose work function is 2.15 eV. The wavelength is then changed to 200 nm while the intensity is kept the same. Take hc = 1240 eV·nm. What is the stopping potential with the 200 nm light?

A 0.95 V

B 1.90 V

C 4.05 V

D 6.20 V

Answer ☐

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-10.04 **[1]**

An electron in a hydrogen atom has orbital angular momentum quantum number $l = 3$. What is the smallest angle that its orbital angular momentum vector can make with the z-axis?

A 0°

B 30.0°

C 54.7°

D 60.0°

Answer ☐

10
CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-11.09 **[1]**

Rest energies: proton 938.3 MeV, neutron 939.6 MeV, electron 0.511 MeV, muon 105.7 MeV, charged pion 139.6 MeV; neutrino masses are negligible. Which of the following processes is allowed by every conservation law?

A $n \to p + e^- + \nu_e$

B $p \to n + e^+ + \nu_e$, for a free proton at rest

C $\mu^- \to e^- + \gamma$

D $\pi^+ \to \mu^+ + \nu_\mu$

Answer

11 CORE COURSE

STRUCTURED · ROUTINE · UPIII-04.08 · UPIII-05.03 · UPIII-05.08 **[11]**

An aerial-mapping camera photographs a field from a height of 120 m. Its lens is a thin converging lens of focal length 8.80 mm and aperture diameter 3.14 mm, and its image sensor has square pixels 2.40 µm wide. Take the wavelength of the light as 550 nm in air. To reduce reflections, the lens glass, of refractive index 1.52, carries a single coating of magnesium fluoride, of refractive index 1.38, designed for 550 nm at normal incidence.

(a) CALCULATE **[2]**

Calculate how far behind the lens the sensor must be placed to focus the field, and how far this is from the lens's focal plane.

(b) CALCULATE **[2]**

Calculate the width of the strip of ground that is imaged onto one pixel.

(c) JUSTIFY **[3]**

Justify whether diffraction at the lens aperture or the size of the pixels sets the smallest ground detail this camera can resolve at 550 nm.

(d) DETERMINE **[2]**

Determine the minimum thickness of the coating that minimises the reflection of 550 nm light at normal incidence.

(e) EXPLAIN **[2]**

When white light is reflected from the coated lens, the reflection looks purple. Explain why.

12 CORE COURSE

STRUCTURED · DEMANDING · UPIII-01.04 · UPIII-01.06 · UPIII-01.08 **[12]**

The pendulum of a longcase clock is a light rigid rod carrying a solid brass sphere of mass 2.30 kg and radius r = 0.0400 m, whose centre is L = 0.990 m below the pivot. The moment of inertia of a solid sphere about an axis through its centre is $(2/5)mr^2$. Take g = 9.81 m/s². The clock's escapement gives the pendulum a small push once each swing to keep its amplitude at 4.00°.

(a) DERIVE **[3]**

Derive, from the rotational form of Newton's second law, an expression for the period of small oscillations of this pendulum in terms of L, r and g, stating the approximation you make.

(b) CALCULATE **[2]**

Calculate the period, and the fraction by which it exceeds the period of a simple pendulum of length 0.990 m.

(c) DETERMINE **[3]**

With the escapement disconnected, the amplitude falls from 4.00° to 2.00° in 12.0 minutes. Determine the average power the escapement must supply to keep the amplitude at 4.00°, stating the assumption about the damping that your method requires.

(d) CALCULATE **[2]**

For a finite amplitude θ_0 the period is $T \approx T_0(1 + \theta_0^2/16)$, with θ_0 in radians. The clock is adjusted to keep perfect time at an amplitude of 4.00°, but friction in its gear train lets the amplitude wander between 3.50° and 4.50°. Calculate the largest time the clock gains or loses in one day as a result.

(e) EVALUATE **[2]**

The clock must keep time to within 1 s per day. Evaluate which idealisation of the simple-pendulum model, the point-mass bob or the small-angle limit, limits this clock's timekeeping.

13 CORE COURSE

STRUCTURED · DEMANDING · UPIII-02.06 · UPIII-02.07 · UPIII-03.01 · UPIII-03.02 · UPIII-03.03 · UPIII-03.05 **[10]**

A pair of extraction fans of the same model stand 6.0 m apart on a flat roof. Each has 8 blades and radiates a steady tone at its blade-passing frequency, the number of blades multiplied by the fan's rotation rate in revolutions per second, and the controller of fan A shows a rotation rate of 24.50 revolutions per second. With both fans running, a technician's sound level meter, filtered to reject everything except a narrow band around this tone and quick enough to follow any change in the tone's amplitude, records a level that rises and falls smoothly and regularly between a maximum of 79.5 dB and a minimum of 71.9 dB, with successive maxima 5.00 s apart. The meter responds to the pressure at its microphone and reports the sound intensity level of a single plane wave with the same pressure amplitude. Treat the fans as small sources in open air and ignore reflections. Take the reference intensity as 1.00×10^{-12} W/m², the density of air as 1.20 kg/m³, the speed of sound as 343 m/s and atmospheric pressure as 1.01×10^{5} Pa.

(a) CALCULATE **[2]**

Calculate the pressure amplitude of the sound at the meter when the level is at a maximum and when it is at a minimum.

(b) EXPLAIN **[2]**

The sound from the two fans reaches the meter travelling in different directions. Explain why the excess pressures of the two waves at the meter can be added as numbers, whereas the displacements of the air in the two waves cannot, and why the superposition principle holds for these waves.

(c) DETERMINE **[4]**

Determine the sound intensity level that the louder fan alone would produce at the meter, the level that the quieter fan alone would produce, and the difference between the two fans' rotation rates.

(d) DESCRIBE **[2]**

The technician adjusts the controller of fan B until the level at the meter stops rising and falling. Describe how the steady level now depends on the fans and on where the meter is placed, giving the range of steady levels the meter could read at its present position, and the change in the difference between its distances from the two fans that takes the level from its highest to its lowest value, ignoring any change in each fan's own amplitude.

14 CORE COURSE

STRUCTURED · DEMANDING · UPIII-08.02 · UPIII-08.04 · UPIII-08.07 **[11]**

In the laboratory frame S, three events are recorded on the x-axis. Event A: a detector at x = 0 fires at t = 0. Event B: a detector at x = 900 m fires at t = 2.00 μs. Event C: the same detector at x = 900 m fires again at t = 4.00 μs. Frame S′ moves at +0.800c along the x-axis relative to S, and its origin coincides with that of S at event A. Take $c = 3.00 \times 10^8$ m/s.

(a) CALCULATE **[2]**

Calculate the invariant interval $s^2 = (c\Delta t)^2 - (\Delta x)^2$ between events A and B, and classify the separation.

(b) DETERMINE **[3]**

Determine the time interval $\Delta t'$ and the spatial separation $\Delta x'$ between A and B measured in S′, and state which event occurs first in S′.

(c) JUSTIFY **[2]**

Justify the claim that event A cannot have caused event B, giving two separate arguments.

(d) DETERMINE **[2]**

Determine the velocity, relative to S, of the inertial frame in which events A and B are simultaneous.

(e) PREDICT **[2]**

Predict the order of events A and C in S′ and in every other inertial frame, giving the reason.

15 CORE COURSE

STRUCTURED · DEMANDING · UPIII-11.03 · UPIII-11.04 · UPIII-11.05 **[12]**

A radioisotope power source for a deep-space probe contains 3.50 kg of plutonium-238 (^{238}Pu, Z = 94), which decays by alpha emission to uranium-234 with a half-life of 87.7 years. Atomic masses: ^{238}Pu 238.049 560 u, ^{234}U 234.040 952 u, ^{4}He 4.002 603 u. Take 1 u = 931.494 MeV/c^2, 1 MeV = 1.602 18 × 10^{-13} J, N_A = 6.022 × 10^{23} mol^{-1}, 1 year = 3.156 × 10^{7} s, and the molar mass of ^{238}Pu as 238.05 g/mol.

(a) STATE **[1]**

State the decay equation, showing the nucleon number and proton number of every particle.

(b) EXPLAIN **[2]**

Explain why the Q value of this decay can be calculated directly from atomic masses, even though the decay is a nuclear process.

(c) CALCULATE **[2]**

Calculate the Q value of the decay in MeV.

(d) SHOW THAT **[2]**

Show that, for a parent nucleus at rest, the alpha particle carries about 98% of the Q value as kinetic energy.

(e) CALCULATE **[3]**

Calculate the initial thermal power of the source, assuming that all of the decay energy is absorbed within it.

(f) DETERMINE **[2]**

Determine how long after the source is made its thermal power falls to 1500 W.

16 CORE COURSE

STRUCTURED · DEMANDING · UPIII-07.04 · UPIII-07.05 · UPIII-07.06 · UPIII-07.09 **[13]**

An engine takes 0.500 mol of an ideal diatomic gas ($C_V = 5R/2$, $\gamma = 1.40$) around a three-stage cycle. A → B: reversible isothermal compression at 300 K, in contact with a cold reservoir at 300 K, until the volume is one quarter of its value at A. B → C: heating at constant volume, by contact with a hot reservoir at 600 K, until the gas reaches temperature T_C, when contact is broken. C → A: reversible adiabatic expansion back to state A. Take $R = 8.314$ J/(mol·K).

(a) SHOW THAT **[2]**

Show that $T_C = 522$ K.

(b) CALCULATE **[4]**

Calculate the heat Q absorbed by the gas and the work W done by the gas in each of the three stages, and the net work done by the gas per cycle.

(c) COMPARE **[2]**

Compare the thermal efficiency of this cycle with the Carnot efficiency of an engine operating between the same two reservoirs.

(d) DETERMINE **[3]**

Determine the total entropy change of the gas and the two reservoirs per cycle, and identify the stage in which entropy is generated.

(e) SHOW THAT **[2]**

Show that the work this cycle loses, compared with a Carnot engine that absorbs the same heat Q_{BC} from the 600 K reservoir, equals $T_{cold} \times S_{gen}$, where $T_{cold} = 300$ K and S_{gen} is the entropy generated per cycle.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIII-09.04 **[10]**

In a low-energy electron diffraction experiment, electrons accelerated from rest through a potential difference V strike a clean copper crystal surface along its normal. The rows of surface atoms, a distance a apart, act as a reflection grating: first-order diffracted beams leave the surface at an angle φ to the normal, where a sin φ = λ and λ is the de Broglie wavelength of the electrons. The angle φ is measured at several accelerating potential differences, and the energy scale of the electron gun is calibrated to ±2%. Take h = 6.626 × 10^{-34} J·s, m_e = 9.109 × 10^{-31} kg and e = 1.602 × 10^{-19} C. In bulk copper the atoms of a plane of this orientation form a square array of side 0.2556 nm.

Angle of a first-order diffracted beam from the copper surface at several accelerating potential differences

accelerating potential difference V / V	first-order beam angle φ / °
50	42.9
75	33.4
100	28.9
125	25.2
150	23.2
175	21.1
200	19.9

(a) SHOW THAT **[2]**

Show that the model predicts a straight line through the origin when sin φ is plotted against 1/√V, and give its gradient in terms of h, m_e, e and a.

(b) DETERMINE **[3]**

Using all of the readings, determine the gradient of the graph of sin φ against 1/√V.

(c) CALCULATE **[3]**

Calculate the row spacing a, with its uncertainty arising from the calibration of the energy scale.

(d) EVALUATE **[2]**

A calculation predicts that the outermost layer of copper atoms relaxes towards the layer beneath it, reducing the spacing between those two layers by about 2%. Evaluate whether these data support a surface whose atoms keep their bulk arrangement within the layer, and whether this experiment could detect the predicted relaxation.

18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPIII-10.03 · UPIII-10.05 **[14]**

In muonic hydrogen a negative muon, of mass 1.8835×10^{-28} kg, takes the place of the electron and is bound to a proton of mass 1.6726×10^{-27} kg; its energy levels are measured to find the size of the proton. The relative motion obeys the hydrogen-atom Schrödinger equation with the reduced mass $\mu = m_\mu m_p/(m_\mu + m_p)$ in place of the electron mass. For a state with no angular dependence ($l = 0$) the equation reads $-(\hbar^2/2\mu)(1/r^2)\,d/dr(r^2\,d\psi/dr) - (e^2/(4\pi\varepsilon_0 r))\psi = E\psi$. Take $\hbar = 1.0546 \times 10^{-34}$ J·s, $e = 1.602 \times 10^{-19}$ C, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, the proton radius as 0.84 fm and the Bohr radius of ordinary hydrogen as 5.29×10^{-11} m. You may use $\int_0^\infty r^n e^{-\beta r}\,dr = n!/\beta^{n+1}$.

(a) DERIVE **[5]**

By substituting the trial solution $\psi(r) = Ce^{-r/a}$ into the Schrödinger equation, derive expressions for the length a and the energy E, and evaluate both for muonic hydrogen.

(b) SHOW THAT **[2]**

Show that the normalised wavefunction has $C = 1/\sqrt{(\pi a^3)}$.

(c) DETERMINE **[2]**

Determine the most probable distance of the muon from the proton, in metres.

(d) SKETCH **[2]**

Sketch the radial probability density $P(r) = 4\pi r^2|\psi(r)|^2$ against r/a, marking the most probable radius found in (c) and the mean radius $\langle r\rangle = 1.5a$.

(e) ESTIMATE **[3]**

Estimate the probability that the muon is found inside the proton, stating the assumption you make, and compare it with the corresponding probability for the electron in ordinary hydrogen.

19 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE PRECISION · DISCRIMINATING · UPIII-08.09 · UPIII-11.02 · UPIII-11.04 **[15]**

The neutron's mass is determined most precisely from the gamma ray emitted when a slow neutron is captured by a proton at rest: $n + p \to d + \gamma$. A crystal-diffraction spectrometer measures the photon energy as $E_\gamma = 2.223\ 25$ MeV, with an uncertainty of ± 0.4 eV. Take the rest energies $m_p c^2 = 938.2721$ MeV and $m_d c^2 = 1875.6129$ MeV, and treat the neutron's kinetic energy before capture as negligible. Write $M = m_n + m_p$ for the total rest mass before capture, and $B = (M - m_d)c^2$ for the binding energy of the deuteron.

(a) DERIVE **[4]**

Derive, from conservation of energy and momentum, an exact expression for E_γ in terms of M, m_d and c.

(b) SHOW THAT **[3]**

Show that the result of (a) can be written exactly as $E_\gamma = B - B^2/(2Mc^2)$, and explain the physical meaning of the term $B^2/(2Mc^2)$.

(c) DETERMINE **[3]**

Determine the rest energy of the neutron, $m_n c^2$, from the measured photon energy.

(d) EVALUATE **[2]**

Evaluate whether the recoil term must be included when the neutron's rest energy is quoted from this measurement.

(e) COMPARE **[3]**

In the reverse process, photodisintegration $\gamma + d \to n + p$, a photon strikes a deuteron at rest. Compare the minimum photon energy for this process with the photon energy emitted in capture, and explain the difference.

MARK SCHEME

University Physics III · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-01.07 **[1]**

A particle of mass 0.150 kg moves along the positive x-axis in the potential $U(x) = A/x^2 - B/x$, where $A = 0.360$ J·m² and $B = 1.20$ J·m. What is the angular frequency of small oscillations about its stable equilibrium position?

KEY **B** **6.09 rad/s**

Equilibrium is where $dU/dx = -2A/x^3 + B/x^2 = 0$, at $x_0 = 2A/B = 0.600$ m, and the effective spring constant is the curvature there: $U''(x_0) = 6A/x_0^4 - 2B/x_0^3 = 16.67 - 11.11 = 5.56$ J/m², so $\omega = \sqrt{(U''/m)} = 6.09$ rad/s. A is the same result converted to a frequency, $f = \omega/2\pi = 0.969$ Hz, and then labelled in rad/s, a units slip that costs the whole mark. C keeps only the curvature of the repulsive term, $6A/x_0^4$, as if the attractive term were a constant offset; but $-B/x$ has a curvature of its own, $-2B/x^3$, and it softens the well. D differentiates the attractive term with the wrong sign, adding 11.11 J/m² instead of subtracting it. A candidate who remembers that the attraction is what creates the minimum, and so must reduce the stiffness below that of the repulsion alone, rejects both C and D at once.

U2

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-02.02 **[1]**

The one-dimensional wave equation for waves of speed v is $\partial^2y/\partial x^2 = (1/v^2)\,\partial^2y/\partial t^2$. In each function below, A, k and b are positive constants and $\omega = kv$. Which function is NOT a solution of this equation?

KEY **D** **$y = A\exp[-(x^2 + v^2t^2)/b^2]$**

D contains vt, but only inside $x^2 + v^2t^2$, which is not a function of $x - vt$ or of $x + vt$. Differentiating, $\partial^2y/\partial x^2 = (4x^2/b^4 - 2/b^2)y$ while $(1/v^2)\partial^2y/\partial t^2 = (4v^2t^2/b^4 - 2/b^2)y$, and the two agree only on the lines $x = \pm vt$, so D fails. A is tempting to reject because it is a product that does not travel, but it is a standing wave: $\partial^2y/\partial x^2 = -k^2y$ and $(1/v^2)\partial^2y/\partial t^2 = -(\omega^2/v^2)y = -k^2y$, so it satisfies the equation. B is not sinusoidal, and a candidate who has met only harmonic waves may reject it, but any twice-differentiable function of $x + vt$ is a solution; this one is a pulse moving in the $-x$ direction. C is the sum of two travelling-wave solutions, and because the wave equation is linear any sum of solutions is a solution; it equals $2A\sin(kx)\cos(\omega t)$, another standing wave. The item rewards testing each function against the equation rather than judging it by its appearance.

U2

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-04.09 **[1]**

A Galilean beam expander widens a collimated laser beam of diameter 2.00 mm. The beam passes first through a thin diverging lens of focal length −25.0 mm and then through a thin converging lens of focal length +100 mm on the same axis. Which lens separation makes the beam emerge collimated, and what is its diameter then?

KEY **C** **Separation 75.0 mm; diameter 8.00 mm**

The diverging lens spreads the collimated beam so that it appears to come from that lens's focal point, a virtual point source 25.0 mm in front of it. The converging lens collimates light from a point source only if the source sits at its own focal point, 100 mm in front of it, so the lenses must be 100 mm − 25.0 mm = 75.0 mm apart; similar triangles from the virtual source then give a diameter of 2.00 mm × 100 mm/25.0 mm = 8.00 mm. A adds the magnitudes of the focal lengths, which is the separation of a Keplerian expander made from two converging lenses; with a diverging first lens the virtual source would then be 150 mm from the converging lens, beyond its focal point, and the beam would leave converging. B has the right separation but inverts the ratio of the focal lengths, as though the light ran backwards through the system. D assumes that a beam which is collimated going in and coming out keeps its width, forgetting that the geometry which fixes the separation also scales the beam by $f_2/|f_1| = 4$.

U3

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-05.09 **[1]**

Sunlight is reflected from the calm surface of a lake, of refractive index 1.33, towards an angler whose sunglasses have polarising lenses with vertical transmission axes. At what angle below the horizontal must the angler look at the water for the reflected glare to be completely polarised, so that the sunglasses block it entirely?

KEY **B** **36.9°**

Reflected light is completely polarised, parallel to the surface, when the angle of incidence is Brewster's angle, tan θ_B = 1.33, so θ_B = 53.1° measured from the normal to the water, which is vertical. The reflected ray therefore rises at 90° − 53.1° = 36.9° above the horizontal, and the angler looks down along it at 36.9° below the horizontal; the horizontally polarised glare is then perpendicular to the vertical transmission axes and is blocked. A is Brewster's angle itself, reported without converting from the normal to the horizontal. D is the critical angle, arcsin(1/1.33) = 48.8°, which governs light trying to leave the water rather than light reflected from above it, and C is its complement, 41.2°. A check on the key: at Brewster's angle the reflected and refracted rays are perpendicular, and 53.1° + arcsin(sin 53.1°/1.33) = 53.1° + 36.9° = 90°.

U3

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-06.05 **[1]**

A sample of argon (molar mass 39.9 g/mol) is at 320 K. At what temperature would a sample of helium (molar mass 4.00 g/mol) have the same root-mean-square atomic speed? Treat both as ideal monatomic gases.

KEY **A** **32.1 K**

Since v_rms = √(3RT/M), equal speeds require equal T/M: T_He = 320 K × 4.00/39.9 = 32.1 K. C is the answer of a candidate who remembers that temperature fixes the mean translational kinetic energy per atom and slides from equal kinetic energy to equal speed; at a common temperature the lighter helium atoms move faster, so helium must be colder to match argon. D inverts the mass ratio, heating the lighter gas when it must be cooled. B scales the temperature by the square root of the mass ratio, which would be right only if v_rms were proportional to T/√M rather than to √(T/M).

U3

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-07.08 **[1]**

A rigid, thermally insulated container is divided by a partition into two equal halves. One half holds 1.00 mol of an ideal gas and the other is evacuated. The partition is punctured and the gas fills the whole container. Take R = 8.314 J/(mol·K). Which statement is correct?

KEY **C** **The internal energy of the gas is unchanged, and its entropy rises by 5.76 J/K.**

In a free expansion the gas pushes against nothing, so W = 0; with Q = 0 as well, ΔU = 0 and an ideal gas keeps its temperature. Entropy is a state function, so ΔS can be found along any reversible path between the same end states, here an isothermal doubling of volume: ΔS = nR ln 2 = 5.76 J/K. A applies ΔS = ∫dQ/T to the actual process, but that integral gives the entropy change only along a reversible path, and a free expansion is irreversible. B imports the result for a reversible adiabatic expansion, in which the gas does work on a piston; expanding into a vacuum, there is nothing to do work on. D gets the gas right and then concludes that insulation keeps the entropy of the universe constant. Insulation means only that the surroundings' entropy is unchanged, so the universe gains the full 5.76 J/K, which is exactly what makes the process irreversible.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-08.03 **[1]**

A bunch of protons travels at constant speed 0.800c along a straight section of a linear accelerator, between two beam-position monitors that are fixed in the laboratory 36.0 m apart. Take c = 3.00 × 10^8 m/s. Which pair gives the distance between the monitors and the time the bunch takes to travel from one to the other, both as measured in the rest frame of the bunch?

KEY **A** **21.6 m and 90.0 ns**

The monitors are at rest in the laboratory, so 36.0 m is the proper length, and the bunch measures it contracted by γ = 1/√(1 − 0.800²) = 5/3 to 21.6 m. In the laboratory the trip takes 36.0 m/(2.40 × 10^8 m/s) = 150 ns, timed by two separated clocks; a clock riding with the bunch is present at both events, so it records the proper time, 150 ns/γ = 90.0 ns. The pair is self-consistent: in the bunch frame the second monitor approaches at 2.40 × 10^8 m/s and covers 21.6 m in 90.0 ns. B pairs the proper length with the proper time, but 36.0 m at 0.800c takes 150 ns, so the two cannot hold in one frame. C contracts the length but keeps the laboratory time. D is self-consistent, since 60.0 m at 0.800c takes 250 ns, and is the answer of a candidate who multiplied by γ throughout. The remedy is to name the proper quantity first: a proper length is the longest length any inertial frame measures, and a proper time the shortest time between the two events.

U1

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-09.02 **[1]**

Light of wavelength 400 nm ejects photoelectrons from a metal surface whose work function is 2.15 eV. The wavelength is then changed to 200 nm while the intensity is kept the same. Take hc = 1240 eV·nm. What is the stopping potential with the 200 nm light?

KEY **C** **4.05 V**

Each photon now carries hc/λ = 1240 eV·nm/200 nm = 6.20 eV, and the most energetic photoelectron leaves with 6.20 eV − 2.15 eV = 4.05 eV, so the stopping potential is 4.05 V. A is the stopping potential for the 400 nm light, from 3.10 eV − 2.15 eV = 0.95 eV, chosen by a candidate who reasons that an unchanged intensity leaves the electron energies unchanged; intensity sets how many photoelectrons are emitted, not their maximum energy. B doubles the old stopping potential because the photon energy doubled, forgetting that the work function is subtracted once from each photon's energy rather than scaling with it. D is the photon energy itself, with the work function left out altogether.

U3

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIII-10.04 **[1]**

An electron in a hydrogen atom has orbital angular momentum quantum number l = 3. What is the smallest angle that its orbital angular momentum vector can make with the z-axis?

KEY **B** **30.0°**

The magnitude is |L| = √(l(l + 1)) ħ = √12 ħ and the largest z-component is L_z = lħ = 3ħ, so cos θ = 3/√12 and θ = 30.0°. A is the classical picture, and the answer of a candidate who takes |L| = lħ; because √(l(l + 1)) always exceeds l, the vector can never lie along the axis, which is the uncertainty principle forbidding L_x and L_y from both vanishing when L_z has a definite value. D takes the inverse sine of the same ratio instead of the inverse cosine. C is the angle for a spin-½ vector, from cos θ = (1/2)/(√3/2), a memorised number that belongs to s = ½ rather than to l = 3.

U1

10

CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIII-11.09 **[1]**

Rest energies: proton 938.3 MeV, neutron 939.6 MeV, electron 0.511 MeV, muon 105.7 MeV, charged pion 139.6 MeV; neutrino masses are negligible. Which of the following processes is allowed by every conservation law?

KEY **D** $\pi^+ \to \mu^+ + \nu_\mu$

D conserves charge (+1 → +1), baryon number (0 → 0) and muon lepton number (0 → −1 + 1), and the pion's 139.6 MeV rest energy exceeds the muon's 105.7 MeV, so energy allows it too; it is the pion's dominant decay. A has the right particles apart from the neutrino: lepton number goes from 0 to +2, and the real decay emits an electron antineutrino. B is what happens inside proton-rich nuclei, which is why it tempts, but a free proton at rest has 938.3 MeV, less than the 940.1 MeV of the products, so energy conservation forbids it; inside a nucleus the difference in binding energy pays. C conserves charge, energy and total lepton number, but it turns a muon-type lepton into an electron-type lepton, violating muon and electron lepton numbers separately, and it has never been observed.

U1

11

CORE COURSE

STRUCTURED · ROUTINE · UPIII-04.08 · UPIII-05.03 · UPIII-05.08 **[11]**

An aerial-mapping camera photographs a field from a height of 120 m. Its lens is a thin converging lens of focal length 8.80 mm and aperture diameter 3.14 mm, and its image sensor has square pixels 2.40 µm wide. Take the wavelength of the light as 550 nm in air. To reduce reflections, the lens glass, of refractive index 1.52, carries a single coating of magnesium fluoride, of refractive index 1.38, designed for 550 nm at normal incidence.

(a) CALCULATE · U3 **[2]**

Calculate how far behind the lens the sensor must be placed to focus the field, and how far this is from the lens's focal plane.

1. $1/s' = 1/f - 1/s = 1/(8.80 \times 10^{-3}\ \text{m}) - 1/(120\ \text{m})$, so $s' = 8.800\,65 \times 10^{-3}$ m behind the lens
2. $s' - f = 6.5 \times 10^{-7}$ m = 0.65 µm, about a quarter of a pixel width, so the field is effectively at infinity and the sensor sits in the focal plane

   Mixing millimetres with metres, or writing $1/s' = 1/f + 1/s$, puts the image well away from the focal plane; the tiny value of $s' - f$ is itself the check that the units and the sign are right.

(b) CALCULATE · U3 **[2]**

Calculate the width of the strip of ground that is imaged onto one pixel.

1. The magnification has magnitude $s'/s = 8.8006 \times 10^{-3}\ \text{m}/120\ \text{m} = 7.334 \times 10^{-5}$
2. Ground width $= 2.40 \times 10^{-6}\ \text{m}/(7.334 \times 10^{-5}) = 3.27 \times 10^{-2}$ m = 32.7 mm

   Multiplying the pixel width by the magnification instead of dividing gives a ground width of order 10^{-10} m; the image is smaller than the object, so the strip of ground must be far wider than the pixel.

(c) JUSTIFY · U3 **[3]**

Justify whether diffraction at the lens aperture or the size of the pixels sets the smallest ground detail this camera can resolve at 550 nm.

1. Rayleigh criterion for a circular aperture: $\theta_{min} = 1.22\lambda/D = 1.22 \times 550 \times 10^{-9}\ \text{m}/(3.14 \times 10^{-3}\ \text{m}) = 2.14 \times 10^{-4}$ rad
2. On the ground this is 120 m × 2.14×10^{-4} rad = 25.6 mm; equivalently $f\theta_{min}$ = 1.88 µm on the sensor
3. One pixel covers 32.7 mm of ground (2.40 µm on the sensor), coarser than the 25.6 mm diffraction limit, so the pixels limit the resolution: two features need at least two pixels, about 65 mm, to be separated

   Comparing an angle with a pixel width directly is the error; both limits must be put on the ground, or both on the sensor, before they can be compared.

(d) DETERMINE · U2 **[2]**

Determine the minimum thickness of the coating that minimises the reflection of 550 nm light at normal incidence.

1. Both reflections, air to coating (1.00 to 1.38) and coating to glass (1.38 to 1.52), go from lower to higher index, so both waves are shifted by π and cancellation needs a path difference $2n_c t = \lambda/2$
2. $t = \lambda/(4n_c) = 550\ \text{nm}/(4 \times 1.38) = 99.6\ \text{nm}$

   Giving only one reflection a phase shift leads to $2n_c t = \lambda$ and $t = 199$ nm, the thickness that maximises reflection; the index sequence has to be checked at both surfaces.

(e) EXPLAIN · U1 **[2]**

When white light is reflected from the coated lens, the reflection looks purple. Explain why.

1. The path difference $2n_c t = 275$ nm is half of 550 nm, but only 0.39 of a wavelength at 700 nm and 0.69 of a wavelength at 400 nm, so the two reflected waves cancel best near the middle of the visible spectrum and only partly at its ends
2. The reflected light is therefore depleted in green and yellow and relatively strong at the red and violet ends, which together look purple

   Attributing the colour to dispersion in the coating misses the mechanism: the coating is a quarter-wave layer for one wavelength only, and it is the phase condition that changes across the spectrum.

12

CORE COURSE

STRUCTURED · DEMANDING · UPIII-01.04 · UPIII-01.06 · UPIII-01.08 **[12]**

The pendulum of a longcase clock is a light rigid rod carrying a solid brass sphere of mass 2.30 kg and radius r = 0.0400 m, whose centre is L = 0.990 m below the pivot. The moment of inertia of a solid sphere about an axis through its centre is $(2/5)mr^2$. Take $g = 9.81\ \text{m/s}^2$. The clock's escapement gives the pendulum a small push once each swing to keep its amplitude at 4.00°.

(a) DERIVE · U2 **[3]**

Derive, from the rotational form of Newton's second law, an expression for the period of small oscillations of this pendulum in terms of L, r and g, stating the approximation you make.

1. Parallel-axis theorem: the moment of inertia about the pivot is $I = (2/5)mr^2 + mL^2$
2. Gravity exerts a torque $-mgL \sin\theta$ about the pivot, so $I\,d^2\theta/dt^2 = -mgL\sin\theta \approx -mgL\theta$, using $\sin\theta \approx \theta$ (θ in radians) for small swings
3. This is simple harmonic motion with $\omega_0^2 = mgL/I$, so $T_0 = 2\pi\sqrt{(I/(mgL))} = 2\pi\sqrt{((L^2 + 2r^2/5)/(gL))}$

   Treating the bob as a point mass gives $I = mL^2$ and the simple-pendulum period; the sphere also turns about its own centre as it swings, and the parallel-axis term is the step that captures it.

(b) CALCULATE · U3 **[2]**

Calculate the period, and the fraction by which it exceeds the period of a simple pendulum of length 0.990 m.

1. $T_0 = 2\pi\sqrt{((0.990^2 + 0.4 \times 0.0400^2)\ \text{m}^2/(9.81\ \text{m/s}^2 \times 0.990\ \text{m}))} = 1.9967$ s, against $2\pi\sqrt{(0.990\ \text{m}/9.81\ \text{m/s}^2)} = 1.9960$ s for a simple pendulum
2. Fractional excess $= \sqrt{(1 + 2r^2/(5L^2))} - 1 \approx r^2/(5L^2) = 3.3 \times 10^{-4}$, about 28 s per day

   Rounding both periods to 2.00 s hides the whole effect; the difference has to come from the ratio $r^2/(5L^2)$, not from subtracting rounded periods.

(c) DETERMINE · U4 **[3]**

With the escapement disconnected, the amplitude falls from 4.00° to 2.00° in 12.0 minutes. Determine the average power the escapement must supply to keep the amplitude at 4.00°, stating the assumption about the damping that your method requires.

1. For light damping proportional to angular velocity the amplitude decays exponentially, so the energy, proportional to the square of the amplitude, falls by a factor of 4 with time constant $\tau_E = 720\ \text{s}/\ln 4 = 519\ \text{s}$
2. Energy of the swing at 4.00° (0.0698 rad): $E = \tfrac{1}{2}I\omega_0^2\theta_0^2 = \tfrac{1}{2}mgL\theta_0^2 = \tfrac{1}{2} \times 2.30\ \text{kg} \times 9.81\ \text{m/s}^2 \times 0.990\ \text{m} \times (0.0698\ \text{rad})^2 = 5.44 \times 10^{-2}\ \text{J}$
3. $P = E/\tau_E = 5.44 \times 10^{-2}\ \text{J}/519\ \text{s} = 1.05 \times 10^{-4}\ \text{W}$, replacing the 0.38% of the energy lost in each cycle

   Using the amplitude ratio of 2 instead of the energy ratio of 4 doubles the time constant and halves the power; the energy goes as the square of the amplitude.

(d) CALCULATE · U3 **[2]**

For a finite amplitude θ_0 the period is $T \approx T_0(1 + \theta_0^2/16)$, with θ_0 in radians. The clock is adjusted to keep perfect time at an amplitude of 4.00°, but friction in its gear train lets the amplitude wander between 3.50° and 4.50°. Calculate the largest time the clock gains or loses in one day as a result.

1. Relative to 4.00° (0.06981 rad), the period at 4.50° (0.07854 rad) is longer by the fraction $(0.07854^2 - 0.06981^2)/16 = 8.09 \times 10^{-5}$, and at 3.50° (0.06109 rad) it is shorter by 7.14×10^{-5}
2. The largest error is $8.09 \times 10^{-5} \times 86\,400\ \text{s} = 7.0\ \text{s}$ lost per day at 4.50° (the clock gains 6.2 s per day at 3.50°)

   Evaluating the whole correction at 4.50°, 33 s per day, ignores that the clock was regulated at 4.00°; only the change in amplitude matters, and it enters as a difference of squares.

(e) EVALUATE · U1 **[2]**

The clock must keep time to within 1 s per day. Evaluate which idealisation of the simple-pendulum model, the point-mass bob or the small-angle limit, limits this clock's timekeeping.

1. The point-mass error, 3.3×10^{-4} or about 28 s per day, is constant, so it is removed once when the clock is regulated by raising or lowering the bob, and it does not limit the timekeeping
2. The amplitude-dependent error cannot be regulated out: it varies by up to 7 s per day as the amplitude wanders, seven times the tolerance, so the small-angle idealisation is the one that fails, and the remedy is a steadier or smaller amplitude rather than a smaller bob

   Ranking the two corrections by size alone picks the bob; the discriminating point is that a constant error is calibrated away, while a varying one is not.

13

CORE COURSE

STRUCTURED · DEMANDING · UPIII-02.06 · UPIII-02.07 · UPIII-03.01 · UPIII-03.02 · UPIII-03.03 · UPIII-03.05 **[10]**

A pair of extraction fans of the same model stand 6.0 m apart on a flat roof. Each has 8 blades and radiates a steady tone at its blade-passing frequency, the number of blades multiplied by the fan's rotation rate in revolutions per second, and the controller of fan A shows a rotation rate of 24.50 revolutions per second. With both fans running, a technician's sound level meter, filtered to reject everything except a narrow band around this tone and quick enough to follow any change in the tone's amplitude, records a level that rises and falls smoothly and regularly between a maximum of 79.5 dB and a minimum of 71.9 dB, with successive maxima 5.00 s apart. The meter responds to the pressure at its microphone and reports the sound intensity level of a single plane wave with the same pressure amplitude. Treat the fans as small sources in open air and ignore reflections. Take the reference intensity as $1.00 \times 10^{-12}\ \text{W/m}^2$, the density of air as $1.20\ \text{kg/m}^3$, the speed of sound as 343 m/s and atmospheric pressure as $1.01 \times 10^5\ \text{Pa}$.

(a) CALCULATE · U3 **[2]**

Calculate the pressure amplitude of the sound at the meter when the level is at a maximum and when it is at a minimum.

1. At the maximum, $I = 1.00 \times 10^{-12}$ W/m² × 10^(79.5/10) = 8.91×10^{-5} W/m², so $p_0 = \sqrt{(2\rho v I)} = \sqrt{(2 \times 1.20\ \text{kg/m}^3 \times 343\ \text{m/s} \times 8.91 \times 10^{-5}\ \text{W/m}^2)} = 0.271$ Pa
2. At the minimum, $I = 1.00 \times 10^{-12}$ W/m² × 10^(71.9/10) = 1.55×10^{-5} W/m², so $p_0 = \sqrt{(2\rho v I)} = 0.113$ Pa, a pressure-amplitude ratio of 2.40 between maximum and minimum

   The 7.6 dB swing is an intensity ratio of 5.75 but a pressure-amplitude ratio of only √5.75 = 2.40; candidates who scale pressure by 10^(β/10) instead of 10^(β/20) carry an amplitude ratio 2.4 times too large into (c).

(b) EXPLAIN · U1 **[2]**

The sound from the two fans reaches the meter travelling in different directions. Explain why the excess pressures of the two waves at the meter can be added as numbers, whereas the displacements of the air in the two waves cannot, and why the superposition principle holds for these waves.

1. Excess pressure is a scalar, so the two waves' pressures add algebraically at each instant; sound is longitudinal, so each wave displaces the air along its own direction of travel, and displacements arriving from different directions add as vectors, which is why a pressure-sensing meter, like the ear, responds to the summed pressure
2. Superposition needs a linear medium: even the largest pressure amplitude, 0.271 Pa, is only 2.7×10^{-6} of atmospheric pressure, so the air responds linearly and each wave passes through the other unchanged

   Candidates who add displacement amplitudes as numbers have assumed, without noticing, that the waves travel along the same line; and asserting that sound waves superpose earns nothing without the comparison with atmospheric pressure that makes the medium linear.

(c) DETERMINE · U2 **[4]**

Determine the sound intensity level that the louder fan alone would produce at the meter, the level that the quieter fan alone would produce, and the difference between the two fans' rotation rates.

1. The two tones differ slightly in frequency, so their relative phase advances steadily: the pressures add in step at a maximum and in opposition at a minimum, $p_{max} = p_1 + p_2$ and $p_{min} = p_1 - p_2$
2. $p_1 = (0.271\ \text{Pa} + 0.113\ \text{Pa})/2 = 0.192$ Pa and $p_2 = (0.271\ \text{Pa} - 0.113\ \text{Pa})/2 = 0.0790$ Pa, so the quieter fan's pressure amplitude is about 0.41 of the louder fan's
3. $I_1 = p_1^2/(2\rho v) = 4.47 \times 10^{-5}$ W/m², a level of 76.5 dB, and $I_2 = p_2^2/(2\rho v) = 7.58 \times 10^{-6}$ W/m², a level of 68.8 dB
4. The level peaks once for each 2π of relative phase, so the blade-passing frequencies differ by 1/(5.00 s) = 0.200 Hz; dividing by the 8 blades, the rotation rates differ by 0.0250 revolutions per second (the beats alone cannot show which fan is faster)

   The two amplitudes come from the half-sum and half-difference of the pressure amplitudes. Averaging the readings in decibels (75.7 dB), halving the sum and the difference of the intensities (77.2 dB and 75.7 dB), or reading the 71.9 dB minimum as the quieter fan alone gives neither fan's level: it is amplitudes, not intensities, that add, and at a minimum the quieter fan's sound is cancelling part of the louder fan's. Candidates who stop at 0.200 Hz have found the difference in blade-passing frequency, which is 8 times the difference in rotation rate.

(d) DESCRIBE · U1 **[2]**

The technician adjusts the controller of fan B until the level at the meter stops rising and falling. Describe how the steady level now depends on the fans and on where the meter is placed, giving the range of steady levels the meter could read at its present position, and the change in the difference between its distances from the two fans that takes the level from its highest to its lowest value, ignoring any change in each fan's own amplitude.

1. With equal frequencies the two tones keep a fixed phase difference, set by the relative angular positions of the two fans' blades and by the difference between the meter's distances from the fans, so the level is steady at some value between 71.9 dB (antiphase) and 79.5 dB (in phase), not necessarily at the 77.2 dB of the two intensities added
2. Changing the difference in distance by half a wavelength, v/(2f) = 343 m/s/(2 × 196 Hz) = 0.875 m, turns in-phase arrival into antiphase and takes the level from its highest to its lowest value, so the roof now carries a fixed pattern of loud and quiet places, and the 77.2 dB sum of intensities is only the average over them

   Candidates who expect synchronised fans to be quieter everywhere, or the steady level to be the sum of the intensities, have missed that equal frequencies make the two sources coherent: the beat in time becomes an interference pattern in space.

14 CORE COURSE

STRUCTURED · DEMANDING · UPIII-08.02 · UPIII-08.04 · UPIII-08.07 **[11]**

In the laboratory frame S, three events are recorded on the x-axis. Event A: a detector at x = 0 fires at t = 0. Event B: a detector at x = 900 m fires at t = 2.00 μs. Event C: the same detector at x = 900 m fires again at t = 4.00 μs. Frame S′ moves at +0.800c along the x-axis relative to S, and its origin coincides with that of S at event A. Take $c = 3.00 \times 10^8$ m/s.

(a) CALCULATE · U3 **[2]**

Calculate the invariant interval $s^2 = (c\Delta t)^2 - (\Delta x)^2$ between events A and B, and classify the separation.

1. $c\Delta t = 3.00 \times 10^8 \text{ m/s} \times 2.00 \times 10^{-6} \text{ s} = 600 \text{ m}$, so $s^2 = (600 \text{ m})^2 - (900 \text{ m})^2 = -4.50 \times 10^5 \text{ m}^2$
2. $s^2 < 0$, so the separation is spacelike

Converting the time interval to a length, $c\Delta t$, before comparing it with Δx is the step that makes the classification visible; candidates who compare 2.00 μs with 900 m directly have nothing to compare.

(b) DETERMINE · U3 **[3]**

Determine the time interval $\Delta t'$ and the spatial separation $\Delta x'$ between A and B measured in S′, and state which event occurs first in S′.

1. $\gamma = 1/\sqrt{(1 - 0.800^2)} = 1.667$
2. $\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = 1.667 \times (2.00 \text{ μs} - 2.40 \text{ μs}) = -0.667 \text{ μs}$, so B occurs before A in S′
3. $\Delta x' = \gamma(\Delta x - v\Delta t) = 1.667 \times (900 \text{ m} - 480 \text{ m}) = 700 \text{ m}$, consistent with $(200 \text{ m})^2 - (700 \text{ m})^2 = -4.50 \times 10^5 \text{ m}^2$

Writing $\Delta t' = \gamma\Delta t$ is the error this part is set on: neither interval is a proper time, because the events occur at different places in both frames, so only the full Lorentz transformation will do.

(c) JUSTIFY · U1 **[2]**

Justify the claim that event A cannot have caused event B, giving two separate arguments.

1. A signal from A would have to cover 900 m in 2.00 μs, a speed of 4.50×10^8 m/s = 1.5c, faster than light; equivalently, the interval is spacelike
2. The time order of A and B depends on the frame (B precedes A in S′), whereas a cause must precede its effect in every inertial frame

Saying only that B comes first in S′ does not yet show why that matters. The two arguments are one fact seen from the interval and from the transformation, and both must be stated.

(d) DETERMINE · U2 **[2]**

Determine the velocity, relative to S, of the inertial frame in which events A and B are simultaneous.

1. Setting $\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = 0$ gives $v = c^2\Delta t/\Delta x$
2. $v = c \times (600 \text{ m}/900 \text{ m}) = 0.667c = 2.00 \times 10^8$ m/s, in the +x direction

Only a spacelike pair has such a frame, since $c^2\Delta t/\Delta x$ is less than c only when $c\Delta t < \Delta x$; a candidate should notice that the result is automatically below c here.

(e) PREDICT · U1 **[2]**

Predict the order of events A and C in S′ and in every other inertial frame, giving the reason.

1. For A and C, $c\Delta t = 1200$ m exceeds $\Delta x = 900$ m, so $s^2 = +6.30 \times 10^5 \text{ m}^2$ and the separation is timelike
2. C follows A in every inertial frame: in S′, $\Delta t' = 1.667 \times (4.00 \text{ μs} - 2.40 \text{ μs}) = +2.67 \text{ μs}$, and no $v < c$ can make $\Delta t - v\Delta x/c^2$ negative when $c\Delta t > \Delta x$

The contrast with (b) is the point. Candidates who conclude that relativity makes every time order frame-dependent have missed that the order of timelike pairs, and so causal order, is the same in all frames.

15 CORE COURSE

STRUCTURED · DEMANDING · UPIII-11.03 · UPIII-11.04 · UPIII-11.05 **[12]**

A radioisotope power source for a deep-space probe contains 3.50 kg of plutonium-238 (^{238}Pu, Z = 94), which decays by alpha emission to uranium-234 with a half-life of 87.7 years. Atomic masses: ^{238}Pu 238.049 560 u, ^{234}U 234.040 952 u, ^{4}He 4.002 603 u. Take 1 u = 931.494 MeV/c², 1 MeV = 1.602 18 × 10^{-13} J, N_A = 6.022 × 10^{23} mol^{-1}, 1 year = 3.156 × 10^{7} s, and the molar mass of ^{238}Pu as 238.05 g/mol.

(a) STATE · U1 **[1]**

State the decay equation, showing the nucleon number and proton number of every particle.

1 $^{238}_{94}Pu \rightarrow {}^{234}_{92}U + {}^{4}_{2}He$, with nucleon numbers 238 = 234 + 4 and proton numbers 94 = 92 + 2

Writing the daughter with proton number 94 or 93 shows that the candidate has balanced nucleons but not charge.

(b) EXPLAIN · U1 **[2]**

Explain why the Q value of this decay can be calculated directly from atomic masses, even though the decay is a nuclear process.

1 An atomic mass includes the atom's electrons: the ^{238}Pu atom carries 94 electron masses, and the products carry 92 in the uranium atom plus 2 in the helium atom, also 94

2 The electron masses therefore cancel in the mass difference, and the change in the atoms' total electron binding energy, at most a few tens of keV, is under 1% of a Q value of several MeV

This is a count, not a rule: in β^+ decay the same count leaves two electron masses uncancelled, and candidates who have memorised the alpha case as a rule are caught out there.

(c) CALCULATE · U3 **[2]**

Calculate the Q value of the decay in MeV.

1 $\Delta m = 238.049\,560\ u - 234.040\,952\ u - 4.002\,603\ u = 0.006\,005\ u$

2 $Q = 0.006\,005\ u \times 931.494\ MeV/u = 5.59\ MeV$

Rounding the masses before subtracting destroys the answer: the difference appears in the sixth significant figure, so every digit given must be carried.

(d) SHOW THAT · U2 **[2]**

Show that, for a parent nucleus at rest, the alpha particle carries about 98% of the Q value as kinetic energy.

1 Momentum conservation gives the alpha particle and the daughter equal and opposite momenta, so with $K = p^2/2m$ their kinetic energies are inversely proportional to their masses

2 $K_\alpha = Q\, m_U/(m_U + m_\alpha) = 5.59\ MeV \times 234.04/238.04 = 5.50\ MeV$, which is 98.3% of Q

Assuming the daughter takes no energy, or that Q is shared equally, are the errors; the non-relativistic treatment is justified because 5.5 MeV is tiny beside the alpha particle's rest energy of about 3.7 GeV.

(e) CALCULATE · U3 **[3]**

Calculate the initial thermal power of the source, assuming that all of the decay energy is absorbed within it.

1 $N = (3500\ g/238.05\ g/mol) \times 6.022 \times 10^{23}\ mol^{-1} = 8.85 \times 10^{24}$ nuclei

2 $\lambda = \ln 2/(87.7 \times 3.156 \times 10^{7}\ s) = 2.50 \times 10^{-10}\ s^{-1}$, so the activity is $A = \lambda N = 2.22 \times 10^{15}$ Bq

3 $P = AQ = 2.22 \times 10^{15}\ s^{-1} \times 5.59\ MeV \times 1.602 \times 10^{-13}\ J/MeV = 1.99 \times 10^{3}$ W

Using K_α instead of Q undercounts by the recoil energy of the daughter, which is also absorbed in the source; the other common slip is leaving the half-life in years when the power is wanted in watts.

(f) DETERMINE · U3 **[2]**

Determine how long after the source is made its thermal power falls to 1500 W.

1. P(t) = P₀e^(−λt), so t = (T½/ln 2) ln(P₀/P) = 87.7 years × log₂(1987 W/1500 W)
2. t = 87.7 years × 0.406 = 35.6 years

Assuming a linear decline at the initial rate gives 31 years; the power follows the exponential decay of the activity, and a fall to 75% of the initial value takes 0.406 half-lives.

16

CORE COURSE

STRUCTURED · DEMANDING · UPIII-07.04 · UPIII-07.05 · UPIII-07.06 · UPIII-07.09 **[13]**

An engine takes 0.500 mol of an ideal diatomic gas ($C_V = 5R/2$, $\gamma = 1.40$) around a three-stage cycle. A → B: reversible isothermal compression at 300 K, in contact with a cold reservoir at 300 K, until the volume is one quarter of its value at A. B → C: heating at constant volume, by contact with a hot reservoir at 600 K, until the gas reaches temperature T_C, when contact is broken. C → A: reversible adiabatic expansion back to state A. Take $R = 8.314$ J/(mol·K).

(a) SHOW THAT · U2 **[2]**

Show that $T_C = 522$ K.

1. Along the reversible adiabat C → A, $TV^{(\gamma-1)}$ is constant, and $V_C = V_B = V_A/4$
2. $T_C = T_A(V_A/V_C)^{(\gamma-1)} = 300\ \text{K} \times 4^{0.40} = 522\ \text{K}$

Using the exponent γ instead of $\gamma - 1$, as if temperature followed the pV^{γ} law, gives $300\ \text{K} \times 4^{1.40} = 2090$ K; this part is set to catch that substitution.

(b) CALCULATE · U3 **[4]**

Calculate the heat Q absorbed by the gas and the work W done by the gas in each of the three stages, and the net work done by the gas per cycle.

1. A → B: $\Delta U = 0$, so $Q = W = nRT \ln(V_B/V_A) = 0.500\ \text{mol} \times 8.314\ \text{J/(mol·K)} \times 300\ \text{K} \times \ln(1/4) = -1.73 \times 10^3$ J
2. B → C: $W = 0$, so $Q = nC_V(T_C - T_B) = 0.500\ \text{mol} \times 20.79\ \text{J/(mol·K)} \times 222\ \text{K} = +2.31 \times 10^3$ J
3. C → A: $Q = 0$, so $W = -\Delta U = nC_V(T_C - T_A) = +2.31 \times 10^3$ J
4. Net work = −1729 J + 0 + 2311 J = +582 J per cycle, equal to the net heat absorbed

Sign discipline is what is examined. A candidate who records the isothermal work as positive, or gives the adiabatic stage a heat term, cannot close the energy ledger, which requires the net work to equal 2311 J − 1729 J.

(c) COMPARE · U3 **[2]**

Compare the thermal efficiency of this cycle with the Carnot efficiency of an engine operating between the same two reservoirs.

1. $\eta = W/Q_{in} = 582\ \text{J}/2311\ \text{J} = 0.252$ (25.2%), where only the B → C heat is heat input
2. $\eta_{Carnot} = 1 - 300\ \text{K}/600\ \text{K} = 0.500$, so the cycle achieves about half of the Carnot bound

Counting $|Q_{AB}|$ as input, or taking 522 K as the hot temperature (which gives 42.6%) when the heat actually comes from a 600 K reservoir, both misstate the bound the engine is measured against.

(d) DETERMINE · U3 **[3]**

Determine the total entropy change of the gas and the two reservoirs per cycle, and identify the stage in which entropy is generated.

1. Over a complete cycle $\Delta S_{gas} = 0$, because entropy is a state function
2. $\Delta S_{hot} = -2311\ \text{J}/600\ \text{K} = -3.85\ \text{J/K}$ and $\Delta S_{cold} = +1729\ \text{J}/300\ \text{K} = +5.76\ \text{J/K}$, so $\Delta S_{total} = +1.91\ \text{J/K}$
3. Entropy is generated in B → C, where heat crosses a finite temperature difference from the 600 K reservoir into gas at 300 K to 522 K; A → B and C → A are reversible

   Candidates often calculate the gas's entropy gain in B → C (+5.76 J/K) and call it the entropy generated. The generation is that gain minus the hot reservoir's loss, and it happens only where heat flows across a temperature gap.

(e) SHOW THAT · U2 **[2]**

Show that the work this cycle loses, compared with a Carnot engine that absorbs the same heat Q_{BC} from the 600 K reservoir, equals $T_{cold} \times S_{gen}$, where $T_{cold} = 300$ K and S_{gen} is the entropy generated per cycle.

1. Over a cycle $W = Q_{BC} - |Q_{AB}|$ and $S_{gen} = |Q_{AB}|/T_{cold} - Q_{BC}/T_{hot}$; eliminating $|Q_{AB}|$ gives $W = Q_{BC}(1 - T_{cold}/T_{hot}) - T_{cold}\,S_{gen}$
2. Numerically, $\eta_{Carnot}\,Q_{BC} - W = 1155\ \text{J} - 582\ \text{J} = 574\ \text{J}$, and $T_{cold}\,S_{gen} = 300\ \text{K} \times 1.91\ \text{J/K} = 574\ \text{J}$

   This is the capstone of the entropy audit: each joule per kelvin of entropy generated costs T_{cold} joules of work. Showing only the numerical agreement, without the general algebra, earns the second mark but not the first.

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17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIII-09.04 **[10]**

In a low-energy electron diffraction experiment, electrons accelerated from rest through a potential difference V strike a clean copper crystal surface along its normal. The rows of surface atoms, a distance a apart, act as a reflection grating: first-order diffracted beams leave the surface at an angle φ to the normal, where a sin φ = λ and λ is the de Broglie wavelength of the electrons. The angle φ is measured at several accelerating potential differences, and the energy scale of the electron gun is calibrated to ±2%. Take h = 6.626 × 10⁻³⁴ J·s, m_e = 9.109 × 10⁻³¹ kg and e = 1.602 × 10⁻¹⁹ C. In bulk copper the atoms of a plane of this orientation form a square array of side 0.2556 nm.

(a) SHOW THAT · U2 **[2]**

Show that the model predicts a straight line through the origin when sin φ is plotted against 1/√V, and give its gradient in terms of h, m_e, e and a.

1. The kinetic energy eV = p²/(2m_e) gives the de Broglie wavelength λ = h/p = h/√(2m_e eV)
2. Then sin φ = λ/a = [h/(a√(2m_e e))] × (1/√V), a straight line through the origin with gradient G = h/(a√(2m_e e))

Using the photon relation λ = hc/(eV) gives sin φ proportional to 1/V and a curved plot; for a massive particle at these energies the wavelength goes as the inverse square root of the energy.

(b) DETERMINE · U4 **[3]**

Using all of the readings, determine the gradient of the graph of sin φ against 1/√V.

1. Values of 1/√V (in V^−½) and sin φ: 0.1414 and 0.681, 0.1155 and 0.550, 0.1000 and 0.483, 0.0894 and 0.426, 0.0816 and 0.394, 0.0756 and 0.360, 0.0707 and 0.340
2. A best-fit straight line through all seven points and the origin (or a least-squares fit), not a gradient taken from a single reading
3. Gradient G = 4.80 V^½ (least-squares line through the origin 4.798 V^½; accept 4.75 V^½ to 4.85 V^½)

Plotting φ in degrees, or sin φ against 1/V, gives a curve whose apparent gradient depends on which readings are chosen; only sin φ against 1/√V is linear in this model.

(c) CALCULATE · U4 **[3]**

Calculate the row spacing a, with its uncertainty arising from the calibration of the energy scale.

1. a = h/(G√(2m_e e)), with √(2m_e e) = √(2 × 9.109 × 10⁻³¹ kg × 1.602 × 10⁻¹⁹ C) = 5.402 × 10⁻²⁵ kg^½ C^½
2. a = 6.626 × 10⁻³⁴ J·s/(4.80 V^½ × 5.402 × 10⁻²⁵ kg^½ C^½) = 2.56 × 10⁻¹⁰ m = 0.256 nm (accept 0.253 nm to 0.258 nm)
3. A ±2% error in V changes 1/√V, and so G and a, by ±1%, so a = 0.256 ± 0.003 nm

Carrying ±2% straight across to a misses the square root: the wavelength, and so the inferred spacing, depends on V^−½, which halves the relative uncertainty.

(d) EVALUATE · U4 **[2]**

A calculation predicts that the outermost layer of copper atoms relaxes towards the layer beneath it, reducing the spacing between those two layers by about 2%. Evaluate whether these data support a surface whose atoms keep their bulk arrangement within the layer, and whether this experiment could detect the predicted relaxation.

1. The measured a = 0.256 ± 0.003 nm agrees with the bulk value of 0.2556 nm within its uncertainty, so the data support a surface layer whose atoms keep the bulk spacing within the plane
2. The predicted relaxation changes only the spacing perpendicular to the surface, which does not enter a sin φ = λ, so the beam angles cannot detect it at any precision; it would have to be sought in how the beam intensities vary with electron energy

The discriminating step is to ask what the measurement is sensitive to. An answer that compares the 2% relaxation with the ±1% uncertainty and calls it detectable has not used the model it was given.

18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPIII-10.03 · UPIII-10.05 **[14]**

In muonic hydrogen a negative muon, of mass 1.8835×10^{-28} kg, takes the place of the electron and is bound to a proton of mass 1.6726×10^{-27} kg; its energy levels are measured to find the size of the proton. The relative motion obeys the hydrogen-atom Schrödinger equation with the reduced mass $\mu = m_\mu m_p/(m_\mu + m_p)$ in place of the electron mass. For a state with no angular dependence ($l = 0$) the equation reads $-(\hbar^2/2\mu)(1/r^2)\, d/dr(r^2\, d\psi/dr) - (e^2/(4\pi\varepsilon_0 r))\psi = E\psi$. Take $\hbar = 1.0546 \times 10^{-34}$ J·s, $e = 1.602 \times 10^{-19}$ C, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, the proton radius as 0.84 fm and the Bohr radius of ordinary hydrogen as 5.29×10^{-11} m. You may use $\int_0^\infty r^n e^{-\beta r}\, dr = n!/\beta^{n+1}$.

(a) DERIVE · U2 **[5]**

By substituting the trial solution $\psi(r) = Ce^{-r/a}$ into the Schrödinger equation, derive expressions for the length a and the energy E, and evaluate both for muonic hydrogen.

1. $d\psi/dr = -\psi/a$, so $r^2\, d\psi/dr = -(r^2/a)\psi$
2. $(1/r^2)\, d/dr(r^2\, d\psi/dr) = (1/a^2 - 2/(ar))\psi$
3. Substituting and dividing by ψ: $-(\hbar^2/2\mu)(1/a^2 - 2/(ar)) - e^2/(4\pi\varepsilon_0 r) = E$, which must hold for every r, so the 1/r terms must cancel: $a = 4\pi\varepsilon_0\hbar^2/(\mu e^2)$
4. The remaining constant terms give $E = -\hbar^2/(2\mu a^2)$
5. $\mu = 1.693 \times 10^{-28}$ kg (185.8 electron masses), $a = 2.85 \times 10^{-13}$ m and $E = -4.05 \times 10^{-16}$ J $= -2.53$ keV

The step that separates candidates is recognising that an equation true for every r fixes two constants at once, one from the 1/r terms and one from the constant terms. Candidates who set r = a and solve a single equation cannot find both.

(b) SHOW THAT · U2 **[2]**

Show that the normalised wavefunction has $C = 1/\sqrt{(\pi a^3)}$.

1. Normalisation over all space: $\int_0^\infty |\psi|^2\, 4\pi r^2\, dr = 4\pi C^2 \int_0^\infty r^2 e^{-2r/a}\, dr = 1$
2. The integral is $2!/(2/a)^3 = a^3/4$, so $\pi a^3 C^2 = 1$ and $C = 1/\sqrt{(\pi a^3)}$

Omitting the volume element $4\pi r^2\, dr$ and normalising $\int|\psi|^2\, dr$ instead gives a constant with the wrong dimensions, which a units check exposes at once.

(c) DETERMINE · U3 **[2]**

Determine the most probable distance of the muon from the proton, in metres.

1. $dP/dr \propto (2r - 2r^2/a)e^{-2r/a} = 0$ gives r = a, the maximum (r = 0 is a minimum)
2. $r = a = 2.85 \times 10^{-13}$ m, about 186 times smaller than the corresponding distance in ordinary hydrogen

Setting $d\psi/dr = 0$ finds no interior maximum at all; the maximum belongs to P(r), not to ψ.

(d) SKETCH · U1 **[2]**

Sketch the radial probability density $P(r) = 4\pi r^2|\psi(r)|^2$ against r/a, marking the most probable radius found in (c) and the mean radius $\langle r\rangle = 1.5a$.

1. Axes labelled P(r) and r/a; the curve is zero at r = 0, rises initially as r^2, and has a single maximum marked at r/a = 1
2. A long exponential tail towards large r, so the curve is asymmetric, with $\langle r\rangle = 1.5a$ marked to the right of the peak

Drawing $|\psi|^2$, which is largest at r = 0, is the commonest error: the probability per unit volume peaks at the nucleus, but the probability per unit radius vanishes there because the volume of a thin shell does.

(e) ESTIMATE · U4 **[3]**

Estimate the probability that the muon is found inside the proton, stating the assumption you make, and compare it with the corresponding probability for the electron in ordinary hydrogen.

1. Because the proton radius R = 0.84 fm is far smaller than a, take $|\psi|^2$ as constant at its value at r = 0 across the proton: $P \approx (4/3)\pi R^3|\psi(0)|^2 = (4/3)(R/a)^3$
2. For muonic hydrogen $P \approx (4/3)(0.84 \times 10^{-15}\ \text{m}/2.85 \times 10^{-13}\ \text{m})^3 = 3.4 \times 10^{-8}$
3. For ordinary hydrogen $P \approx (4/3)(0.84 \times 10^{-15}\ \text{m}/5.29 \times 10^{-11}\ \text{m})^3 = 5.3 \times 10^{-15}$, so the muon is about 6×10^6 times more likely to be inside the proton, which is why muonic energy levels are far more sensitive to the proton's size

Integrating the exact probability costs time without changing the answer: with $R/a \approx 3 \times 10^{-3}$ the exponential factor is 1 to better than 1%. The skill examined is choosing that approximation and saying so.

19

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE PRECISION · DISCRIMINATING · UPIII-08.09 · UPIII-11.02 · UPIII-11.04 **[15]**

The neutron's mass is determined most precisely from the gamma ray emitted when a slow neutron is captured by a proton at rest: n + p → d + γ. A crystal-diffraction spectrometer measures the photon energy as $E_\gamma = 2.223\ 25$ MeV, with an uncertainty of ±0.4 eV. Take the rest energies $m_p c^2 = 938.2721$ MeV and $m_d c^2 = 1875.6129$ MeV, and treat the neutron's kinetic energy before capture as negligible. Write $M = m_n + m_p$ for the total rest mass before capture, and $B = (M - m_d)c^2$ for the binding energy of the deuteron.

(a) DERIVE · U2 **[4]**

Derive, from conservation of energy and momentum, an exact expression for E_γ in terms of M, m_d and c.

1. The initial system is at rest, so the deuteron recoils with momentum equal and opposite to the photon's, of magnitude E_γ/c
2. Energy conservation: $Mc^2 = E_\gamma + \sqrt{((m_d c^2)^2 + E_\gamma^2)}$
3. Isolating the square root and squaring: $(Mc^2 - E_\gamma)^2 = (m_d c^2)^2 + E_\gamma^2$, so $M^2c^4 - 2Mc^2E_\gamma = m_d^2c^4$
4. $E_\gamma = (M^2 - m_d^2)c^2/(2M)$

Writing $E_\gamma = (M - m_d)c^2$ assumes the deuteron stays at rest, which momentum conservation forbids; the whole question turns on the keV that this assumption throws away.

(b) SHOW THAT · U2 **[3]**

Show that the result of (a) can be written exactly as $E_\gamma = B - B^2/(2Mc^2)$, and explain the physical meaning of the term $B^2/(2Mc^2)$.

1. $M^2 - m_d^2 = (M - m_d)(M + m_d)$, with $(M - m_d)c^2 = B$ and $M + m_d = 2M - B/c^2$
2. $E_\gamma = B(2Mc^2 - B)/(2Mc^2) = B - B^2/(2Mc^2)$, with no approximation
3. By energy conservation $Mc^2 = E_\gamma + m_d c^2 + K_d$, so $B^2/(2Mc^2) = B - E_\gamma$ is exactly the kinetic energy of the recoiling deuteron, about $E_\gamma^2/(2m_d c^2) = 1.32$ keV

Candidates who expand to first order and stop have shown an approximation where the question asks for an identity; the factorisation $M^2 - m_d^2 = (M - m_d)(M + m_d)$ makes the result exact in one line.

(c) DETERMINE · U3 **[3]**

Determine the rest energy of the neutron, $m_n c^2$, from the measured photon energy.

1. $B = E_\gamma + B^2/(2Mc^2)$; the correction is small, so evaluate it with $B \approx E_\gamma$ and $Mc^2 \approx 1877.8$ MeV: $(2.2233\ \text{MeV})^2/(2 \times 1877.8\ \text{MeV}) = 1.32$ keV
2. $B = 2.223\ 25\ \text{MeV} + 0.001\ 32\ \text{MeV} = 2.224\ 57\ \text{MeV}$
3. $m_n c^2 = m_d c^2 + B - m_p c^2 = 1875.6129\ \text{MeV} + 2.2246\ \text{MeV} - 938.2721\ \text{MeV} = 939.5654\ \text{MeV}$

One step of iteration suffices: the recoil correction changes by only about 1.6 eV between the first and second estimates, far below the 0.1 keV precision of the rest energies given.

(d) EVALUATE · U4 **[2]**

Evaluate whether the recoil term must be included when the neutron's rest energy is quoted from this measurement.

1. Neglecting recoil gives $m_n c^2 = 939.5641$ MeV, lower by 1.32 keV, which is about 3×10^3 times the ±0.4 eV uncertainty in E_γ and larger than the 0.1 keV rounding of the rest energies given
2. So the recoil term is essential: without it the result is wrong by thousands of standard deviations of the photon measurement, and once it is included the precision of m_n is limited by the proton and deuteron rest energies rather than by the spectrometer

   A correction of one part in 1700 of E_γ looks negligible until it is set against the measurement's own uncertainty; the judgement must be made against the error bar, not against the size of the quantity.

(e) COMPARE · U1 **[3]**

In the reverse process, photodisintegration $\gamma + d \rightarrow n + p$, a photon strikes a deuteron at rest. Compare the minimum photon energy for this process with the photon energy emitted in capture, and explain the difference.

1. At threshold the neutron and proton move off together with the photon's momentum, so the invariant mass must just reach M: $(E_{th} + m_d c^2)^2 - E_{th}^2 = M^2 c^4$, giving $E_{th} = (M^2 - m_d^2)c^2/(2m_d) = B + B^2/(2m_d c^2)$
2. $E_{th} = 2.224\,57$ MeV $+ 0.001\,32$ MeV $= 2.225\,89$ MeV, which is 2.64 keV more than the 2.223 25 MeV emitted in capture
3. In capture the photon carries the binding energy less the deuteron's recoil energy; in photodisintegration the photon must supply the binding energy plus the kinetic energy that momentum conservation forces the n–p pair to carry, so the two energies differ by about two recoil energies, $B^2/(m_d c^2)$

   Candidates who expect the two energies to be equal, because one process is the reverse of the other, have treated the deuteron as infinitely heavy; the asymmetry is momentum conservation acting in opposite directions.

University Physics IV · Midterm Examination

Where classical physics stops and what replaces it: Planck's quanta and the photoelectric evidence, the Lorentz boost and four-momentum, photons and matter waves, and the Schrödinger equation solved for the square well and the harmonic oscillator.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**73**	**13**	**7**

SECTION	**UPIV · Midterm · University Physics IV — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Routine 6 · Demanding 6 · Discriminating 1** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UPIV-01 · UPIV-02 · UPIV-03 · UPIV-04 · UPIV-05 · UPIV-06 · UPIV-07 · UPIV-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Physics IV · Midterm Examination

Answer all questions. 73 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-01.03 · UPIV-01.04 **[1]**

Cavity radiation at temperature 300 K is examined at wavelength 1.00 mm. By what factor does the Rayleigh–Jeans spectral energy density exceed the Planck spectral energy density at this wavelength? Take $h = 6.63 \times 10^{-34}$ J s, $c = 3.00 \times 10^{8}$ m s^{-1} and $k_B = 1.38 \times 10^{-23}$ J K^{-1}.

A 1.00, because the two laws agree exactly at long wavelength

B 1.02

C 1.05

D 20.8

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-03.06 **[1]**

A proton moves at a speed for which $\gamma = 2.00$. A force of magnitude $F_{\parallel}$ applied parallel to its velocity gives it an acceleration of magnitude a. A force of magnitude $F_{\perp}$ applied perpendicular to its velocity gives it an acceleration of the same magnitude a. What is $F_{\parallel}/F_{\perp}$ at that instant?

A 0.250

B 1.00

C 4.00

D 8.00

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-04.07 **[1]**

A wave packet for a free, non-relativistic electron is built from de Broglie waves with angular frequency $\omega(k) = E/\hbar$, where $E = \hbar^2k^2/(2m) + V_0$ and V_0 is a constant potential energy whose value depends on where the zero of energy is chosen. The packet is centred on wavenumber k_0, and $v = \hbar k_0/m$ is the electron's classical speed. Which statement is correct?

A The phase velocity is v/2 and the group velocity is v, whatever value of V_0 is chosen.

B The phase velocity is v and the group velocity is v/2.

C Both the phase velocity and the group velocity change when V_0 is changed.

D The group velocity is v for every choice of V_0, but the phase velocity depends on V_0.

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04 **[1]**

Two identical narrow wave packets, ψ_L centred at $x = -d$ and ψ_R centred at $x = +d$, do not overlap anywhere. Many copies of a particle are prepared either (I) in the superposition $(\psi_L + \psi_R)/\sqrt{2}$, or (II) as a statistical mixture in which half the copies are in ψ_L and half are in ψ_R. Which measurement, repeated on many copies, distinguishes preparation I from preparation II?

A Momentum: for I the momentum distribution is modulated by fringes of spacing $h/(2d)$, and for II it is not.

B Position: for I the position distribution shows interference fringes between the two packets, and for II it does not.

C The mean position $\langle x \rangle$, which is zero for I but not for II.

D None, because in both preparations the particle is found near each centre with probability ½.

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-06.08 **[1]**

Ehrenfest's theorem gives $d\langle x \rangle/dt = \langle p \rangle/m$ and $d\langle p \rangle/dt = -\langle \partial V/\partial x \rangle$. For which potential does $\langle x \rangle$ obey exactly the classical equation $m\, d^2\langle x \rangle/dt^2 = -V'(\langle x \rangle)$ for every state of the particle?

A $V(x) = \alpha x^4$ with $\alpha > 0$, because it is a symmetric confining well.

B Any potential, provided the wave packet is narrow at $t = 0$.

C $V(x) = V_0 \cos(qx)$, because the force it exerts is bounded.

D $V(x) = \tfrac{1}{2} m\omega^2 x^2 + Fx$, with F a constant.

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-07.07 **[1]**

Electrons of kinetic energy 8.00 eV travel in the +x direction towards a potential step at which the potential energy rises abruptly from 0 to 6.00 eV. What fraction of the incident probability current is reflected?

A 0

B 0.111

C 0.333

D 0.360

Answer

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.05 · UPIV-08.07 **[1]**

For a particle of mass m in a harmonic oscillator of angular frequency ω, $\hat{x} = \sqrt{(\hbar/(2m\omega))}\,(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$. What is $\langle x^2\rangle$ in the energy eigenstate $|n\rangle$?

A $(n + \tfrac{1}{2})\hbar/(m\omega)$

B $n\hbar/(m\omega)$

C $(2n + 1)\hbar/(m\omega)$

D Zero

Answer

8 CORE COURSE

STRUCTURED · DEMANDING · UPIV-02.05 · UPIV-02.06 · UPIV-02.07 · UPIV-02.08 **[10]**

An inspection gantry at rest in frame S has two laser gates: G_1 at $x = -60.0$ m and G_2 at $x = +60.0$ m, so the gantry's rest length is 120 m. A cargo shuttle of rest length 150 m moves in the +x direction at 0.800c. Frame S′ is the shuttle's rest frame, in standard configuration with S: the origins coincide at $t = t' = 0$, when the centre of the shuttle passes $x = 0$. Each gate sits at the edge of the track and can fire a brief pulse straight across it, taking negligible time to cross; a pulse that meets the shuttle leaves a mark on its hull. The gantry controller fires both gates at $t = 0$ in S. Take $c = 3.00 \times 10^8$ m s^{-1}.

(a) CALCULATE **[2]**

Calculate the length of the shuttle measured in S, and the length of the gantry measured in S′.

(b) CALCULATE **[3]**

The firing of G_1 is event E_1 and the firing of G_2 is event E_2. Calculate the coordinates x' and t' of both events in S′, and state which gate fires first in S′ and by how much.

(c) EXPLAIN **[3]**

In S′ the shuttle is 150 m long and the gantry is only 72.0 m long. Explain, using your results from (b), why neither pulse strikes the shuttle in S′ either.

(d) DETERMINE **[2]**

A drone is launched forwards from the shuttle's nose at 0.600c relative to the shuttle. By composing rapidities, determine the drone's speed in S.

9 CORE COURSE

STRUCTURED · DEMANDING · UPIV-03.01 · UPIV-03.08 · UPIV-03.09 **[10]**

A secondary beam of positive kaons, each with momentum 2.00 GeV/c, travels along an evacuated decay tunnel. A K^+ at rest has mean lifetime $\tau = 1.238 \times 10^{-8}$ s and decays mainly by $K^+ \to \mu^+ + \nu_\mu$. Take $m_K c^2 = 493.7$ MeV, $m_\mu c^2 = 105.7$ MeV, $m_p c^2 = 938.3$ MeV, $m_\Lambda c^2 = 1115.7$ MeV and $c = 3.00 \times 10^8$ m s^{-1}, and treat the neutrino as massless.

(a) CALCULATE **[2]**

Calculate the mean decay length of the kaons in the laboratory.

(b) DERIVE **[3]**

Starting from conservation of four-momentum, derive expressions for the muon's energy E^* and momentum p^* in the kaon rest frame, and evaluate both.

(c) DETERMINE **[2]**

Determine the largest energy a muon from this beam can have in the laboratory, and state the direction of emission in the kaon rest frame that gives it.

(d) CALCULATE **[3]**

The kaons are produced when a proton beam strikes protons at rest, through $p + p \to p + \Lambda + K^+$. Calculate the minimum kinetic energy of a beam proton for this reaction.

10 CORE COURSE

STRUCTURED · DEMANDING · UPIV-01.07 · UPIV-04.03 · UPIV-04.04 · UPIV-04.05 **[11]**

Technetium-99m, used in medical gamma imaging, emits photons of energy 140.5 keV. Before reaching the camera, many of these photons Compton scatter in soft tissue from electrons that may be treated as free and at rest. The camera rejects scattered photons only by their energy: it accepts any photon whose energy is at least 90.0% of 140.5 keV. Take $m_e c^2 = 511$ keV, $h = 6.63 \times 10^{-34}$ J s, $c = 3.00 \times 10^{8}$ m s^{-1} and $e = 1.60 \times 10^{-19}$ C.

(a) DERIVE **[4]**

A photon of wavelength λ scatters through angle θ from a free electron at rest. By conserving relativistic energy and both components of momentum, derive $\lambda' - \lambda = (h/(m_e c))(1 - \cos\theta)$ for the scattered wavelength λ'.

(b) CALCULATE **[3]**

Calculate the largest angle through which a photon can be scattered once and still be accepted by the camera.

(c) CALCULATE **[2]**

Calculate the de Broglie wavelength of the recoil electron for the scattering in (b).

(d) COMPARE **[2]**

A PET camera detects 511 keV annihilation photons and uses the same fractional window, accepting photons of at least 90.0% of 511 keV. Compare its largest accepted single-scattering angle with your answer to (b), and explain the difference.

11 CORE COURSE

STRUCTURED · DEMANDING · UPIV-05.06 · UPIV-06.03 · UPIV-06.04 · UPIV-07.02 · UPIV-07.03 **[11]**

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.067m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, between $x = 0$ and $x = L$. The energy eigenfunctions are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ the electron is prepared in the state $\Psi(x, 0) = A[\sqrt{3}\,\psi_1(x) + \psi_2(x)]$, with A real and positive. You may use $\int_0^L x\sin(\pi x/L)\sin(2\pi x/L)\,dx = -8L^2/(9\pi^2)$. Take $m_e = 9.11 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $h = 6.63 \times 10^{-34}$ J s.

(a) DETERMINE **[2]**

Determine A, and the probabilities that a measurement of the energy at $t = 0$ returns E_1 and returns E_2.

(b) CALCULATE **[2]**

The mean position $\langle x\rangle$ of the electron oscillates. Calculate the period T of this oscillation.

(c) SHOW THAT **[3]**

Show that $\langle x\rangle(t) = L/2 - (8\sqrt{3}L/(9\pi^2))\cos\omega t$, where $\omega = (E_2 - E_1)/\hbar$.

(d) SKETCH **[2]**

On one set of axes, sketch $|\Psi(x, t)|^2$ for $0 \le x \le L$ at $t = 0$ and at $t = T/2$.

(e) PREDICT **[2]**

At $t = T/2$ the energy of the electron is measured and the result is E_2. State the probability of this result, and predict $\langle x\rangle$ at all later times.

12 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIV-01.06 · UPIV-01.08 **[10]**

A vacuum photocell has a cathode whose surface work function is quoted by its manufacturer as 2.00 eV, and an anode made of a different metal. Monochromatic light of each wavelength in the table falls on the cathode. An external supply applies a retarding potential difference between anode and cathode, which is increased until the photocurrent just falls to zero; that reading is the stopping potential V_s. Each V_s reading has an uncertainty of ±0.02 V, and the wavelengths may be treated as exact. The frequencies were calculated with $c = 3.00 \times 10^{8}$ m s^{-1}. Take $e = 1.60 \times 10^{-19}$ C.

Stopping potential against wavelength and frequency for the photocell

Setting	Wavelength λ / nm	Frequency f / 10^{14} Hz	Stopping potential V_s / V
1	300	10.00	1.83
2	340	8.82	1.36
3	380	7.89	0.95
4	420	7.14	0.66
5	460	6.52	0.39
6	500	6.00	0.19

(a) STATE **[2]**

State the relation between V_s and f predicted by Einstein's photon model, and what the gradient and the vertical intercept of a graph of V_s against f represent.

(b) DETERMINE **[4]**

Using a straight-line fit of V_s against f, determine the Planck constant h and the work function φ indicated by the data.

(c) ESTIMATE **[2]**

Assuming the ±0.02 V reading uncertainty is the only significant source of error, estimate the uncertainty in your value of h.

(d) JUSTIFY **[2]**

A student concludes that the cathode's true work function is the value found in (b), and that the manufacturer's 2.00 eV is wrong. Justify the alternative conclusion that the intercept measures the work function of the anode, not the cathode.

13 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION, THEN TEST THE MODEL ON A MOLECULE · DISCRIMINATING · UPIV-05.08 · UPIV-08.04 · UPIV-08.05 · UPIV-08.06 · UPIV-08.07 · UPIV-08.09 **[14]**

A particle of mass m moves in the potential $V(x) = \tfrac{1}{2}m\omega^2x^2$, so $\hat{H} = \hat{p}^2/(2m) + \tfrac{1}{2}m\omega^2\hat{x}^2$, with $[\hat{x}, \hat{p}] = i\hbar$. Define $\hat{a} = \sqrt{(m\omega/(2\hbar))}\,(\hat{x} + i\hat{p}/(m\omega))$ and $\hat{a}\dagger = \sqrt{(m\omega/(2\hbar))}\,(\hat{x} - i\hat{p}/(m\omega))$. In parts (d) and (e) the stretching vibration of a nitric oxide (NO) molecule is modelled as such an oscillator, with reduced mass $\mu = 1.24 \times 10^{-26}$ kg, vibrational frequency $f = 5.64 \times 10^{13}$ Hz and equilibrium bond length 115 pm. Take $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.63 \times 10^{-34}$ J s and $k_B = 1.38 \times 10^{-23}$ J K^{-1}. You may use $\int \exp(-\alpha x^2)\,dx = \sqrt{(\pi/\alpha)}$ and $\int x^2 \exp(-\alpha x^2)\,dx = \sqrt{\pi}/(2\alpha\hat{}(3/2))$, both taken over all x.

(a) SHOW THAT **[3]**

Show that $[\hat{a}, \hat{a}\dagger] = 1$ and that $\hat{H} = \hbar\omega(\hat{a}\dagger\hat{a} + \tfrac{1}{2})$.

(b) DERIVE **[3]**

Using only the two results of (a), derive the energy spectrum $E_n = (n + \tfrac{1}{2})\hbar\omega$, n = 0, 1, 2, …

(c) DERIVE **[3]**

Derive the normalised ground-state wavefunction $\psi_0(x)$ from the condition $\hat{a}|0\rangle = 0$.

(d) CALCULATE **[3]**

For the NO ground state, calculate σ_x and σ_p, and compare their product with the lower bound set by the commutator $[\hat{x}, \hat{p}]$.

(e) EVALUATE **[2]**

Evaluate whether the harmonic model can be trusted for (i) the fraction of NO molecules in the first excited vibrational level at 300 K, and (ii) the energies of vibrational levels close to dissociation.

MARK SCHEME

University Physics IV · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-01.03 · UPIV-01.04 **[1]**

Cavity radiation at temperature 300 K is examined at wavelength 1.00 mm. By what factor does the Rayleigh–Jeans spectral energy density exceed the Planck spectral energy density at this wavelength? Take h = 6.63 × 10⁻³⁴ J s, c = 3.00 × 10⁸ m s⁻¹ and k_B = 1.38 × 10⁻²³ J K⁻¹.

KEY **B** **1.02**

The number that decides the regime is x = hν/k_BT = hc/(λk_BT) = 0.0480. Both laws use the same mode density, 8πν²/c³; they differ only in the mean energy per mode, k_BT against hν/(eˣ − 1), so the ratio is (eˣ − 1)/x = 1.024. A is tempting because x ≪ 1 is the classical regime, but the Rayleigh–Jeans law is the limit Planck's law approaches, not one it reaches: the leading correction is x/2, about 2.4%. C is 1 + x = 1.048, the result of expanding eˣ − 1 as x + x² instead of x + x²/2. D is 1/x = 20.8, which compares k_BT with a single quantum hν, as if every mode held exactly one quantum rather than a thermal average.

U3

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-03.06 **[1]**

A proton moves at a speed for which γ = 2.00. A force of magnitude F∥ applied parallel to its velocity gives it an acceleration of magnitude a. A force of magnitude F⊥ applied perpendicular to its velocity gives it an acceleration of the same magnitude a. What is F∥/F⊥ at that instant?

KEY **C** **4.00**

Differentiating p = γmv, a force along the motion changes the speed and therefore γ as well, giving F∥ = γ³ma; a force across the motion changes only the direction, so γ stays fixed and F⊥ = γma. The ratio is γ² = 4.00. B is the Newtonian answer, and also the answer of anyone who treats γm as a single relativistic mass with the same inertia in every direction, which is exactly why that idea was abandoned. D is γ³, from using the parallel result correctly but forgetting that the perpendicular force also carries a factor of γ. A inverts the ratio: it is chosen by a candidate who remembers that the two directions differ by γ² but assumes that deflecting a fast particle is the harder job.

U2

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-04.07 **[1]**

A wave packet for a free, non-relativistic electron is built from de Broglie waves with angular frequency ω(k) = E/ħ, where E = ħ²k²/(2m) + V₀ and V₀ is a constant potential energy whose value depends on where the zero of energy is chosen. The packet is centred on wavenumber k₀, and v = ħk₀/m is the electron's classical speed. Which statement is correct?

KEY **D** **The group velocity is v for every choice of V₀, but the phase velocity depends on V₀.**

The group velocity is dω/dk = ħk₀/m = v, and the constant V₀/ħ vanishes on differentiation. The phase velocity is ω/k = ħk₀/(2m) + V₀/(ħk₀), which depends on the arbitrary energy zero, so it cannot be a physical property of the electron; only the group velocity, the speed of the packet, can be measured. A is the tempting half-truth: v/2 is the phase velocity only in the convention V₀ = 0, and the words 'whatever value of V₀' make it false. B swaps the two, usually because the candidate assumes the wave crests must travel with the particle. C is tempting because both velocities are built from the same ω(k), so changing ω seems bound to change both; but moving the zero of energy shifts ω by a constant, which changes ω/k and leaves dω/dk untouched.

U1

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04 **[1]**

Two identical narrow wave packets, ψ_L centred at x = −d and ψ_R centred at x = +d, do not overlap anywhere. Many copies of a particle are prepared either (I) in the superposition (ψ_L + ψ_R)/√2, or (II) as a statistical mixture in which half the copies are in ψ_L and half are in ψ_R. Which measurement, repeated on many copies, distinguishes preparation I from preparation II?

KEY **A** **Momentum: for I the momentum distribution is modulated by fringes of spacing h/(2d), and for II it is not.**

*In momentum space the displaced packets carry phase factors e^(±ipd/ħ), so the superposition has amplitude √2 φ₀(p) cos(pd/ħ) and a distribution proportional to cos²(pd/ħ), with fringes of period πħ/d = h/(2d); the mixture adds the two separate distributions, which are identical and carry no fringes. B is the double-slit picture misapplied: the interference term in position is ψ_L*ψ_R, which is zero everywhere when the packets do not overlap, so the two position distributions are identical. C is tempting because a mixture sounds less even-handed than a superposition, but both preparations put probability ½ near each centre, so both give ⟨x⟩ = 0. D mistakes equal position probabilities for equal states: amplitudes add before they are squared, and the relative phase that a mixture lacks shows up as soon as the measurement is in a representation where the two amplitudes overlap.*

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-06.08 **[1]**

Ehrenfest's theorem gives d⟨x⟩/dt = ⟨p⟩/m and d⟨p⟩/dt = −⟨∂V/∂x⟩. For which potential does ⟨x⟩ obey exactly the classical equation m d²⟨x⟩/dt² = −V′(⟨x⟩) for every state of the particle?

KEY **D** **V(x) = ½mω²x² + Fx, with F a constant.**

The mean follows Newton's law only if ⟨V′(x)⟩ = V′(⟨x⟩), and that holds for every state only when V′ is linear in x, which means V is at most quadratic. D qualifies because V′ = mω²x + F. A fails because V′ = 4αx³ and ⟨x³⟩ is not in general equal to ⟨x⟩³: for a packet of spread σ distributed symmetrically about its mean, ⟨x³⟩ = ⟨x⟩³ + 3⟨x⟩σ², so a displaced packet does not follow the classical equation. The symmetry of the well is what makes A tempting, and it is irrelevant. B describes an approximation, not an identity: it holds only while the packet stays narrow compared with the distance over which V′ changes, and a packet that starts narrow spreads. C fails for the same reason as A, since the cosine is non-linear, and a bounded force does nothing to make ⟨V′(x)⟩ equal V′(⟨x⟩).

U1

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-07.07 **[1]**

Electrons of kinetic energy 8.00 eV travel in the +x direction towards a potential step at which the potential energy rises abruptly from 0 to 6.00 eV. What fraction of the incident probability current is reflected?

KEY **B** **0.111**

Above the step the wavenumber is $k_2 = k_1\sqrt{(E - V_0)/E} = k_1\sqrt{(2.00/8.00)} = k_1/2$. Matching ψ and ψ′ at the step gives a reflected amplitude $(k_1 - k_2)/(k_1 + k_2) = 1/3$, and since current goes as the modulus squared, R = 1/9 = 0.111 and T = 0.889. A is the classical expectation, because $E > V_0$ means every particle could pass; the continuity of ψ′ at an abrupt step forces a reflected wave anyway. C is the reflected amplitude left unsquared. D is $((E - (E - V_0))/(E + (E - V_0)))^2 = (6/10)^2$, the result of putting kinetic energies where the wavenumbers belong, when each wavenumber goes as the square root of its kinetic energy.

U3

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.05 · UPIV-08.07 **[1]**

For a particle of mass m in a harmonic oscillator of angular frequency ω, $\hat{x} = \sqrt{(\hbar/(2m\omega))}\,(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$. What is $\langle x^2\rangle$ in the energy eigenstate $|n\rangle$?

KEY **A** **$(n + \tfrac{1}{2})\hbar/(m\omega)$**

Expanding, $\hat{x}^2 = (\hbar/(2m\omega))(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$. In $|n\rangle$ the first two terms have zero diagonal elements, and $\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}$ gives $(n + 1) + n = 2n + 1$, so $\langle x^2\rangle = (n + \tfrac{1}{2})\hbar/(m\omega)$. As a check, $\tfrac{1}{2}m\omega^2\langle x^2\rangle = \tfrac{1}{2}E_n$, the virial split. C drops the ½ from the prefactor, which would make the potential energy alone equal to the whole of E_n. B treats $\hat{a}\hat{a}^\dagger$ as if it were $\hat{a}^\dagger\hat{a}$, forgetting $[\hat{a}, \hat{a}^\dagger] = 1$; it would give zero spread in the ground state, which the uncertainty principle forbids. D confuses $\langle x\rangle$, which is zero by parity, with $\langle x^2\rangle$, which is the spread and is never zero.

U2

8 CORE COURSE

STRUCTURED · DEMANDING · UPIV-02.05 · UPIV-02.06 · UPIV-02.07 · UPIV-02.08 **[10]**

An inspection gantry at rest in frame S has two laser gates: G_1 at $x = -60.0$ m and G_2 at $x = +60.0$ m, so the gantry's rest length is 120 m. A cargo shuttle of rest length 150 m moves in the +x direction at 0.800c. Frame S′ is the shuttle's rest frame, in standard configuration with S: the origins coincide at $t = t' = 0$, when the centre of the shuttle passes $x = 0$. Each gate sits at the edge of the track and can fire a brief pulse straight across it, taking negligible time to cross; a pulse that meets the shuttle leaves a mark on its hull. The gantry controller fires both gates at $t = 0$ in S. Take $c = 3.00 \times 10^8$ m s^{-1}.

(a) CALCULATE · U3 **[2]**

Calculate the length of the shuttle measured in S, and the length of the gantry measured in S′.

1. $\gamma = 1/\sqrt{(1 - 0.800^2)} = 5/3$, so the shuttle's length in S is $150 \text{ m}/\gamma = 90.0$ m; at $t = 0$ it spans $x = -45.0$ m to $+45.0$ m, clear of both gates
2. The gantry's length in S′ is $120 \text{ m}/\gamma = 72.0$ m

Each frame measures the other's object as the shorter one. A candidate who divides in one case and multiplies in the other has not asked which frame owns each proper length.

(b) CALCULATE · U3 **[3]**

The firing of G_1 is event E_1 and the firing of G_2 is event E_2. Calculate the coordinates x' and t' of both events in S′, and state which gate fires first in S′ and by how much.

1. Uses $x' = \gamma(x - vt)$ and $t' = \gamma(t - vx/c^2)$ with $t = 0$ for both events
2. E_1: $x' = -100$ m, $t' = +\gamma(0.800)(60.0 \text{ m})/c = +2.67 \times 10^{-7}$ s; E_2: $x' = +100$ m, $t' = -2.67 \times 10^{-7}$ s
3. G_2, the gate the shuttle is heading towards, fires first in S′, 5.33×10^{-7} s before G_1

The sign of vx/c^2 is the whole question. A candidate who drops it, or who applies the time-dilation formula to two events that are not at the same place in either frame, gets the order wrong.

(c) EXPLAIN · U1 **[3]**

In S′ the shuttle is 150 m long and the gantry is only 72.0 m long. Explain, using your results from (b), why neither pulse strikes the shuttle in S′ either.

1. In S′ the shuttle occupies $-75.0 \text{ m} \le x' \le +75.0$ m at all times; E_2 happens at $x' = +100$ m, 25.0 m ahead of the nose, and E_1 at $x' = -100$ m, 25.0 m behind the tail, so neither event is on the hull
2. Between the two firings the gantry moves $(0.800c)(5.33 \times 10^{-7} \text{ s}) = 128$ m in the $-x'$ direction, so G_2 fires before it reaches the nose and G_1 fires after it has passed the tail
3. Whether a pulse meets the hull is a coincidence of events at one place, on which every frame agrees; the statement that the shuttle is between the gates when they fire depends on E_1 and E_2 being simultaneous, which is true in S and false in S′, so there is no contradiction

Answers that stop at saying that length contraction is relative earn nothing here. The marks are for placing both events relative to the hull in S′, for following the gantry's motion between the two firings, and for naming simultaneity as the one statement the frames do not share.

(d) DETERMINE · U2 **[2]**

A drone is launched forwards from the shuttle's nose at 0.600c relative to the shuttle. By composing rapidities, determine the drone's speed in S.

1. Rapidities $\varphi = \operatorname{artanh}\beta$ add along a line: $\operatorname{artanh} 0.800 = \ln 3 = 1.099$ and $\operatorname{artanh} 0.600 = \ln 2 = 0.693$, so $\varphi = \ln 6 = 1.792$
2. $\beta = \tanh(1.792) = 35/37 = 0.946$, so the speed is $0.946c = 2.84 \times 10^8$ m s^{-1}, in agreement with $(0.800 + 0.600)/(1 + 0.480)$ and below c

Rapidity is the parameter that adds; velocity is not. A Galilean 1.40c, or $\tanh(1.40) = 0.885$ from treating β itself as the rapidity, both show the difference has not been understood.

9 CORE COURSE

STRUCTURED · DEMANDING · UPIV-03.01 · UPIV-03.08 · UPIV-03.09 **[10]**

A secondary beam of positive kaons, each with momentum 2.00 GeV/c, travels along an evacuated decay tunnel. A K^+ at rest has mean lifetime $\tau = 1.238 \times 10^{-8}$ s and decays mainly by $K^+ \to \mu^+ + \nu_\mu$. Take $m_K c^2 = 493.7$ MeV, $m_\mu c^2 = 105.7$ MeV, $m_p c^2 = 938.3$ MeV, $m_\Lambda c^2 = 1115.7$ MeV and $c = 3.00 \times 10^8$ m s^{-1}, and treat the neutrino as massless.

(a) CALCULATE · U3 **[2]**

Calculate the mean decay length of the kaons in the laboratory.

1. $\beta\gamma = pc/(m_K c^2) = 2000$ MeV/493.7 MeV $= 4.05$ ($\gamma = 4.17$, $\beta = 0.971$)
2. Mean decay length $= \beta\gamma c\tau = 4.05 \times (3.00 \times 10^8 \text{ m s}^{-1})(1.238 \times 10^{-8} \text{ s}) = 15.0$ m

   τ is the proper lifetime, owned by the kaon's rest frame. $\beta c\tau = 3.61$ m ignores time dilation, and $\gamma c\tau = 15.5$ m uses c where the kaon's speed belongs.

(b) DERIVE · U2 **[3]**

Starting from conservation of four-momentum, derive expressions for the muon's energy E^* and momentum p^* in the kaon rest frame, and evaluate both.

1. Writes $P_K = p_\mu + p_\nu$ and isolates the neutrino: $p_\nu^2 = (P_K - p_\mu)^2 = 0$ because the neutrino is massless
2. Expands in the kaon rest frame: $m_K^2c^4 + m_\mu^2c^4 - 2m_K c^2 E^* = 0$, so $E^* = (m_K^2 + m_\mu^2)c^2/(2m_K) = 258.2$ MeV
3. $p^*c = \sqrt{(E^{*2} - m_\mu^2c^4)} = (m_K^2 - m_\mu^2)c^2/(2m_K) = 235.5$ MeV, so $p^* = 235.5$ MeV/c, equal to the neutrino's energy divided by c

   Squaring the four-vector that contains the unknown you do not want removes it in one line. Candidates who conserve energy and momentum separately usually lose the algebra, or assume the muon is non-relativistic when its kinetic energy is 152.5 MeV.

(c) DETERMINE · U3 **[2]**

Determine the largest energy a muon from this beam can have in the laboratory, and state the direction of emission in the kaon rest frame that gives it.

1. Boosting, $E = \gamma(E^* + \beta p^* c \cos\theta^*)$, which is greatest for emission along the beam direction, $\theta^* = 0$
2. $E_{max} = 4.173 \times (258.2 \text{ MeV} + 0.9709 \times 235.5 \text{ MeV}) = 2031 \text{ MeV} \approx 2.03$ GeV (accept 2.02 to 2.04 GeV), just below the kaon's energy of 2060 MeV

   Adding the rest-frame energy to the kaon's energy, or multiplying E^* by γ alone, ignores the momentum term in the boost. A result above 2060 MeV should have been caught as impossible.

(d) CALCULATE · U3 **[3]**

The kaons are produced when a proton beam strikes protons at rest, through $p + p \to p + \Lambda + K^+$. Calculate the minimum kinetic energy of a beam proton for this reaction.

1. At threshold the products are at rest in the centre-of-momentum frame, so the invariant $(E_{beam} + m_p c^2)^2 - (p_{beam} c)^2$ equals $(m_p + m_\Lambda + m_K)^2c^4 = (2547.7 \text{ MeV})^2$
2. The invariant expands to $2m_p^2c^4 + 2E_{beam} m_p c^2$, so $E_{beam} = [(2547.7)^2 - 2(938.3)^2]/(2 \times 938.3)$ MeV $= 2520$ MeV
3. $K_{th} = E_{beam} - m_p c^2 = 1582$ MeV ≈ 1.58 GeV, far more than the 671 MeV of rest energy created, because momentum conservation requires the products to move in the laboratory

   The commonest wrong answer is 671 MeV, the change in rest energy, which would be correct only if the target were infinitely massive. The invariant reaches the threshold without first having to work out how fast the products move in the laboratory.

10 CORE COURSE

STRUCTURED · DEMANDING · UPIV-01.07 · UPIV-04.03 · UPIV-04.04 · UPIV-04.05 **[11]**

Technetium-99m, used in medical gamma imaging, emits photons of energy 140.5 keV. Before reaching the camera, many of these photons Compton scatter in soft tissue from electrons that may be treated as free and at rest. The camera rejects scattered photons only by their energy: it accepts any photon whose energy is at least 90.0% of 140.5 keV. Take $m_e c^2 = 511$ keV, $h = 6.63 \times 10^{-34}$ J s, $c = 3.00 \times 10^{8}$ m s^{-1} and $e = 1.60 \times 10^{-19}$ C.

(a) DERIVE · U2 **[4]**

A photon of wavelength λ scatters through angle θ from a free electron at rest. By conserving relativistic energy and both components of momentum, derive $\lambda' - \lambda = (h/(m_e c))(1 - \cos\theta)$ for the scattered wavelength λ'.

1. Energy: $pc + m_e c^2 = p'c + E_e$, with $p = h/\lambda$ and $p' = h/\lambda'$ for the incident and scattered photons
2. Momentum as a vector: $p_e = p - p'$, so $p_e^2 = p^2 + p'^2 - 2pp' \cos\theta$, which uses both components
3. Eliminates the electron with $E_e^2 = p_e^2c^2 + m_e^2c^4$: $(pc - p'c + m_e c^2)^2 = (p^2 + p'^2 - 2pp' \cos\theta)c^2 + m_e^2c^4$, which reduces to $m_e c(p - p') = pp'(1 - \cos\theta)$
4. Divides by $m_e c\, pp'$ to get $1/p' - 1/p = (1 - \cos\theta)/(m_e c)$, and substitutes $1/p = \lambda/h$ to reach $\lambda' - \lambda = (h/(m_e c))(1 - \cos\theta)$

   Using the non-relativistic electron kinetic energy $p_e^2/(2m_e)$ gives an approximate result, not this one, and conserving only the momentum component along the beam loses the $\cos\theta$ term. The second line removes the electron's unknown recoil angle by squaring the momentum vector; the mark for the third line is for eliminating the electron's energy and momentum through $E_e^2 = p_e^2c^2 + m_e^2c^4$ and reaching the reduced form, not for asserting the result.

(b) CALCULATE · U3 **[3]**

Calculate the largest angle through which a photon can be scattered once and still be accepted by the camera.

1. In energy form $1/E' - 1/E = (1 - \cos\theta)/(m_e c^2)$, with $E' = 0.900 \times 140.5$ keV $= 126.45$ keV
2. $1 - \cos\theta = 511 \text{ keV} \times (1/126.45 \text{ keV} - 1/140.5 \text{ keV}) = 0.404$
3. $\cos\theta = 0.596$, so $\theta_{max} = 53.4°$

   Subtracting energies as if the shift were linear in energy gives the wrong angle. The relation is linear in wavelength, equivalently in 1/E.

(c) CALCULATE · U3 **[2]**

Calculate the de Broglie wavelength of the recoil electron for the scattering in (b).

1. $K = 140.5 \text{ keV} - 126.45 \text{ keV} = 14.05$ keV, and $pc = \sqrt{(K^2 + 2K m_e c^2)} = 120.7$ keV
2. $\lambda_e = hc/(pc) = (6.63 \times 10^{-34} \text{ J s} \times 3.00 \times 10^{8} \text{ m s}^{-1})/(120.7 \times 10^{3} \times 1.60 \times 10^{-19} \text{ J}) = 1.03 \times 10^{-11}$ m $= 10.3$ pm

   Writing $\lambda = hc/K$ treats the electron as a photon and gives 88 pm. The non-relativistic momentum $\sqrt{(2m_eK)}$ gives 10.4 pm, under 1% away, which is itself worth noticing at 14 keV.

(d) COMPARE · U1 **[2]**

A PET camera detects 511 keV annihilation photons and uses the same fractional window, accepting photons of at least 90.0% of 511 keV. Compare its largest accepted single-scattering angle with your answer to (b), and explain the difference.

1. For 511 keV, $1 - \cos\theta = (511 \text{ keV}/511 \text{ keV})(1/0.900 - 1) = 0.111$, giving $\theta_{max} = 27.3°$, about half the 53.4° accepted at 140.5 keV
2. In general $1 - \cos\theta_{max} = (m_e c^2/E)(1/0.900 - 1)$: at a given angle a photon with E well below $m_e c^2$ loses a smaller fraction of its energy, so it must scatter through a larger angle before falling below 90.0%, and the same fractional window accepts a much wider cone at 140.5 keV than at 511 keV

   Candidates often expect the same fractional window to accept the same cone of scattering angles at every photon energy. The comparison turns on the ratio $E/m_e c^2$, which sets the fraction of energy lost at a given angle.

11 CORE COURSE

STRUCTURED · DEMANDING · UPIV-05.06 · UPIV-06.03 · UPIV-06.04 · UPIV-07.02 · UPIV-07.03 **[11]**

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.067m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, between $x = 0$ and $x = L$. The energy eigenfunctions are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ the electron is prepared in the state $\Psi(x, 0) = A[\sqrt{3}\,\psi_1(x) + \psi_2(x)]$, with A real and positive. You may use $\int_0^L x\sin(\pi x/L)\sin(2\pi x/L)\,dx = -8L^2/(9\pi^2)$. Take $m_e = 9.11 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $h = 6.63 \times 10^{-34}$ J s.

(a) DETERMINE · U2 **[2]**

Determine A, and the probabilities that a measurement of the energy at $t = 0$ returns E_1 and returns E_2.

1. Orthonormality of ψ_1 and ψ_2 gives $|A|^2(3 + 1) = 1$, so $A = 1/2$
2. $P(E_1) = |A\sqrt{3}|^2 = 3/4$ and $P(E_2) = |A|^2 = 1/4$

Normalising by integrating the square of the sum term by term, without using orthogonality, wastes time; forgetting that the cross terms vanish gives the wrong A.

(b) CALCULATE · U3 **[2]**

The mean position $\langle x\rangle$ of the electron oscillates. Calculate the period T of this oscillation.

1. $E_1 = \pi^2\hbar^2/(2m^*L^2) = 6.25 \times 10^{-21}$ J (39.1 meV), so $E_2 - E_1 = 3E_1 = 1.87 \times 10^{-20}$ J
2. $T = 2\pi\hbar/(E_2 - E_1) = h/(E_2 - E_1) = 3.54 \times 10^{-14}$ s

The beat is set by the energy difference, not by E_1 or E_2 alone. Using the free-electron mass instead of m^* gives a period 15 times too long.

(c) SHOW THAT · U2 **[3]**

Show that $\langle x\rangle(t) = L/2 - (8\sqrt{3}L/(9\pi^2))\cos\omega t$, where $\omega = (E_2 - E_1)/\hbar$.

1. $\Psi(x, t) = \tfrac{1}{2}[\sqrt{3}\,\psi_1\,e^{(-iE_1t/\hbar)} + \psi_2\,e^{(-iE_2t/\hbar)}]$, with each stationary state carrying its own phase
2. $\langle x\rangle = \tfrac{3}{4}\langle\psi_1|x|\psi_1\rangle + \tfrac{1}{4}\langle\psi_2|x|\psi_2\rangle + (\sqrt{3}/2)\langle\psi_1|x|\psi_2\rangle\cos\omega t$, and both diagonal terms equal $L/2$ because each $|\psi_n|^2$ is symmetric about the centre of the well
3. $\langle\psi_1|x|\psi_2\rangle = (2/L)(-8L^2/(9\pi^2)) = -16L/(9\pi^2)$, so $\langle x\rangle = L/2 - (8\sqrt{3}L/(9\pi^2))\cos\omega t$, an oscillation of amplitude $0.156L = 1.87$ nm

Candidates who give both components the same phase find $\langle x\rangle$ constant and then force the stated answer. Any response that does not show the two cross terms combining into a cosine earns at most the first mark.

(d) SKETCH · U1 **[2]**

On one set of axes, sketch $|\Psi(x, t)|^2$ for $0 \le x \le L$ at $t = 0$ and at $t = T/2$.

1. $t = 0$: zero at both walls, with a single main peak left of centre near $x \approx 0.32L$, falling to zero at $x = 5L/6$ and staying very small between there and $x = L$
2. $t = T/2$: the mirror image of the $t = 0$ curve about $x = L/2$, with its main peak near $x \approx 0.68L$ and a zero at $x = L/6$; the two curves enclose equal areas

The $t = 0$ density must lean towards the side where ψ_1 and ψ_2 have the same sign, which is consistent with $\langle x\rangle = 0.344L$ from (c). A symmetric sketch at either time shows that the relative phase has been ignored.

(e) PREDICT · U1 **[2]**

At $t = T/2$ the energy of the electron is measured and the result is E_2. State the probability of this result, and predict $\langle x\rangle$ at all later times.

1. The probability is 1/4, the same at every time, since the modulus of each coefficient $c_n\,e^{(-iE_nt/\hbar)}$ does not change
2. The state is updated to ψ_2 (up to a phase), which is stationary, so there is no cross term and $\langle x\rangle = L/2$ at all later times: the oscillation stops

Many candidates expect the probability to oscillate along with $\langle x\rangle$. Only quantities with a cross term between different energies can change in time.

12 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIV-01.06 · UPIV-01.08 **[10]**

A vacuum photocell has a cathode whose surface work function is quoted by its manufacturer as 2.00 eV, and an anode made of a different metal. Monochromatic light of each wavelength in the table falls on the cathode. An external supply applies a retarding potential difference between anode and cathode, which is increased until the photocurrent just falls to zero; that reading is the stopping potential V_s. Each V_s reading has an uncertainty of ±0.02 V, and the wavelengths may be treated as exact. The frequencies were calculated with $c = 3.00 \times 10^{8}$ m s^{-1}. Take $e = 1.60 \times 10^{-19}$ C.

(a) STATE · U1 **[2]**

State the relation between V_s and f predicted by Einstein's photon model, and what the gradient and the vertical intercept of a graph of V_s against f represent.

1. $eV_s = hf - \varphi$, so $V_s = (h/e)f - \varphi/e$
2. The gradient is h/e, and the intercept on the V_s axis is $-\varphi/e$ (equivalently, the intercept on the f axis is the threshold frequency φ/h)

   Writing the maximum kinetic energy as ½mv² and never connecting it to eV_s leaves no straight line to fit.

(b) DETERMINE · U4 **[4]**

Using a straight-line fit of V_s against f, determine the Planck constant h and the work function φ indicated by the data.

1. Plots or fits V_s against the tabulated f; all six points lie on one straight line within ±0.02 V
2. Gradient $= 4.12 \times 10^{-15}$ V s (accept 4.05×10^{-15} to 4.20×10^{-15} V s)
3. $h = e \times$ gradient $= 6.6 \times 10^{-34}$ J s (accept 6.5×10^{-34} to 6.7×10^{-34} J s)
4. Intercept on the V_s axis = −2.29 V (accept −2.19 V to −2.35 V), so $\varphi = 2.29$ eV; or threshold frequency 5.55×10^{14} Hz with $\varphi = hf_0 = 2.29$ eV

   Reading the intercept requires extrapolating well beyond the data, since the lowest frequency is 6.00×10^{14} Hz. A candidate who plots V_s against λ has no straight line and so no gradient to convert into h; one who fits correctly but reads the work function off the last data point instead of extrapolating loses the final mark.

(c) ESTIMATE · U4 **[2]**

Assuming the ±0.02 V reading uncertainty is the only significant source of error, estimate the uncertainty in your value of h.

1. Steepest and shallowest lines through the ends of the ±0.02 V error bars at 300 nm and 500 nm have gradients 4.20×10^{-15} V s and 4.00×10^{-15} V s
2. $\Delta h \approx \tfrac{1}{2}(6.72 - 6.40) \times 10^{-34}$ J s $= 0.16 \times 10^{-34}$ J s, so $h = (6.6 \pm 0.2) \times 10^{-34}$ J s, a 2 to 3% uncertainty (a justified least-squares standard error is also accepted: about 0.10×10^{-34} J s from σ = 0.02 V on each reading, or about 0.05×10^{-34} J s from the scatter of the residuals)

   An uncertainty quoted to more decimal places than the value, or an uncertainty in h of 0.02×10^{-34} J s obtained by copying the ±0.02 V, shows that the uncertainty has not been carried through the gradient.

(d) JUSTIFY · U4 **[2]**

A student concludes that the cathode's true work function is the value found in (b), and that the manufacturer's 2.00 eV is wrong. Justify the alternative conclusion that the intercept measures the work function of the anode, not the cathode.

1. An electron leaves the cathode with at most $hf - \varphi_C$, but between two different metals there is also a contact potential difference $(\varphi_A - \varphi_C)/e$, so the external stopping potential satisfies $eV_s = (hf - \varphi_C) - (\varphi_A - \varphi_C) = hf - \varphi_A$
2. The intercept range 2.19 to 2.35 eV excludes 2.00 eV, so the 0.3 eV difference is significant and is accounted for as the contact potential; this does not depend on f, so the gradient and h are unaffected and the manufacturer's value can stand

   Blaming the discrepancy on random error ignores the steepest and shallowest lines of (c), which put the intercept's own uncertainty at only about ±0.08 V, far smaller than the 0.3 V gap. The point being tested is the difference between what the voltmeter reads and what the cathode does.

13

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION, THEN TEST THE MODEL ON A MOLECULE · DISCRIMINATING · UPIV-05.08 · UPIV-08.04 · UPIV-08.05 · UPIV-08.06 · UPIV-08.07 · UPIV-08.09 **[14]**

A particle of mass m moves in the potential $V(x) = \tfrac{1}{2}m\omega^2x^2$, so $\hat{H} = \hat{p}^2/(2m) + \tfrac{1}{2}m\omega^2\hat{x}^2$, with $[\hat{x}, \hat{p}] = i\hbar$. Define $\hat{a} = \sqrt{(m\omega/(2\hbar))}\,(\hat{x} + i\hat{p}/(m\omega))$ and $\hat{a}^\dagger = \sqrt{(m\omega/(2\hbar))}\,(\hat{x} - i\hat{p}/(m\omega))$. In parts (d) and (e) the stretching vibration of a nitric oxide (NO) molecule is modelled as such an oscillator, with reduced mass $\mu = 1.24 \times 10^{-26}$ kg, vibrational frequency $f = 5.64 \times 10^{13}$ Hz and equilibrium bond length 115 pm. Take $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.63 \times 10^{-34}$ J s and $k_B = 1.38 \times 10^{-23}$ J K^{-1}. You may use $\int \exp(-\alpha x^2)\,dx = \sqrt{(\pi/\alpha)}$ and $\int x^2 \exp(-\alpha x^2)\,dx = \sqrt{\pi}/(2\alpha\text{^}(3/2))$, both taken over all x.

(a) SHOW THAT · U2 **[3]**

Show that $[\hat{a}, \hat{a}^\dagger] = 1$ and that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$.

1. $\hat{a}\hat{a}^\dagger = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - (i/(2\hbar))[\hat{x}, \hat{p}] = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] + \tfrac{1}{2}$
2. Similarly $\hat{a}^\dagger\hat{a} = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - \tfrac{1}{2}$, so $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
3. $(m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] = \hat{H}/(\hbar\omega)$, so $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$

   Treating $\hat{x}$ and $\hat{p}$ as numbers makes the cross terms cancel and loses the ½, which is the zero-point energy. The commutator must appear explicitly in the working.

(b) DERIVE · U2 **[3]**

Using only the two results of (a), derive the energy spectrum $E_n = (n + \tfrac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$

1. $[\hat{H}, \hat{a}] = \hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}] = \hbar\omega[\hat{a}^\dagger, \hat{a}]\hat{a} = -\hbar\omega\hat{a}$, so if $\hat{H}|E\rangle = E|E\rangle$ then $\hat{H}(\hat{a}|E\rangle) = (E - \hbar\omega)\hat{a}|E\rangle$: $\hat{a}$ lowers the energy by $\hbar\omega$, and likewise $\hat{a}^\dagger$ raises it by $\hbar\omega$
2. $E = \langle E|\hat{H}|E\rangle = \hbar\omega(\|\hat{a}|E\rangle\|^2 + \tfrac{1}{2}) \geq \tfrac{1}{2}\hbar\omega$, so repeated lowering cannot continue and must end on a state $|0\rangle$ with $\hat{a}|0\rangle = 0$
3. Then $\hat{H}|0\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})|0\rangle = \tfrac{1}{2}\hbar\omega|0\rangle$; every eigenvalue lies a whole number of steps $\hbar\omega$ above this, and raising $|0\rangle$ n times reaches each one, so $E_n = (n + \tfrac{1}{2})\hbar\omega$

   The step that separates answers is the second: without the norm argument there is no reason for the ladder to stop, and asserting $\hat{a}|0\rangle = 0$ earns nothing on its own.

(c) DERIVE · U2 **[3]**

Derive the normalised ground-state wavefunction $\psi_0(x)$ from the condition $\hat{a}|0\rangle = 0$.

1. In the position representation $\hat{p} = -i\hbar\,d/dx$, so $\hat{a}|0\rangle = 0$ becomes $(x + (\hbar/(m\omega))\,d/dx)\psi_0 = 0$
2. Separating, $d\psi_0/\psi_0 = -(m\omega/\hbar)x\,dx$, so $\psi_0 = N\exp(-m\omega x^2/(2\hbar))$
3. $\int|\psi_0|^2\,dx = N^2\sqrt{(\pi\hbar/(m\omega))} = 1$, so $N = (m\omega/(\pi\hbar))\text{^}(1/4)$

   A sign error in $i\hat{p}/(m\omega)$ gives $\exp(+m\omega x^2/(2\hbar))$, which cannot be normalised. A candidate who checks normalisability catches it.

(d) CALCULATE · U3 **[3]**

For the NO ground state, calculate σ_x and σ_p, and compare their product with the lower bound set by the commutator $[\hat{x}, \hat{p}]$.

1. $|\psi_0|^2 \propto \exp(-\mu\omega x^2/\hbar)$ is a Gaussian of variance $\hbar/(2\mu\omega)$; with $\omega = 2\pi f = 3.54 \times 10^{14}$ rad s^{-1}, $\sigma_x = \sqrt{(\hbar/(2\mu\omega))} = 3.46 \times 10^{-12}$ m = 3.46 pm, about 3% of the bond length
2. $\langle p \rangle = 0$ and $\langle p^2 \rangle/(2\mu) = \tfrac{1}{2}\langle \hat{H} \rangle = \tfrac{1}{4}\hbar\omega$ (equal kinetic and potential shares), so $\sigma_p = \sqrt{(\mu\hbar\omega/2)} = 1.52 \times 10^{-23}$ kg m s^{-1}
3. $\sigma_x\sigma_p = 5.27 \times 10^{-35}$ J s $= \hbar/2$, equal to the Robertson bound $\tfrac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle| = \hbar/2$, so the Gaussian ground state saturates it

   Reading σ_x off the exponent of ψ_0 as $\sqrt{(\hbar/(\mu\omega))} = 4.90$ pm confuses the width of ψ_0 with the width of $|\psi_0|^2$ and gives $\sigma_x\sigma_p = \hbar/\sqrt{2} = 7.46 \times 10^{-35}$ J s (or $\hbar$ if σ_p is misread in the same way), which would put the ground state above the bound it actually meets.

(e) EVALUATE · U4 **[2]**

Evaluate whether the harmonic model can be trusted for (i) the fraction of NO molecules in the first excited vibrational level at 300 K, and (ii) the energies of vibrational levels close to dissociation.

1. (i) hf/(k_BT) = $(6.63 \times 10^{-34}$ J s $\times$ 5.64×10^{13} Hz$)/(1.38 \times 10^{-23}$ J $K^{-1} \times 300$ K) = 9.03, so $N_1/N_0 = e\text{^}(-9.03) = 1.2 \times 10^{-4}$, which is also the fraction in n = 1 because almost every molecule is in n = 0; this depends only on the lowest spacing, and with σ_x only about 3% of the bond length the lowest levels sample just the bottom of the well, where the quadratic approximation holds, so the model can be trusted
2. (ii) High levels have large amplitudes that reach the anharmonic part of the potential, which flattens towards dissociation; the real levels crowd together and end at a finite dissociation energy, while the harmonic ladder stays evenly spaced for ever, so the model cannot be trusted there

   A verdict of 'good' or 'bad' with no criterion earns nothing. The criterion is the amplitude a level explores compared with the distance over which the real potential departs from a parabola, and (d) has already supplied it for n = 0.

University Physics IV · Final Examination

The whole of modern physics and introductory quantum mechanics, weighted towards the second half of the course: angular momentum and spin, the hydrogen atom, many-electron atoms, solids, nuclei, particles and the frontier, with the evidence for quanta, special relativity and the one-dimensional Schrödinger equation sampled throughout. Every result is expected to survive its classical or non-relativistic limit.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**116**	**19**	**10**

SECTION — **UPIV · Final · University Physics IV — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 1 · Routine 5 · Demanding 9 · Discriminating 4**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPIV-01 · UPIV-02 · UPIV-03 · UPIV-04 · UPIV-05 · UPIV-06 · UPIV-07 · UPIV-08 · UPIV-09 · UPIV-10 · UPIV-11 · UPIV-12 · UPIV-13 · UPIV-14 · UPIV-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Physics IV · Final Examination

Answer all questions. 116 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-01.06 · UPIV-01.08 **[1]**

A clean tungsten surface of work function $\varphi = 4.50$ eV is illuminated in vacuum by femtosecond laser pulses of wavelength 800 nm, and electrons are emitted even though each photon carries less energy than the work function. An electron spectrometer shows that the most energetic electrons leave with 0.15 eV at every pulse intensity used, and doubling the peak intensity I multiplies the number of electrons emitted per pulse by 7.9 ± 0.4. Take $hc = 1240$ eV nm. Which conclusion do these measurements justify?

A The intense pulses heat the surface and the electrons are emitted thermally, so these data say nothing about how light exchanges energy with electrons.

B Einstein's photoelectric equation is refuted, because it forbids emission whenever the photon energy is below the work function, however intense the light.

C Each emitted electron absorbs three quanta of 1.55 eV, since $3h\nu - \varphi = 0.15$ eV and an emission rate proportional to I^3 grows by $2^3 = 8$ when I doubles; energy is exchanged with the light in quanta of $h\nu$, but these data do not by themselves prove that the light field is quantised.

D Each emitted electron absorbs three quanta, which proves that the light field consists of photons, because a classical wave cannot deliver energy to an electron in whole multiples of $h\nu$.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-04.05 · UPIV-04.06 **[1]**

At a pulsed neutron source, each pulse releases neutrons with a wide spread of speeds from the moderator at the same instant. They travel a total flight path of 12.0 m, by way of a powder sample, to a detector that records neutrons scattered elastically through 90.0° by the sample. A Bragg peak reaches the detector 6.00 ms after the pulse. Treating it as a first-order reflection, what is the spacing of the lattice planes that produce it? Take $h = 6.626 \times 10^{-34}$ J s and $m_n = 1.675 \times 10^{-27}$ kg.

A 0.140 nm

B 0.0989 nm

C 0.280 nm

D 0.0223 nm

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04 **[1]**

An electron is held in a one-dimensional trap whose potential energy is not symmetric about any point. It is in the trap's ground state, described by a real, normalised wavefunction u(x) whose probability density is lopsided. A uniform force then acts for a time so short that u(x) does not change while it acts, delivering an impulse $\hbar k_0$ in the +x direction, so that immediately afterwards the wavefunction is u(x)e^(ik₀x). With φ(p) the momentum-space wavefunction, which statement is correct?

A Before the impulse ⟨p⟩ is not zero, because the lopsided density $|u(x)|^2$ makes the momentum distribution lopsided too; the impulse then adds $\hbar k_0$ to ⟨p⟩.

B The factor e^(ik₀x) has modulus 1 and leaves the position density unchanged, so no measurement made immediately afterwards, of position or of momentum, can detect the impulse.

C Before the impulse $|\varphi(p)|^2$ is symmetric in p only if u(x) is an even function; immediately afterwards the momentum distribution is shifted by $\hbar k_0$ and also broadened, because the factor e^(ik₀x) adds momenta near $\hbar k_0$ to those already present.

D Before the impulse $|\varphi(p)|^2 = |\varphi(-p)|^2$, so ⟨p⟩ = 0 even though $|u(x)|^2$ is lopsided; immediately afterwards the momentum distribution is the same curve shifted by $\hbar k_0$, while the position density is unchanged.

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.04 · UPIV-08.09 **[1]**

A single ion held in a radio-frequency trap oscillates along the trap axis as a quantum harmonic oscillator of frequency f = 4.00 MHz, with energy levels $E_n = (n + \tfrac{1}{2})hf$. After laser cooling, its axial motion is in thermal equilibrium at T = 0.600 mK. What is the probability that the axial motion is found in its ground state, n = 0? Take h = 6.626 × 10^{-34} J s and k_B = 1.381 × 10^{-23} J K^{-1}.

A 0.726

B 0.274

C 0.579

D 0.852

Answer

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-09.01 · UPIV-09.02 **[1]**

A particle is in the simultaneous eigenstate of J^2 and J_z with quantum numbers j = 3/2 and m = 3/2. What is the expectation value $\langle J_x^2 \rangle$?

A $3\hbar^2/4$

B 0

C $3\hbar^2/2$

D $15\hbar^2/4$

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-11.04 · UPIV-11.06 **[1]**

A free ion has the ground configuration [Ar]$3d^3$. Applying all three of Hund's rules, what is its ground-state term symbol?

A $^4F_{9/2}$

B $^2H_{9/2}$

C $^4P_{1/2}$

D $^4F_{3/2}$

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-12.06 **[1]**

In a one-dimensional tight-binding model an electron band has dispersion $E(k) = E_0 - 2t\cos(ka)$, with hopping energy $t = 0.50$ eV and lattice spacing $a = 0.30$ nm. What is the effective mass of an electron near the bottom of the band, at $k = 0$? Take $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C and $m_e = 9.109 \times 10^{-31}$ kg.

A $0.42m_e$

B $0.85m_e$

C $1.69m_e$

D $-0.85m_e$

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-13.03 **[1]**

In the spin-orbit shell model the proton single-particle levels fill in the order $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, each holding $2j + 1$ protons, and neutrons fill the same sequence. What spin and parity does the model predict for the ground state of ^{39}K, with $Z = 19$ and $N = 20$?

A $3/2^+$

B $1/2^+$

C $3/2^-$

D $7/2^-$

Answer ☐

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-14.01 **[1]**

A neutral meson decays in flight into two photons, which an electromagnetic calorimeter records with energies 1.50 GeV and 0.500 GeV and an angle of 36.9° between their directions of travel. Treating the photons as massless, what is the rest energy of the meson?

A 2.00 GeV

B 1.64 GeV

C 0.388 GeV

D 0.548 GeV

Answer

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10 CORE COURSE

MULTIPLE CHOICE · RECALL · UPIV-15.02 · UPIV-15.09 **[1]**

Entangled photon pairs are measured at two stations whose analyser settings are chosen randomly and independently, with the detection events at the two stations spacelike separated and the detection efficiency high enough that no fair-sampling assumption is needed. The measured CHSH combination is $S = 2.70 \pm 0.02$. Which statement correctly describes what this result establishes?

A The choice of setting at one station changes the outcomes recorded at the other, so a message could be sent faster than light.

B Every hidden-variable theory, local or non-local, is ruled out, which proves that quantum mechanics is complete.

C All models in which each outcome is fixed by variables carried locally from the source, independent of the settings, are excluded; the correlations still cannot be used to send a signal faster than light, and non-local hidden-variable models are not excluded.

D The source emitted a statistical mixture, since a pure entangled state would give S exactly equal to 2.

Answer

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11 CORE COURSE

STRUCTURED · DEMANDING · UPIV-11.03 · UPIV-11.09 **[10]**

A handheld X-ray fluorescence analyser is aimed at a corroded brass fitting. Its primary X-rays eject inner-shell electrons from atoms in the sample, and the detector records two strong lines at 8.04 keV and 8.63 keV, together with weaker lines, one of which lies at 8.90 keV. Model each K line with hydrogen-like levels and a screening constant σ: a transition from shell n to the K shell (n = 1) emits a photon of energy $E = (13.6\ \text{eV})(Z - \sigma)^2(1 - 1/n^2)$, so that the Kα line (n = 2) follows Moseley's law, $E = (3/4)(13.6\ \text{eV})(Z - \sigma)^2$.

(a) DESCRIBE **[2]**

Describe the sequence of events, beginning with the absorption of a primary X-ray photon, that leads to the emission of a Kα photon, and say why the primary photons must exceed a threshold energy.

(b) CALCULATE **[3]**

Taking σ = 1, calculate the atomic numbers of the elements that produce the 8.04 keV and 8.63 keV lines, and identify the elements.

(c) EXPLAIN **[2]**

Explain why σ ≈ 1 is a reasonable screening constant for a Kα line.

(d) EVALUATE **[3]**

A colleague suggests that the 8.90 keV line is the Kα line of a third element. Evaluate this suggestion against the alternative that it is the Kβ line (n = 3 → n = 1) of copper, and state what your conclusion implies about σ for a Kβ line.

12 CORE COURSE

STRUCTURED · DEMANDING · UPIV-06.04 · UPIV-07.02 · UPIV-07.03 · UPIV-07.05 **[11]**

A single lithium-7 atom, of mass $m = 1.165 \times 10^{-26}$ kg, is confined along one direction between two sheets of laser light that act as impenetrable walls at $x = 0$ and $x = L$, with $L = 10.0$ μm. Its stationary states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$ for $0 \le x \le L$, with energies $E_n = n^2\pi^2\hbar^2/(2mL^2)$, and it is prepared in the ground state ψ_1. At $t = 0$ a pulse of light lasting far less than any timescale of the atom's motion illuminates only the half of the box $0 < x < L/2$, multiplying the wavefunction there by $e^{i\pi} = -1$ and leaving the half $L/2 < x < L$ untouched. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K^{-1}. You may use $\int_0^{L/2} \sin(n\pi x/L)\sin(\pi x/L)\,dx = (L/(2\pi))[\sin((n-1)\pi/2)/(n-1) - \sin((n+1)\pi/2)/(n+1)]$ for $n \ge 2$.

(a) CALCULATE **[2]**

Calculate E_1, and the temperature T at which k_BT equals $E_2 - E_1$, the temperature scale that the atom must be cooled well below for its ground state to be prepared reliably.

(b) DERIVE **[4]**

Starting from the expansion of $\Psi(x, 0)$ in the stationary states and the Born rule, derive the probability P_n that a measurement of the energy made just after the pulse returns E_n. Find every n for which $P_n = 0$, and evaluate P_2 and P_4.

(c) EXPLAIN **[3]**

The pulse leaves the probability density $|\Psi(x, 0)|^2$ equal to $|\psi_1(x)|^2$. Explain why the density nevertheless changes after the pulse, why it remains symmetric about the centre of the box at all later times, and why it is zero at $x = L/2$ at every $t > 0$.

(d) PREDICT **[2]**

In the real trap the light sheets form barriers of finite height $V_0 = 400E_1$, with E_1 the value found in (a), and the same pulse, which reverses the sign of the wavefunction everywhere to the left of $x = L/2$, is applied to the ground state of this finite well. Predict, with reasons, whether a measurement of the energy just after the pulse could return the ground-state energy of the finite well, and whether the atom is certain to remain in the trap.

13 CORE COURSE

STRUCTURED · DEMANDING · UPIV-09.04 · UPIV-09.07 · UPIV-09.09 **[11]**

The resonance lines of the alkali atoms come from an np → ns transition that spin-orbit coupling splits into D_1 ($^2P_{1/2} \to {}^2S_{1/2}$) and D_2 ($^2P_{3/2} \to {}^2S_{1/2}$). The fine-structure splitting between $^2P_{3/2}$ and $^2P_{1/2}$ is 7.16×10^{-3} eV for the 4p level of potassium and 4.16×10^{-5} eV for the 2p level of lithium. Both vapours are placed in a uniform magnetic field of 0.500 T. Ignore hyperfine structure throughout. Take $\mu_B = 5.788 \times 10^{-5}$ eV T^{-1} and $g_s = 2$, so that $g_J = 1 + [j(j+1) + s(s+1) - l(l+1)]/[2j(j+1)]$.

(a) CALCULATE **[2]**

Calculate g_J for the $^2S_{1/2}$, $^2P_{1/2}$ and $^2P_{3/2}$ levels.

(b) DETERMINE **[3]**

In the weak-field limit each sublevel shifts by $g_J m_j \mu_B B$. Using the selection rule $\Delta m_j = 0, \pm 1$, determine the number of components into which the D_1 line splits and the shift of each component from the zero-field line, in units of $\mu_B B$.

(c) COMPARE **[2]**

Compare the fine-structure splitting with $\mu_B B$ at 0.500 T for each atom.

(d) JUSTIFY **[2]**

Justify, for each atom, whether the D_1 pattern you found in (b) describes its spectrum at 0.500 T.

(e) PREDICT **[2]**

Predict the appearance of the lithium resonance line at 5.00 T, naming the good quantum numbers.

14 CORE COURSE

STRUCTURED · DEMANDING · UPIV-14.03 · UPIV-14.04 · UPIV-14.05 · UPIV-14.07 **[12]**

The muon decays by $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ with a branching fraction of essentially 1 and a mean lifetime $\tau_\mu = 2.197 \times 10^{-6}$ s. The τ^- lepton decays through the same charged weak current into $e^- + \bar{\nu}_e + \nu_\tau$ with branching fraction $B_e = 0.178$, and also into $\mu^- + \bar{\nu}_\mu + \nu_\tau$ and into hadrons. Take $m_\mu c^2 = 105.66$ MeV, $m_\tau c^2 = 1776.9$ MeV and the Fermi constant $G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV^{-2}. In natural units, with $\hbar = c = 1$, a decay width $\Gamma = \hbar/\tau$ has the dimensions of energy.

(a) STATE **[2]**

State the values of L_e, L_μ and L_τ before and after the decay $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, and state which conservation law forbids $\tau^- \to e^- + \gamma$.

(b) DESCRIBE **[2]**

Describe the lowest-order Feynman diagram for $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, and give the power of the weak coupling g_W in its amplitude and in its rate.

(c) SHOW THAT **[2]**

Show that if the width depends only on G_F and on the mass m of the decaying lepton, the result of (b) requires $\Gamma \propto G_F^2 m^5$.

(d) ESTIMATE **[3]**

Estimate the mean lifetime of the τ, stating the assumptions you make.

(e) EVALUATE **[3]**

The measured τ lifetime is 2.903×10^{-13} s. Evaluate what the agreement with your estimate tests, and use a count of the final states available to the virtual W^- to explain why B_e is close to 0.2.

15 CORE COURSE

STRUCTURED · DISCRIMINATING · UPIV-13.02 · UPIV-13.04 **[11]**

^{210}Po decays by α emission to ^{206}Pb with a half-life of 138.4 days. The atomic masses are ^{210}Po 209.982874 u, ^{206}Pb 205.974465 u and ^{4}He 4.002603 u, with 1 u = 931.494 MeV/c². In the Gamow model the α particle, of reduced mass μ, meets a Coulomb barrier V(r) = Z_dZ_ααħc/r that begins at the nuclear radius R, and tunnels out to the classical turning point b at which V(b) = Q. The WKB exponent is 2G = 2αZ_dZ_α√(2μc²/Q)[arccos√(R/b) − √((R/b)(1 − R/b))], and the half-life is proportional to e^(2G) divided by an assault frequency. Take αħc = 1.440 MeV fm, α = 1/137.0, μc² = 3.66 × 10³ MeV, R = 9.0 fm, Z_d = 82 and Z_α = 2.

(a) CALCULATE **[2]**

Calculate the Q value of the decay.

(b) CALCULATE **[1]**

Calculate the outer turning point b.

(c) SHOW THAT **[3]**

Using arccos√x ≈ π/2 − √x and √(x(1 − x)) ≈ √x for x = R/b much less than 1, show that 2G ≈ παZ_dZ_α√(2μc²/Q) − 4√(2αZ_dZ_αμc²R/ħc), and hence that the second term does not depend on Q.

(d) PREDICT **[3]**

^{214}Po also decays by α emission to a lead isotope (Z_d = 82), with Q = 7.833 MeV. Assuming the same R, μ and assault frequency for both nuclei, predict the half-life of ^{214}Po from that of ^{210}Po using the result of (c).

(e) EVALUATE **[2]**

The measured half-life of ^{214}Po is 164 μs. Evaluate the model in the light of this comparison.

16 CORE COURSE

STRUCTURED · DISCRIMINATING · UPIV-02.07 · UPIV-03.02 · UPIV-03.03 **[11]**

A sample-return probe leaves a space station, which defines an inertial frame S, and travels in a straight line at 0.600c to a relay beacon at rest in S at a distance of 3.00 light-years. At the beacon it reverses in a negligibly short time and returns at 0.800c. Work in units in which c = 1 light-year per year, label events by their S coordinates (t, x), and take the departure as the event (0, 0). Let S′ be the inertial frame moving with the probe on the outbound leg and S″ the inertial frame moving with it on the return leg.

(a) CALCULATE **[3]**

Calculate the time that elapses on the station clock, and the proper time that elapses on the probe's clock, for the round trip.

(b) DETERMINE **[4]**

Determine the reading of the station clock that is simultaneous with the turnaround event, first in S′ and then in S″.

(c) SHOW THAT **[2]**

Show that the probe's account, with the station clock running slow on each leg plus the jump found in (b), reproduces the station's total of 8.75 y.

(d) EVALUATE **[2]**

Evaluate the claim: “The probe ages less because it accelerated, and the harder it accelerates at the turnaround, the more time it loses.”

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIV-12.07 · UPIV-12.08 **[11]**

The intrinsic carrier density n_i of an undoped semiconductor bar is found from Hall-effect and conductivity measurements at six temperatures, and each value of n_i carries a random uncertainty of ±3%. For a non-degenerate intrinsic semiconductor with a temperature-independent band gap E_g, $n_i = CT^{3/2}\exp(-E_g/2kT)$, where C is a constant. Take $k = 8.617 \times 10^{-5}$ eV K^{-1}.

Intrinsic carrier density of the bar at six temperatures

Temperature T (K)	Intrinsic carrier density n_i (m^{-3})
300	2.03×10^{19}
340	1.40×10^{20}
380	6.89×10^{20}
420	2.41×10^{21}
460	7.32×10^{21}
500	1.78×10^{22}

(a) EXPLAIN **[2]**

Explain why a graph of $\ln(n_i/T^{3/2})$ against $1/T$ is expected to be a straight line, and what its gradient gives.

(b) DETERMINE **[3]**

Determine E_g from the data.

(c) ESTIMATE **[2]**

Using the first and last rows only, estimate the uncertainty in E_g that follows from the ±3% uncertainty in n_i, stating the assumption you make about the two uncertainties.

(d) COMPARE **[2]**

A student instead plots $\ln n_i$ against $1/T$ and takes $E_g = -2k \times$ gradient. Compare the gap this gives with your value from (b), and account for the difference.

(e) EVALUATE **[2]**

Optical absorption measured on the same bar at 400 K shows an absorption edge at 0.66 eV. Semiconductor gaps typically narrow as the temperature rises. Evaluate whether this contradicts your value from (b).

18
CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPIV-10.03 · UPIV-10.04 · UPIV-10.05 · UPIV-10.07 · UPIV-10.09 **[15]**

An electron is bound to a nucleus of charge Ze. With μ the reduced mass, the radial function u(r) = rR(r) of a state of orbital quantum number l satisfies the radial Schrödinger equation $-(\hbar^2/2\mu)(d^2u/dr^2) + [-Ze^2/(4\pi\varepsilon_0 r) + l(l+1)\hbar^2/(2\mu r^2)]u = Eu$, with $E < 0$ for a bound state. Write $\kappa = \sqrt{(-2\mu E)}/\hbar$ and $a_\mu = 4\pi\varepsilon_0\hbar^2/(\mu e^2)$. You may use $\int_0^\infty r^k e^{-\beta r}\,dr = k!/\beta^{k+1}$ for integer $k \geq 0$. For the numerical parts take $a_0 = 5.292 \times 10^{-11}$ m and 13.606 eV as the Bohr radius and the ionisation energy of hydrogen for an infinitely heavy nucleus, $m_p/m_e = 1836$ and $h = 4.136 \times 10^{-15}$ eV s.

(a) SHOW THAT **[2]**

Show that a normalisable solution must behave as $e^{-\kappa r}$ at large r and as r^{l+1} near $r = 0$.

(b) DERIVE **[5]**

By substituting $u = r^{l+1}e^{-\kappa r}$ into the radial Schrödinger equation, derive the conditions under which it is an exact solution, and hence derive κ and the energy E of this state. Identify its principal quantum number n.

(c) DETERMINE **[3]**

For this state, determine the most probable radius r_{mp} and the mean radius $\langle r \rangle$ in terms of n, a_μ and Z.

(d) CALCULATE **[3]**

Circular Rydberg states, with $l = n - 1$, are used in microwave cavity experiments. For the $n = 50$ circular state of hydrogen, calculate the most probable radius and the frequency of the photon emitted in the transition to $n = 49$.

(e) JUSTIFY **[2]**

Justify, using the electric-dipole selection rules, the claim that this state can decay by only one electric-dipole transition.

19 CORE COURSE

FREE RESPONSE · DERIVE A MODEL, THEN TEST IT AGAINST EVIDENCE · DISCRIMINATING · UPIV-15.06 · UPIV-15.07 **[14]**

Model the universe as a uniform, pressureless dust of mass density $\rho(t)$ in which every separation grows in proportion to a scale factor $a(t)$, with $a = 1$ today. A particle of mass m moving with the dust lies at distance $r = a(t)x$ from an arbitrary origin, where x is constant, and by the shell theorem it is attracted only by the mass inside the sphere of radius r. Take $G = 6.674 \times 10^{-11}$ N m^2 kg^{-2}, the present expansion rate $H_0 = \dot{a}/a = 67.4$ km s^{-1} Mpc^{-1}, 1 Mpc $= 3.086 \times 10^{22}$ m, the mass of a hydrogen atom as 1.674×10^{-27} kg and 1 yr $= 3.156 \times 10^{7}$ s.

(a) DERIVE **[4]**

Starting from conservation of energy for the particle, derive the Friedmann equation $(\dot{a}/a)^2 = (8\pi G/3)\rho - K/a^2$, where K is a constant, and state what the sign of K means for the future of the expansion in this Newtonian picture.

(b) CALCULATE **[3]**

The critical density ρ_c is the density for which $K = 0$. Calculate ρ_c today, and express it as a number of hydrogen atoms per cubic metre.

(c) DERIVE **[3]**

For $K = 0$ and $\rho \propto a^{-3}$, derive $a(t)$ from the Friedmann equation, taking $a = 0$ at $t = 0$, and hence obtain the age t_0 of the universe in terms of H_0 and in gigayears.

(d) EVALUATE **[2]**

The oldest globular-cluster stars in the Milky Way are dated at about 12.5 Gyr. Evaluate the flat, matter-only model in the light of this, including the value of H_0 the model would need.

(e) CALCULATE **[2]**

The cosmic microwave background was released at redshift $z = 1090$. Calculate the scale factor at that time and the factor by which the matter density then exceeded its present value.

MARK SCHEME

University Physics IV · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-01.06 · UPIV-01.08 **[1]**

A clean tungsten surface of work function φ = 4.50 eV is illuminated in vacuum by femtosecond laser pulses of wavelength 800 nm, and electrons are emitted even though each photon carries less energy than the work function. An electron spectrometer shows that the most energetic electrons leave with 0.15 eV at every pulse intensity used, and doubling the peak intensity I multiplies the number of electrons emitted per pulse by 7.9 ± 0.4. Take hc = 1240 eV nm. Which conclusion do these measurements justify?

KEY **C** **Each emitted electron absorbs three quanta of 1.55 eV, since 3hν − φ = 0.15 eV and an emission rate proportional to I^3 grows by $2^3 = 8$ when I doubles; energy is exchanged with the light in quanta of hν, but these data do not by themselves prove that the light field is quantised.**

Each photon carries hc/λ = (1240 eV nm)/(800 nm) = 1.55 eV. Two quanta (3.10 eV) cannot free an electron, three leave a maximum kinetic energy of 3(1.55 eV) − 4.50 eV = 0.15 eV, exactly as measured, and four would leave 1.70 eV; independently, a rate proportional to I^n grows by 2^n when I doubles, and 7.9 ± 0.4 corresponds to n between 2.91 and 3.05. Two separate measurements pointing to the same integer is what justifies C. B is tempting because the threshold is usually taught as absolute, but Einstein's equation describes one quantum absorbed by one electron: absorbing several at once needs an enormous photon flux, and the rate of n-quantum absorption scales as I^n, negligible for a lamp and measurable only with intense, focused laser pulses, so nothing in the quantum picture forbids it. A appeals to the heating that intense pulses really do cause, but thermal emission would give an energy spread set by the electron temperature, widening as the intensity rises, and a yield growing exponentially with that temperature, not a fixed cut-off at exactly 3hν − φ and a clean third-power law. D draws the right multiphoton picture and then overclaims: a classical field of frequency ν acting on an electron whose states are quantised also transfers energy only in multiples of hν, because the nth-order response to an oscillating field is resonant only when the energy change is nhν. Like the ordinary photoelectric effect, this experiment establishes that energy is exchanged in quanta of hν; showing that the field itself is quantised takes measurements on the light alone, such as photon antibunching.

U4

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-04.05 · UPIV-04.06 **[1]**

At a pulsed neutron source, each pulse releases neutrons with a wide spread of speeds from the moderator at the same instant. They travel a total flight path of 12.0 m, by way of a powder sample, to a detector that records neutrons scattered elastically through 90.0° by the sample. A Bragg peak reaches the detector 6.00 ms after the pulse. Treating it as a first-order reflection, what is the spacing of the lattice planes that produce it? Take h = 6.626 × 10^{-34} J s and m_n = 1.675 × 10^{-27} kg.

KEY **A** **0.140 nm**

The neutrons in the peak left the moderator together and keep their speed through elastic scattering, so v = (12.0 m)/(6.00 × 10^{-3} s) = 2.00 × 10^3 m s^{-1} and λ = h/(m_nv) = 1.98 × 10^{-10} m. A deflection of 90.0° is 2θ, twice the Bragg angle measured from the planes, so θ = 45.0° and d = λ/(2 sin θ) = 1.40 × 10^{-10} m = 0.140 nm. B takes the deflection of the beam as the Bragg angle, d = λ/(2 sin 90.0°); the angle in 2d sin θ = λ lies between the beam and the planes, and a beam turned through 90° meets the planes at 45°. C drops the factor of 2 from 2d sin θ, counting the extra path of the ray reflected from the deeper plane on the way in but not on the way out. D writes the de Broglie relation with ħ in place of h, which makes λ 2π times too short and gives a spacing of 0.0223 nm, far smaller than any distance between neighbouring atoms in a solid.

U3

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04 **[1]**

An electron is held in a one-dimensional trap whose potential energy is not symmetric about any point. It is in the trap's ground state, described by a real, normalised wavefunction u(x) whose probability density is lopsided. A uniform force then acts for a time so short that u(x) does not change while it acts, delivering an impulse $\hbar k_0$ in the +x direction, so that immediately afterwards the wavefunction is u(x)e^(ik₀x). With φ(p) the momentum-space wavefunction, which statement is correct?

KEY **D** **Before the impulse $|\varphi(p)|^2 = |\varphi(-p)|^2$, so $\langle p \rangle = 0$ even though $|u(x)|^2$ is lopsided; immediately afterwards the momentum distribution is the same curve shifted by $\hbar k_0$, while the position density is unchanged.**

Writing φ(p) = (2πℏ)^(−1/2) ∫ u(x)e^(−ipx/ℏ) dx, a real u gives φ(−p) = φ(p), so |φ(p)|² is even in p and ⟨p⟩ = 0: a real wavefunction is a superposition in which the components of momentum +p and −p always carry amplitudes of equal size, whatever the shape of u. After the impulse, u(x)e^(ik₀x) transforms to φ(p − ℏk₀), the same distribution translated by ℏk₀, while |u(x)e^(ik₀x)|² = |u(x)|². A reads the lopsided position density into the momentum distribution, but the ground state is stationary, so ⟨p⟩ = m d⟨x⟩/dt = 0, and a real wavefunction carries no probability current. B is the commonest error: a phase factor that varies with x is not a global phase, and although the position density cannot see it, the Born rule applied in the momentum representation uses |φ(p)|², which has moved by ℏk₀. C gets the symmetry condition wrong, since an even u would make φ itself real and even, which is more than |φ(p)|² needs in order to be even, and it gets the shift wrong too: multiplying by a single plane wave translates φ(p) without reshaping it, adding ℏk₀ to ⟨p⟩ and leaving the variance ⟨p²⟩ − ⟨p⟩² unchanged.*

U1

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.04 · UPIV-08.09 **[1]**

A single ion held in a radio-frequency trap oscillates along the trap axis as a quantum harmonic oscillator of frequency f = 4.00 MHz, with energy levels $E_n = (n + \tfrac{1}{2})hf$. After laser cooling, its axial motion is in thermal equilibrium at T = 0.600 mK. What is the probability that the axial motion is found in its ground state, n = 0? Take $h = 6.626 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K⁻¹.

KEY **B** **0.274**

With x = hf/(k_BT) = (6.626 × 10⁻³⁴ J s)(4.00 × 10⁶ Hz)/[(1.381 × 10⁻²³ J K⁻¹)(6.00 × 10⁻⁴ K)] = 0.320, the Boltzmann weights e^(−(n + ½)x) of the evenly spaced ladder form a geometric series with partition function Z = e^(−x/2)/(1 − e^(−x)), so P₀ = e^(−x/2)/Z = 1 − e^(−x) = 0.274. The zero-point energy multiplies every weight by the same factor e^(−x/2) and cancels. A is e^(−x), the ratio of the populations of n = 1 and n = 0: it is the familiar Boltzmann factor, but it compares two levels instead of giving the share of all the levels that n = 0 holds. C normalises over n = 0 and n = 1 only, as if the motion were a two-level system, giving 1/(1 + e^(−x)); with k_BT about 3.1hf many levels are occupied, and the mean occupation 1/(e^(x) − 1) is 2.65 quanta. D is e^(−x/2), the Boltzmann factor of the zero-point energy alone, taken as a probability without dividing by the partition function.

U3

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-09.01 · UPIV-09.02 **[1]**

A particle is in the simultaneous eigenstate of J² and J_z with quantum numbers j = 3/2 and m = 3/2. What is the expectation value ⟨J_x²⟩?

KEY **A** **$3\hbar^2/4$**

Because J² = J_x² + J_y² + J_z² and the state is an eigenstate of both J² and J_z, ⟨J_x²⟩ + ⟨J_y²⟩ = [j(j + 1) − m²]ℏ² = (15/4 − 9/4)ℏ² = 3ℏ²/2, and symmetry under rotation about the z axis shares this equally, giving 3ℏ²/4. B is the classical picture of a vector lying along z. It is ruled out because j(j + 1) exceeds m² for every j > 0, so the angular momentum can never be fully aligned; it would also violate σ_Jx σ_Jy ≥ (ℏ/2)|⟨J_z⟩| = 3ℏ²/4, a bound this state exactly saturates. C is the sum ⟨J_x²⟩ + ⟨J_y²⟩, not yet shared between the two components. D is j(j + 1)ℏ², the whole of ⟨J²⟩ attributed to a single component.

U2

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-11.04 · UPIV-11.06 **[1]**

A free ion has the ground configuration [Ar]3d³. Applying all three of Hund's rules, what is its ground-state term symbol?

KEY **D** **$^4F_{3/2}$**

The first rule maximises S: three parallel spins give S = 3/2, a quartet. The second then maximises L within that S: with parallel spins the three electrons must occupy different m_l, and the largest sum is 2 + 1 + 0 = 3, an F term. The third rule looks at filling: 3 electrons in a shell that holds 10 is less than half full, so J = |L − S| = 3/2, giving $^4F_{3/2}$. A applies the third rule the wrong way round; J = L + S belongs to a shell more than half full. B maximises L before S: m_l = 2, 2, 1 is possible only if two electrons pair their spins, which gives L = 5 but S = 1/2, so it inverts the order of the first two rules. C is the other quartet term of d³, the choice of a candidate who takes the smallest rather than the largest L once S is fixed.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-12.06 **[1]**

In a one-dimensional tight-binding model an electron band has dispersion $E(k) = E_0 - 2t\cos(ka)$, with hopping energy t = 0.50 eV and lattice spacing a = 0.30 nm. What is the effective mass of an electron near the bottom of the band, at k = 0? Take $\hbar = 1.055 \times 10^{-34}$ J s, e = 1.602×10^{-19} C and $m_e = 9.109 \times 10^{-31}$ kg.

KEY **B** **$0.85m_e$**

The effective mass comes from the curvature of the band, $1/m^* = (1/\hbar^2)d^2E/dk^2$. At k = 0, $d^2E/dk^2 = 2ta^2$, so $m^* = \hbar^2/(2ta^2) = 7.7 \times 10^{-31}$ kg = $0.85m_e$. A halves this, the result of equating $\hbar^2k^2/(2m^*)$ to $2ta^2k^2$ instead of to the ta^2k^2 that the expansion $2t\cos(ka) \approx 2t - ta^2k^2$ actually gives. C doubles it, from dropping the factor of 2 in 2t. D has the right magnitude but the sign that belongs at the top of the band, k = ±π/a, where the curvature is negative; near the bottom of the band an electron accelerates along the applied force, as a particle of positive mass does.

U2

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-13.03 **[1]**

In the spin-orbit shell model the proton single-particle levels fill in the order $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, each holding 2j + 1 protons, and neutrons fill the same sequence. What spin and parity does the model predict for the ground state of ^{39}K, with Z = 19 and N = 20?

KEY **A** **$3/2^+$**

N = 20 fills the neutron levels through $1d_{3/2}$, which contributes zero angular momentum. The protons fill $1s_{1/2}$ (2), $1p_{3/2}$ (6), $1p_{1/2}$ (8), $1d_{5/2}$ (14) and $2s_{1/2}$ (16), then place 3 of a possible 4 in $1d_{3/2}$. A single hole in a j level carries the same j as a single particle would, so J = 3/2, and the parity is (−1)^l with l = 2 for a d orbital, giving $3/2^+$. B takes the last completely filled level, $2s_{1/2}$, as the one that sets the spin. C has the right j but reads the parity from j, or from a p orbital, instead of from l = 2. D places the odd proton in $1f_{7/2}$, a level that begins to fill only beyond 20; it is the prediction for a nucleus with one nucleon outside a closed shell of 20, such as ^{41}Ca, not for ^{39}K.

U1

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-14.01 **[1]**

A neutral meson decays in flight into two photons, which an electromagnetic calorimeter records with energies 1.50 GeV and 0.500 GeV and an angle of 36.9° between their directions of travel. Treating the photons as massless, what is the rest energy of the meson?

KEY **D** **0.548 GeV**

The rest energy is the invariant mass of the photon pair, the same in every frame: $(Mc^2)^2 = (E_1 + E_2)^2 - (p_1c + p_2c)\cdot(p_1c + p_2c)$, and with $\lvert p_1c\rvert = E_1$ and $\lvert p_2c\rvert = E_2$ for massless photons this reduces to $(Mc^2)^2 = 2E_1E_2(1 - \cos\theta) = 2(1.50\ \text{GeV})(0.500\ \text{GeV})(1 - \cos 36.9°) = 0.300\ \text{GeV}^2$, so $Mc^2 = 0.548$ GeV, the rest energy of the η meson. A adds the two photon energies, which gives the meson's energy in the laboratory rather than its rest energy; the two differ because the meson was moving, with momentum 1.92 GeV/c. B has the wrong sign on the cross term: $(p_1c + p_2c)\cdot(p_1c + p_2c) = E_1^2 + E_2^2 + 2E_1E_2\cos\theta$, and because this is subtracted from the energy squared the invariant contains $1 - \cos\theta$, not $1 + \cos\theta$. C drops the factor of 2 from the cross term $2p_1\cdot p_2$, which leaves the result too small by a factor of $\sqrt{2}$.

U3

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10 CORE COURSE

MULTIPLE CHOICE · RECALL · UPIV-15.02 · UPIV-15.09 **[1]**

Entangled photon pairs are measured at two stations whose analyser settings are chosen randomly and independently, with the detection events at the two stations spacelike separated and the detection efficiency high enough that no fair-sampling assumption is needed. The measured CHSH combination is $S = 2.70 \pm 0.02$. Which statement correctly describes what this result establishes?

KEY **C** **All models in which each outcome is fixed by variables carried locally from the source, independent of the settings, are excluded; the correlations still cannot be used to send a signal faster than light, and non-local hidden-variable models are not excluded.**

Any model whose outcomes are fixed by local variables that do not depend on the settings obeys $\lvert S\rvert \le 2$, so a value 35 standard deviations above 2, with the locality and detection loopholes closed as the stem describes, excludes that whole class; quantum mechanics predicts $2\sqrt{2} \approx 2.83$ for a maximally entangled pair at the optimal settings. The violation itself does not prove that signalling is impossible; C says only that these correlations give no means of signalling, because each station's own statistics are independent of the distant setting. A is the most common over-reading: the outcome statistics at either station alone do not depend on the distant setting, so the correlation appears only when the two records are brought together through an ordinary channel. B over-claims: the violation rules out locality combined with predetermined outcomes, not hidden variables as such, and non-local hidden-variable theories of the Bohmian kind reproduce the quantum predictions for this experiment. D inverts the physics: mixing in noise lowers S towards and below 2, and it is the pure maximally entangled state that gives the largest value.

U1

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11 CORE COURSE

STRUCTURED · DEMANDING · UPIV-11.03 · UPIV-11.09 **[10]**

A handheld X-ray fluorescence analyser is aimed at a corroded brass fitting. Its primary X-rays eject inner-shell electrons from atoms in the sample, and the detector records two strong lines at 8.04 keV and 8.63 keV, together with weaker lines, one of which lies at 8.90 keV. Model each K line with hydrogen-like levels and a screening constant σ: a transition from shell n to the K shell (n = 1) emits a photon of energy $E = (13.6\ \text{eV})(Z - \sigma)^2(1 - 1/n^2)$, so that the Kα line (n = 2) follows Moseley's law, $E = (3/4)(13.6\ \text{eV})(Z - \sigma)^2$.

(a) DESCRIBE · U1 **[2]**

Describe the sequence of events, beginning with the absorption of a primary X-ray photon, that leads to the emission of a Kα photon, and say why the primary photons must exceed a threshold energy.

1. 1 mark: a primary photon with energy above the K-shell binding energy is absorbed and ejects a 1s electron, leaving a K-shell vacancy; a photon below this K edge cannot create the vacancy, hence the threshold
2. 1 mark: an electron from the n = 2 (L) shell falls into the vacancy, and the energy difference between the two inner-shell levels is carried away by the Kα photon (the competing Auger process ejects an electron instead)

Candidates who describe an outer electron being excited and falling back are describing optical fluorescence. An X-ray line needs an inner-shell vacancy, which is why its energy identifies the element almost independently of its chemical state.

(b) CALCULATE · U3 **[3]**

Taking σ = 1, calculate the atomic numbers of the elements that produce the 8.04 keV and 8.63 keV lines, and identify the elements.

1. 1 mark: rearranges Moseley's law to $Z = 1 + \sqrt{[E/((3/4)(13.6\ \text{eV}))]} = 1 + \sqrt{(E/10.2\ \text{eV})}$, with E in eV
2. 1 mark: 8.04 keV gives $Z = 1 + \sqrt{788} = 29.1$, so Z = 29, copper
3. 1 mark: 8.63 keV gives $Z = 1 + \sqrt{846} = 30.1$, so Z = 30, zinc, the two constituents of brass

Dividing an energy in keV by 13.6 eV without converting gives $Z \approx 1.9$ for both lines. That both answers land within 0.1 of an integer is itself the check that the units are right.

(c) EXPLAIN · U1 **[2]**

Explain why $\sigma \approx 1$ is a reasonable screening constant for a Kα line.

1. 1 mark: with one 1s electron ejected, one 1s electron remains; it lies almost entirely inside the orbit of the n = 2 electron that makes the jump, and so screens one unit of the nuclear charge
2. 1 mark: the electrons of the n = 2 shell and beyond are spread at the same or larger radii, so by Gauss's law they screen little, leaving an effective charge close to $Z - 1$ in a hydrogen-like energy

The error to catch is crediting the screening to the ejected electron. That electron has left the atom; it is the 1s electron left behind, inside the orbit of the electron that jumps, that reduces the charge it sees.

(d) EVALUATE · U4 **[3]**

A colleague suggests that the 8.90 keV line is the Kα line of a third element. Evaluate this suggestion against the alternative that it is the Kβ line ($n = 3 \to n = 1$) of copper, and state what your conclusion implies about σ for a Kβ line.

1. 1 mark: as a Kα line with σ = 1, 8.90 keV gives $Z = 1 + \sqrt{(8900/10.2)} = 30.5$, not an integer, whereas the two strong lines gave Z within 0.1 of an integer; the neighbouring Kα lines, zinc (observed at 8.63 keV) and gallium (predicted 9.18 keV), bracket it, so the suggestion is rejected
2. 1 mark: copper Kβ with σ = 1 is predicted at $(13.6\ \text{eV})(28)^2(1 - 1/9) = 9.48$ keV, within 7% of 8.90 keV, and copper is known to be present, so the line is copper Kβ
3. 1 mark: matching 8.90 keV exactly needs $\sigma \approx 1.9$, because the n = 3 electron is screened by the remaining 1s electron and also, in part, by the eight n = 2 electrons; σ = 1 does not carry over from Kα to Kβ

The discriminating step is to use the precision shown in (b) as the test. A model that places two lines within 0.1 of an integer cannot then be stretched to accept Z = 30.5, and the 7% misfit for Kβ is a statement about screening, not a failure of the identification.

12 CORE COURSE

STRUCTURED · DEMANDING · UPIV-06.04 · UPIV-07.02 · UPIV-07.03 · UPIV-07.05 **[11]**

A single lithium-7 atom, of mass m = 1.165 × 10⁻²⁶ kg, is confined along one direction between two sheets of laser light that act as impenetrable walls at x = 0 and x = L, with L = 10.0 μm. Its stationary states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$ for $0 \le x \le L$, with energies $E_n = n^2\pi^2\hbar^2/(2mL^2)$, and it is prepared in the ground state ψ_1. At t = 0 a pulse of light lasting far less than any timescale of the atom's motion illuminates only the half of the box 0 < x < L/2, multiplying the wavefunction there by e^(iπ) = −1 and leaving the half L/2 < x < L untouched. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K⁻¹. You may use ∫₀^(L/2) sin(nπx/L) sin(πx/L) dx = (L/(2π))[sin((n − 1)π/2)/(n − 1) − sin((n + 1)π/2)/(n + 1)] for n ≥ 2.

(a) CALCULATE · U3 **[2]**

Calculate E_1, and the temperature T at which k_BT equals $E_2 - E_1$, the temperature scale that the atom must be cooled well below for its ground state to be prepared reliably.

1 1 mark: $E_1 = \pi^2\hbar^2/(2mL^2) = \pi^2(1.055 \times 10^{-34}\ \text{J s})^2/[2(1.165 \times 10^{-26}\ \text{kg})(1.00 \times 10^{-5}\ \text{m})^2] = 4.71 \times 10^{-32}$ J

2 1 mark: $E_2 - E_1 = 3E_1 = 1.41 \times 10^{-31}$ J, so $T = 3E_1/k_B = 1.02 \times 10^{-8}$ K, about 10 nK

Converting 10.0 μm and then forgetting to square it, or taking the gap as $E_2 = 4E_1$ instead of $E_2 - E_1 = 3E_1$, are the slips this part is set to catch. At T itself the atom would still be found in an excited state about 30% of the time, which is why a single atom in a box this size has to be cooled well below 10 nK before its ground state can be prepared reliably.

(b) DERIVE · U2 **[4]**

Starting from the expansion of Ψ(x, 0) in the stationary states and the Born rule, derive the probability P_n that a measurement of the energy made just after the pulse returns E_n. Find every n for which $P_n = 0$, and evaluate P_2 and P_4.

1 1 mark: $\Psi(x, 0) = \Sigma\, c_n\psi_n(x)$, and orthonormality of the ψ_n gives $c_n = \int_0^L \psi_n(x)\Psi(x, 0)\,dx$, with $P_n = |c_n|^2$ by the Born rule

2 1 mark: Ψ(x, 0) = −ψ₁ in the left half and +ψ₁ in the right half, so c_n is the integral of $\psi_n\psi_1$ over the right half minus its integral over the left half; since $\psi_n(L - x) = (-1)^{(n+1)}\psi_n(x)$, the product $\psi_n\psi_1$ is symmetric about x = L/2 for odd n, the two integrals are equal, and $P_n = 0$ for every odd n, the ground state included

3 1 mark: for even n the product is antisymmetric about x = L/2, so $c_n = -2\int_0^{(L/2)} \psi_n\psi_1\,dx = -(2/\pi)[\sin((n - 1)\pi/2)/(n - 1) - \sin((n + 1)\pi/2)/(n + 1)]$, which gives $P_n = 16n^2/[\pi^2(n^2 - 1)^2]$ for even n

4 1 mark: $P_2 = 64/(9\pi^2) = 0.721$ and $P_4 = 256/(225\pi^2) = 0.115$

Candidates who reason that the density has not changed, and so the atom must still be in its ground state, miss that $P_1 = 0$ exactly. The symmetry argument does the work: without it every overlap has to be integrated separately and the zeros look like accidents.

(c) EXPLAIN · U1 **[3]**

The pulse leaves the probability density $|\Psi(x, 0)|^2$ equal to $|\psi_1(x)|^2$. Explain why the density nevertheless changes after the pulse, why it remains symmetric about the centre of the box at all later times, and why it is zero at x = L/2 at every t > 0.

1 1 mark: the pulse changes the relative phase of the two halves, so Ψ(x, 0) is no longer a stationary state but a superposition of ψ₂, ψ₄, ψ₆, …; each term acquires its own phase $e^{(-iE_nt/\hbar)}$, so the cross terms in $|\Psi|^2$ change with time

2 1 mark: every component present is antisymmetric about x = L/2 and keeps that symmetry as its phase evolves, so $\Psi(L - x, t) = -\Psi(x, t)$ and $|\Psi(x, t)|^2 = |\Psi(L - x, t)|^2$ at all times

3 1 mark: antisymmetry requires Ψ(L/2, t) = −Ψ(L/2, t), so Ψ(L/2, t) = 0, as every even ψ_n has a node there; a notch opens at the centre, where the density had its maximum before the pulse

Reasoning from the density alone leads to the answer that a pure change of phase does nothing. The relative phase between parts of a wavefunction is physical: it decides which stationary states are present, and so how the state evolves.

(d) PREDICT · U1 **[2]**

In the real trap the light sheets form barriers of finite height $V_0 = 400E_1$, with E_1 the value found in (a), and the same pulse, which reverses the sign of the wavefunction everywhere to the left of $x = L/2$, is applied to the ground state of this finite well. Predict, with reasons, whether a measurement of the energy just after the pulse could return the ground-state energy of the finite well, and whether the atom is certain to remain in the trap.

1 1 mark: the finite well is still symmetric about $x = L/2$, so its bound states still have definite parity and its ground state is symmetric; the pulsed wavefunction is antisymmetric about the centre, so its overlap with the ground state is zero and the ground-state energy cannot be returned

2 1 mark: the sign jump makes the amplitudes fall off only as $1/n$, so part of the probability lies in unbound states above the barrier top (in the infinite-well estimate the states with $n > 20$, whose energies exceed $400E_1$, carry about 4%), and the atom is not certain to remain trapped

Predicting that finite walls merely lower the levels a little misses both effects that matter here: parity, which a symmetric finite well preserves exactly, and the unbound states above V_0, which a sharp step in the wavefunction always populates.

13

CORE COURSE

STRUCTURED · DEMANDING · UPIV-09.04 · UPIV-09.07 · UPIV-09.09 **[11]**

The resonance lines of the alkali atoms come from an np → ns transition that spin-orbit coupling splits into D_1 ($^2P_{1/2} \to {}^2S_{1/2}$) and D_2 ($^2P_{3/2} \to {}^2S_{1/2}$). The fine-structure splitting between $^2P_{3/2}$ and $^2P_{1/2}$ is 7.16×10^{-3} eV for the 4p level of potassium and 4.16×10^{-5} eV for the 2p level of lithium. Both vapours are placed in a uniform magnetic field of 0.500 T. Ignore hyperfine structure throughout. Take $\mu_B = 5.788 \times 10^{-5}$ eV T^{-1} and $g_s = 2$, so that $g_J = 1 + [j(j+1) + s(s+1) - l(l+1)]/[2j(j+1)]$.

(a) CALCULATE · U2 **[2]**

Calculate g_J for the $^2S_{1/2}$, $^2P_{1/2}$ and $^2P_{3/2}$ levels.

1 1 mark: $^2S_{1/2}$: $g_J = 1 + (3/4 + 3/4)/(3/2) = 2$, the moment being pure spin

2 1 mark: $^2P_{1/2}$: $g_J = 1 + (3/4 + 3/4 - 2)/(3/2) = 2/3$; $^2P_{3/2}$: $g_J = 1 + (15/4 + 3/4 - 2)/(15/2) = 4/3$

$g_s = 2$ is what separates the anomalous Zeeman effect from the normal one. A candidate who sets $g_s = 1$ obtains $g_J = 1$ for every level and predicts the normal triplet for an alkali line, which is exactly what is not observed.

(b) DETERMINE · U2 **[3]**

In the weak-field limit each sublevel shifts by $g_J m_j \mu_B B$. Using the selection rule $\Delta m_j = 0, \pm 1$, determine the number of components into which the D_1 line splits and the shift of each component from the zero-field line, in units of $\mu_B B$.

1 1 mark: the $^2P_{1/2}$ sublevels shift by $\pm(1/3)\mu_B B$ and the $^2S_{1/2}$ sublevels by $\pm\mu_B B$

2 1 mark: all four pairings of $m_j = \pm 1/2$ in the upper level with $m_j = \pm 1/2$ in the lower level satisfy $\Delta m_j = 0, \pm 1$

3 1 mark: four components, at $\pm(2/3)\mu_B B$ for $\Delta m_j = 0$ and $\pm(4/3)\mu_B B$ for $\Delta m_j = \pm 1$

The frequent error is to use one g factor for both levels, which collapses the pattern into the normal triplet. The upper and lower levels shift by different amounts, and each component's shift is their difference.

(c) COMPARE · U3 **[2]**

Compare the fine-structure splitting with $\mu_B B$ at 0.500 T for each atom.

1 1 mark: $\mu_B B = (5.788 \times 10^{-5}\ \text{eV T}^{-1})(0.500\ \text{T}) = 2.89 \times 10^{-5}$ eV

2 1 mark: potassium: 7.16×10^{-3} eV$/2.89 \times 10^{-5}$ eV ≈ 250; lithium: 4.16×10^{-5} eV$/2.89 \times 10^{-5}$ eV ≈ 1.4, so the same field is a small perturbation for potassium but comparable with the spin-orbit energy for lithium

Whether a field is weak or strong is a comparison, not a property of the field. The same 0.500 T is 250 times too weak to disturb potassium's spin-orbit coupling and comparable with lithium's.

(d) JUSTIFY · U1 **[2]**

Justify, for each atom, whether the D_1 pattern you found in (b) describes its spectrum at 0.500 T.

1 1 mark: potassium: the Zeeman energy is about 250 times smaller than the spin-orbit energy, so j and m_j remain good quantum numbers, $^2P_{1/2}$ and $^2P_{3/2}$ do not mix, and the four-component pattern of (b) applies

2 1 mark: lithium: the two energies are comparable, so the field mixes $^2P_{1/2}$ and $^2P_{3/2}$ sublevels of the same m_j; j is no longer a good quantum number, the shifts are not linear in B, and the weak-field pattern does not apply (the intermediate regime)

Candidates who classify by field strength alone give the same answer for both atoms. Which quantum numbers are good is decided by which interaction is larger, and for lithium at this field neither dominates.

(e) PREDICT · U1 **[2]**

Predict the appearance of the lithium resonance line at 5.00 T, naming the good quantum numbers.

1 1 mark: at 5.00 T, $\mu_B B = 2.89 \times 10^{-4}$ eV, about seven times the fine-structure splitting, so the Paschen–Back limit is approached: spin and orbit decouple, and m_l and m_s, not j and m_j, are the good quantum numbers

2 1 mark: each state shifts by $(m_l + 2m_s)\mu_B B$; with $\Delta m_s = 0$ and $\Delta m_l = 0, \pm 1$ the line becomes a triplet at 0 and $\pm\mu_B B = \pm 2.89 \times 10^{-4}$ eV, like the normal Zeeman pattern, with each component slightly split by the residual spin-orbit energy

The error is to keep the Landé g factors at high field. Once spin and orbit decouple g_J describes nothing; each state's shift is $(m_l + 2m_s)\mu_B B$, and the rule $\Delta m_s = 0$ does the rest.

14

CORE COURSE

STRUCTURED · DEMANDING · UPIV-14.03 · UPIV-14.04 · UPIV-14.05 · UPIV-14.07 **[12]**

The muon decays by $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ with a branching fraction of essentially 1 and a mean lifetime $\tau_\mu = 2.197 \times 10^{-6}$ s. The τ^- lepton decays through the same charged weak current into $e^- + \bar{\nu}_e + \nu_\tau$ with branching fraction $B_e = 0.178$, and also into $\mu^- + \bar{\nu}_\mu + \nu_\tau$ and into hadrons. Take $m_\mu c^2 = 105.66$ MeV, $m_\tau c^2 = 1776.9$ MeV and the Fermi constant $G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV^{-2}. In natural units, with $\hbar = c = 1$, a decay width $\Gamma = \hbar/\tau$ has the dimensions of energy.

(a) STATE · U1 **[2]**

State the values of L_e, L_μ and L_τ before and after the decay $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, and state which conservation law forbids $\tau^- \to e^- + \gamma$.

1 1 mark: before, $(L_e, L_\mu, L_\tau) = (0, 0, 1)$; after, e^- (1, 0, 0) + $\bar{\nu}_e$ (−1, 0, 0) + ν_τ (0, 0, 1) = (0, 0, 1), so each lepton-family number is conserved

2 1 mark: $\tau^- \to e^- + \gamma$ changes L_τ from 1 to 0 and L_e from 0 to 1, so it violates conservation of lepton-family number, although charge, energy and total lepton number all balance

Checking only the total lepton number $L_e + L_\mu + L_\tau$ leaves $\tau^- \to e^-\gamma$ allowed. It is the separate family numbers that forbid it, and searches for that decay are searches for exactly this violation.

(b) DESCRIBE · U1 **[2]**

Describe the lowest-order Feynman diagram for $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, and give the power of the weak coupling g_W in its amplitude and in its rate.

1 1 mark: at one vertex the τ^- becomes a ν_τ by emitting a virtual W^-, which propagates to a second vertex where it becomes e^- and $\bar{\nu}_e$

2 1 mark: two vertices give an amplitude proportional to g_W^2 and a rate proportional to its square, g_W^4; at energies far below M_W the propagator is about $1/M_W^2$, so the rate goes as $(g_W^2/M_W^2)^2$, that is as G_F^2

A Z^0 exchange cannot turn a charged lepton into a neutrino, since charge would not balance at the vertex. The rate going as g_W^4 rather than g_W^2 is the step most often missed.

(c) SHOW THAT · U2 **[2]**

Show that if the width depends only on G_F and on the mass m of the decaying lepton, the result of (b) requires $\Gamma \propto G_F^2 m^5$.

1 1 mark: in natural units G_F has dimensions of energy^{-2} and Γ of energy, so writing Γ ∝ G_F^a m^b requires $-2a + b = 1$

2 1 mark: the rate goes as G_F^2, so a = 2 and b = 5, giving $\Gamma \propto G_F^2 m^5$

The m^5 law is why a lepton 16.8 times heavier opens each such channel about 1.3×10^6 times faster. The argument holds only while the final-state masses are negligible, which is true for $\tau \to e\bar{\nu}\nu$ and $\mu \to e\bar{\nu}\nu$ alike.

(d) ESTIMATE · U3 **[3]**

Estimate the mean lifetime of the τ, stating the assumptions you make.

1 1 mark: assumes lepton universality, so the W couples equally to τ and μ, and neglects the electron and neutrino masses, so $\Gamma(\tau \to e\bar{\nu}\nu)/\Gamma(\mu \to e\bar{\nu}\nu) = (m_\tau/m_\mu)^5 = (16.82)^5 = 1.35 \times 10^6$

2 1 mark: the partial width is $\Gamma(\tau \to e\bar{\nu}\nu) = B_e \hbar/\tau_\tau$, so $\tau_\tau = B_e \tau_\mu (m_\mu/m_\tau)^5$

3 1 mark: $\tau_\tau = 0.178 \times 2.197 \times 10^{-6}\ \text{s} \times 7.43 \times 10^{-7} = 2.91 \times 10^{-13}\ \text{s}$

Omitting the branching fraction gives 1.6×10^{-12} s, more than five times too long. The τ is heavy enough to decay into muons and hadrons as well, and every extra channel shortens its life.

(e) EVALUATE · U4 **[3]**

The measured τ lifetime is 2.903×10^{-13} s. Evaluate what the agreement with your estimate tests, and use a count of the final states available to the virtual W^- to explain why B_e is close to 0.2.

1 1 mark: the estimate agrees with the measurement to about 0.2%, so the charged weak coupling of the τ equals that of the μ to within the few-tenths-of-a-percent precision of the inputs: lepton universality holds across generations

2 1 mark: the W^- from a τ can materialise as $e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$ or $d\bar{u}$, and the quark pair counts three times, once for each colour, giving five channels of nearly equal weight and B_e ≈ 1/5

3 1 mark: the measured 0.178 lies below 1/5 because strong-interaction (QCD) corrections raise the hadronic width: 1/B_e = 5.6 requires the quark channels to count as about 3.6 instead of 3, an enhancement of about 20%

Counting the quark channel once gives B_e ≈ 1/3. The factor of three from colour, confirmed by the measured branching fraction, is the same factor that appears in the R ratio.

15

CORE COURSE

STRUCTURED · DISCRIMINATING · UPIV-13.02 · UPIV-13.04 **[11]**

^{210}Po decays by α emission to ^{206}Pb with a half-life of 138.4 days. The atomic masses are ^{210}Po 209.982874 u, ^{206}Pb 205.974465 u and ^{4}He 4.002603 u, with 1 u = 931.494 MeV/c². In the Gamow model the α particle, of reduced mass μ, meets a Coulomb barrier $V(r) = Z_d Z_\alpha \alpha \hbar c/r$ that begins at the nuclear radius R, and tunnels out to the classical turning point b at which V(b) = Q. The WKB exponent is $2G = 2\alpha Z_d Z_\alpha \sqrt{(2\mu c^2/Q)}[\arccos\sqrt{(R/b)} - \sqrt{((R/b)(1 - R/b))}]$, and the half-life is proportional to e^(2G) divided by an assault frequency. Take $\alpha\hbar c = 1.440$ MeV fm, $\alpha = 1/137.0$, $\mu c^2 = 3.66 \times 10^3$ MeV, R = 9.0 fm, $Z_d = 82$ and $Z_\alpha = 2$.

(a) CALCULATE · U3 **[2]**

Calculate the Q value of the decay.

1 1 mark: $Q = [m(^{210}\text{Po}) - m(^{206}\text{Pb}) - m(^4\text{He})]c^2 = (0.005806\ \text{u})(931.494\ \text{MeV/u})$, the electron masses cancelling when atomic masses are used throughout

2 1 mark: Q = 5.41 MeV

Atomic masses balance the electrons, 84 = 82 + 2. Subtracting electron masses on one side only shifts Q by about 1 MeV, which in a barrier exponent changes the half-life by many orders of magnitude.

(b) CALCULATE · U3 **[1]**

Calculate the outer turning point b.

1 1 mark: b = Z_dZ_ααℏc/Q = (82)(2)(1.440 MeV fm)/(5.41 MeV) = 43.7 fm, nearly five times R, so R/b = 0.21

The barrier is thick because Q is only a fifth of the 26 MeV Coulomb energy at R. Candidates who take b to be close to R have not used V(b) = Q, and a thin barrier could never produce half-lives spanning twenty orders of magnitude.

(c) SHOW THAT · U2 **[3]**

Using arccos√x ≈ π/2 − √x and √(x(1 − x)) ≈ √x for x = R/b much less than 1, show that 2G ≈ παZ_dZ_α√(2μc²/Q) − 4√(2αZ_dZ_αμc²R/ℏc), and hence that the second term does not depend on Q.

1 1 mark: the bracket becomes π/2 − 2√(R/b), so the first term is 2αZ_dZ_α√(2μc²/Q)(π/2) = παZ_dZ_α√(2μc²/Q)

2 1 mark: the second term is 4αZ_dZ_α√(2μc²/Q)√(R/b), and with b = Z_dZ_ααℏc/Q, √(R/b) = √(RQ/(Z_dZ_ααℏc))

3 1 mark: multiplying out, the factors of Q cancel: 4√[α²Z_d²Z_α²(2μc²/Q)(RQ/(Z_dZ_ααℏc))] = 4√(2αZ_dZ_αμc²R/ℏc), independent of Q

The algebra turns on substituting b = Z_dZ_ααℏc/Q inside √(R/b). Candidates who treat b as a fixed length miss that the second term is independent of Q, which is what lets the prediction in (d) proceed without R.

(d) PREDICT · U3 **[3]**

^{214}Po also decays by α emission to a lead isotope (Z_d = 82), with Q = 7.833 MeV. Assuming the same R, μ and assault frequency for both nuclei, predict the half-life of ^{214}Po from that of ^{210}Po using the result of (c).

1 1 mark: with R, μ and the assault frequency unchanged only the first term differs, so ln[t½(^{210}Po)/t½(^{214}Po)] = παZ_dZ_α√(2μc²)[1/√Q(^{210}Po) − 1/√Q(^{214}Po)]

2 1 mark: = (3.761)(85.56 MeV^(1/2))(0.4300 − 0.3573) MeV^(−1/2) = 23.4, the first terms being 138.4 and 115.0

3 1 mark: the ratio is e^23.4 = 1.4 × 10^{10}, so t½(^{214}Po) = (138.4 × 86 400 s)/(1.4 × 10^{10}) = 8.3 × 10^{-4} s

The exponent is so sensitive that an error of 0.1 MeV in the Q value of ^{210}Po alone changes the predicted ratio by a factor of nearly 4. Carry Q to at least three significant figures throughout.

(e) EVALUATE · U4 **[2]**

The measured half-life of ^{214}Po is 164 μs. Evaluate the model in the light of this comparison.

1 1 mark: the prediction, 0.83 ms, is about five times the measured 0.164 ms, yet the model spans a half-life ratio of order 10^{10}: the measured ln ratio is 25.0 against the predicted 23.4, so the Q dependence of the barrier accounts for nearly all of the ten orders of magnitude

2 1 mark: agreement to within a factor of a few is all the model can claim, because the half-life depends exponentially on R and on the approximations made; it also omits nuclear structure, and ^{210}Po, with the closed neutron shell N = 126, has a reduced probability of preforming an α cluster, lengthening its half-life relative to ^{214}Po (N = 130), in the direction of the discrepancy

Calling a factor of five a failure ignores the ten orders of magnitude the model spans. Calling it a success without saying what the model omits is not an evaluation either.

16

CORE COURSE

STRUCTURED · DISCRIMINATING · UPIV-02.07 · UPIV-03.02 · UPIV-03.03 **[11]**

A sample-return probe leaves a space station, which defines an inertial frame S, and travels in a straight line at 0.600c to a relay beacon at rest in S at a distance of 3.00 light-years. At the beacon it reverses in a negligibly short time and returns at 0.800c. Work in units in which c = 1 light-year per year, label events by their S coordinates (t, x), and take the departure as the event (0, 0). Let S′ be the inertial frame moving with the probe on the outbound leg and S″ the inertial frame moving with it on the return leg.

(a) CALCULATE · U3 **[3]**

Calculate the time that elapses on the station clock, and the proper time that elapses on the probe's clock, for the round trip.

1 1 mark: station: 3.00/0.600 = 5.00 y out and 3.00/0.800 = 3.75 y back, a total of 8.75 y

2 1 mark: probe, leg by leg with $\Delta\tau = \Delta t\sqrt{(1 - v^2/c^2)}$: 5.00 y × 0.800 = 4.00 y and 3.75 y × 0.600 = 2.25 y

3 1 mark: total proper time 6.25 y, so the probe returns 2.50 y younger than the station clock reads

Using a single γ for the whole trip, or one built from the average speed, is the error this part catches. Proper time accumulates leg by leg along the worldline.

(b) DETERMINE · U2 **[4]**

Determine the reading of the station clock that is simultaneous with the turnaround event, first in S′ and then in S″.

1 1 mark: the turnaround is the event (5.00 y, 3.00 ly); in S′, with v = +0.600c and γ = 1.25, $t' = \gamma(t - vx/c^2) = 1.25(5.00 - 1.80) = 4.00$ y, the probe's own clock reading

2 1 mark: a station event $(t_s, 0)$ has $t' = \gamma t_s$, and setting this equal to 4.00 y gives $t_s = 3.20$ y

3 1 mark: in S″, with v = −0.800c and γ = 5/3, $t'' = \gamma(t + 0.800x) = (5/3)(5.00 + 2.40) = 12.3$ y

4 1 mark: setting $\gamma t_s = 12.3$ y gives $t_s = 7.40$ y, so the station reading simultaneous with the turnaround jumps from 3.20 y to 7.40 y

The trap is the sign of v in S″. Using +0.800c puts the station clock at 2.60 y just after the turnaround, earlier than just before it, and no account of the return leg can then be reconciled with 8.75 y.

(c) SHOW THAT · U2 **[2]**

Show that the probe's account, with the station clock running slow on each leg plus the jump found in (b), reproduces the station's total of 8.75 y.

1 1 mark: in S′ the station clock runs slow by γ = 1.25, advancing 4.00 y/1.25 = 3.20 y; in S″ it runs slow by γ = 5/3, advancing 2.25 y × 0.600 = 1.35 y

2 1 mark: 3.20 y + (7.40 − 3.20) y + 1.35 y = 8.75 y, agreeing with the station's own clock

Candidates who apply time dilation to the station clock on both legs and stop there get 4.55 y and conclude that the station ages less than the probe. The missing 4.20 y is the change of simultaneity at the turnaround.

(d) EVALUATE · U1 **[2]**

Evaluate the claim: “The probe ages less because it accelerated, and the harder it accelerates at the turnaround, the more time it loses.”

1 1 mark: the ageing difference is fixed by the speeds and distances of the two inertial legs and is 2.50 y however brief the turnaround; the size of the acceleration does not appear in it, so the second half of the claim is false

2 1 mark: the first half is right only in a limited sense: the acceleration makes the probe's worldline non-inertial and breaks the symmetry, since the probe changes inertial frame and with it its simultaneity (the 4.20 y jump), while the station stays in one frame, and of all worldlines joining departure and return the inertial one has the greatest proper time

Acceleration is needed to break the symmetry but does not set the size of the effect. A candidate who cannot say where the 2.50 y comes from without mentioning acceleration has not separated the two.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPIV-12.07 · UPIV-12.08 **[11]**

The intrinsic carrier density n_i of an undoped semiconductor bar is found from Hall-effect and conductivity measurements at six temperatures, and each value of n_i carries a random uncertainty of ±3%. For a non-degenerate intrinsic semiconductor with a temperature-independent band gap E_g, n_i = CT^(3/2)exp(−E_g/2kT), where C is a constant. Take k = 8.617 × 10^{-5} eV K^{-1}.

(a) EXPLAIN · U1 **[2]**

Explain why a graph of ln(n_i/T^(3/2)) against 1/T is expected to be a straight line, and what its gradient gives.

1 1 mark: taking logarithms, ln(n_i/T^(3/2)) = ln C − (E_g/2k)(1/T), which is linear in 1/T with the constant intercept ln C

2 1 mark: the gradient is −E_g/2k, so E_g = −2k × gradient; the T^(3/2) comes from the effective densities of states of the two bands and must be divided out, or it curves the line

Candidates who plot ln n_i against 1/T without dividing by T^(3/2) are the subject of part (d). The linearisation must isolate a term that is exactly linear in 1/T.

(b) DETERMINE · U4 **[3]**

Determine E_g from the data.

1 1 mark: computes 1/T = 3.333, 2.941, 2.632, 2.381, 2.174 and 2.000 (× 10^{-3} K^{-1}) and ln(n_i/T^(3/2)) = 35.90, 37.64, 39.07, 40.17, 41.15 and 41.91

2 1 mark: plots or fits a straight line, the points lying on it within the ±0.03 expected from ±3% uncertainties, with gradient −4.5 × 10^3 K (least-squares −4.52 × 10^3 K; end points −4.51 × 10^3 K)

3 1 mark: E_g = 2(8.617 × 10^{-5} eV K^{-1})(4.52 × 10^3 K) = 0.78 eV (accept 0.77 eV to 0.79 eV)

Using $\log_{10}$ in place of ln with E_g = −2k × gradient understates the gap by a factor of 2.30. Checking the scatter about the line against the stated ±3% is what shows the model fits.

(c) ESTIMATE · U4 **[2]**

Using the first and last rows only, estimate the uncertainty in E_g that follows from the ±3% uncertainty in n_i, stating the assumption you make about the two uncertainties.

1 1 mark: assuming the two uncertainties are independent, the difference of the two logarithms is uncertain by √2 × 0.03 = 0.042, over a span in 1/T of 1.333 × 10^{-3} K^{-1}, so the gradient is uncertain by about 32 K

2 1 mark: δE_g = 2(8.617 × 10^{-5} eV K^{-1})(32 K) = 0.0055 eV, so E_g = 0.779 ± 0.005 eV, an uncertainty of about 0.7%

Candidates who add the two 3% uncertainties linearly obtain 0.008 eV, which is acceptable if that assumption is stated. What matters is the size: a few meV, far smaller than the effects in (d) and (e).

(d) COMPARE · U4 **[2]**

A student instead plots ln n_i against 1/T and takes E_g = −2k × gradient. Compare the gap this gives with your value from (b), and account for the difference.

1 1 mark: ln n_i = 44.46, 46.39, 47.98, 49.23, 50.34 and 51.23 gives a gradient of about −5.09 × 10^3 K and an apparent gap of 0.88 eV, about 0.10 eV (13%) larger than 0.78 eV

2 1 mark: ln n_i contains (3/2)ln T, which rises with T and so steepens the line; its contribution to the apparent gap is about 3kT at the mid-range temperature of about 390 K, 3(8.6 × 10^{-5} eV K^{-1})(390 K) ≈ 0.10 eV, some twenty times the random uncertainty from (c)

The discriminating step is recognising that the 0.10 eV difference is systematic and nearly twenty times the random uncertainty. A slowly varying power of T is invisible by eye on a logarithmic plot and still shifts the gap by 13%.

(e) EVALUATE · U4 **[2]**

Optical absorption measured on the same bar at 400 K shows an absorption edge at 0.66 eV. Semiconductor gaps typically narrow as the temperature rises. Evaluate whether this contradicts your value from (b).

1 1 mark: if $E_g(T) = E_{g0} - \beta T$, then $\exp(-E_g/2kT) = \exp(\beta/2k)\exp(-E_{g0}/2kT)$, so the linear term changes only the constant C, and the gradient in (b) returns E_{g0}, the gap extrapolated to 0 K, not the gap at the measurement temperatures

2 1 mark: there is no contradiction: both results fit $E_{g0} = 0.78$ eV with $\beta = (0.78 - 0.66)\ \text{eV}/(400\ \text{K}) = 3.0 \times 10^{-4}\ \text{eV K}^{-1}$, a typical narrowing, so the transport fit and the optical edge together determine both E_{g0} and β

Candidates who put the 0.12 eV difference down to experimental error have not compared it with the ±0.005 eV found in (c). The transport fit and the optical edge measure different things, and together they determine both E_{g0} and β.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPIV-10.03 · UPIV-10.04 · UPIV-10.05 · UPIV-10.07 · UPIV-10.09 **[15]**

An electron is bound to a nucleus of charge Ze. With μ the reduced mass, the radial function $u(r) = rR(r)$ of a state of orbital quantum number l satisfies the radial Schrödinger equation $-(\hbar^2/2\mu)(d^2u/dr^2) + [-Ze^2/(4\pi\varepsilon_0 r) + l(l+1)\hbar^2/(2\mu r^2)]u = Eu$, with $E < 0$ for a bound state. Write $\kappa = \sqrt{(-2\mu E)}/\hbar$ and $a_\mu = 4\pi\varepsilon_0\hbar^2/(\mu e^2)$. You may use $\int_0^\infty r^k e^{-\beta r}\,dr = k!/\beta^{k+1}$ for integer $k \geq 0$. For the numerical parts take $a_0 = 5.292 \times 10^{-11}$ m and 13.606 eV as the Bohr radius and the ionisation energy of hydrogen for an infinitely heavy nucleus, $m_p/m_e = 1836$ and $h = 4.136 \times 10^{-15}$ eV s.

(a) SHOW THAT · U2 **[2]**

Show that a normalisable solution must behave as $e^{-\kappa r}$ at large r and as r^{l+1} near $r = 0$.

1 1 mark: as $r \to \infty$ both potential terms vanish, leaving $d^2u/dr^2 = \kappa^2 u$, with solutions $e^{\pm\kappa r}$; normalisability rejects $e^{+\kappa r}$, so $u \propto e^{-\kappa r}$

2 1 mark: as $r \to 0$ the centrifugal term dominates, $d^2u/dr^2 \approx l(l+1)u/r^2$, and $u = r^s$ requires $s(s-1) = l(l+1)$, so $s = l+1$ or $s = -l$; $u \propto r^{-l}$ makes $R = u/r$ diverge, non-normalisably for $l \geq 1$ and, for $l = 0$, as $1/r$, which fails the Schrödinger equation at the origin, so $u \propto r^{l+1}$

Rejecting $u \propto r^{-l}$ by saying only that it diverges misses the $l = 0$ case, where $R = 1/r$ is still square-integrable. There the solution fails the Schrödinger equation itself, since $\nabla^2(1/r)$ is a delta function.

(b) DERIVE · U2 **[5]**

By substituting $u = r^{l+1}e^{-\kappa r}$ into the radial Schrödinger equation, derive the conditions under which it is an exact solution, and hence derive κ and the energy E of this state. Identify its principal quantum number n.

1 1 mark: $u' = [(l+1)/r - \kappa]u$ and $u'' = [l(l+1)/r^2 - 2\kappa(l+1)/r + \kappa^2]u$

2 1 mark: on substitution the $1/r^2$ terms, $-(\hbar^2/2\mu)l(l+1)/r^2$ and $+l(l+1)\hbar^2/(2\mu r^2)$, cancel identically

3 1 mark: the $1/r$ terms cancel only if $\hbar^2\kappa(l+1)/\mu = Ze^2/(4\pi\varepsilon_0)$, so $\kappa = Z/[(l+1)a_\mu]$

4 1 mark: the constant terms give $E = -\hbar^2\kappa^2/(2\mu) = -Z^2\hbar^2/[2\mu a_\mu^2(l+1)^2] = -\mu Z^2 e^4/[2(4\pi\varepsilon_0)^2\hbar^2(l+1)^2]$

5 1 mark: u has no radial nodes, so $n = n_r + l + 1 = l + 1$, and $E_n = -Z^2e^2/(8\pi\varepsilon_0 a_\mu n^2)$, the Bohr energy with the reduced mass, for the state with $l = n - 1$

The $1/r$ balance is the quantisation condition. A candidate who fixes κ from the constant terms alone and leaves the $1/r$ terms unbalanced has a trial function, not a solution, and no discrete spectrum.

(c) DETERMINE · U2 **[3]**

For this state, determine the most probable radius r_mp and the mean radius ⟨r⟩ in terms of n, a_μ and Z.

1 1 mark: the radial probability density is P(r) ∝ |u|² = r^(2n)e^(−2Zr/(na_μ)), the r² Jacobian being already contained in u = rR

2 1 mark: dP/dr = 0 gives 2n/r = 2Z/(na_μ), so r_mp = n²a_μ/Z

3 1 mark: ⟨r⟩ = ∫r^(2n+1)e^(−βr) dr/∫r^(2n)e^(−βr) dr = (2n + 1)/β with β = 2Z/(na_μ), so ⟨r⟩ = n(n + ½)a_μ/Z

Multiplying |u|² by a further r² counts the Jacobian twice and moves the peak to n(n + 1)a_μ/Z. The r² is already inside u = rR.

(d) CALCULATE · U3 **[3]**

Circular Rydberg states, with l = n − 1, are used in microwave cavity experiments. For the n = 50 circular state of hydrogen, calculate the most probable radius and the frequency of the photon emitted in the transition to n = 49.

1 1 mark: $a_\mu = a_0(1 + m_e/m_p) = (5.292 \times 10^{-11}\ \text{m})(1.000545) = 5.295 \times 10^{-11}\ \text{m}$

2 1 mark: $r_{mp} = 2500 a_\mu = 1.32 \times 10^{-7}\ \text{m}$ (132 nm)

3 1 mark: $\Delta E = (13.606\ \text{eV}/1.000545)(1/49^2 - 1/50^2) = 2.24 \times 10^{-4}\ \text{eV}$, so $f = \Delta E/h = 5.42 \times 10^{10}\ \text{Hz}$ (54.2 GHz)

Omitting the reduced-mass correction shifts the frequency by about 30 MHz, a 0.05% error that a microwave measurement resolves easily.

(e) JUSTIFY · U1 **[2]**

Justify, using the electric-dipole selection rules, the claim that this state can decay by only one electric-dipole transition.

1 1 mark: electric-dipole transitions require Δl = ±1 (with Δm = 0, ±1), and spontaneous emission requires a final level of lower energy, n′ < 50

2 1 mark: Δl = +1 needs l′ = 50 and so n′ ≥ 51, which is higher in energy; Δl = −1 needs l′ = 48 and so n′ ≥ 49, leaving only n′ = 49, l′ = 48, the next circular state

Applying a selection rule to n instead of to l lists every n′ < 50. In hydrogen Δn is unrestricted; it is the constraint l′ ≤ n′ − 1 that leaves a single channel.

19

CORE COURSE

FREE RESPONSE · DERIVE A MODEL, THEN TEST IT AGAINST EVIDENCE · DISCRIMINATING · UPIV-15.06 · UPIV-15.07 **[14]**

Model the universe as a uniform, pressureless dust of mass density ρ(t) in which every separation grows in proportion to a scale factor a(t), with a = 1 today. A particle of mass m moving with the dust lies at distance r = a(t)x from an arbitrary origin, where x is constant, and by the shell theorem it is attracted only by the mass inside the sphere of radius r. Take $G = 6.674 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$, the present expansion rate $H_0 = \dot{a}/a = 67.4\ \text{km s}^{-1}\ \text{Mpc}^{-1}$, $1\ \text{Mpc} = 3.086 \times 10^{22}\ \text{m}$, the mass of a hydrogen atom as $1.674 \times 10^{-27}\ \text{kg}$ and $1\ \text{yr} = 3.156 \times 10^7\ \text{s}$.

(a) DERIVE · U2 **[4]**

Starting from conservation of energy for the particle, derive the Friedmann equation $(\dot{a}/a)^2 = (8\pi G/3)\rho - K/a^2$, where K is a constant, and state what the sign of K means for the future of the expansion in this Newtonian picture.

1 1 mark: the mass inside the sphere, $M = (4\pi/3)r^3\rho$, is fixed because the dust moves with the sphere, so ρa^3 is constant

2 1 mark: energy conservation for the particle: $\tfrac{1}{2}m\dot{r}^2 - GMm/r = E$, with E constant

3 1 mark: substituting $r = ax$, $\dot{r} = \dot{a}x$ and $M = (4\pi/3)a^3x^3\rho$ and dividing by $\tfrac{1}{2}ma^2x^2$ gives $(\dot{a}/a)^2 = (8\pi G/3)\rho + 2E/(mx^2a^2)$, so $K = -2E/(mx^2)$, a constant

4 1 mark: K > 0 (E < 0): the expansion halts and reverses; K < 0 (E > 0): it continues for ever; K = 0: the expansion rate falls to zero only as t → ∞

Candidates who hold ρ constant while the sphere grows lose the a^{-3} dilution, and with it the content of the derivation. The shell theorem is what makes the choice of origin irrelevant.

(b) CALCULATE · U3 **[3]**

The critical density ρ_c is the density for which $K = 0$. Calculate ρ_c today, and express it as a number of hydrogen atoms per cubic metre.

1 1 mark: $K = 0$ at $a = 1$ gives $\rho_c = 3H_0^2/(8\pi G)$, with $H_0 = (67.4 \times 10^3\ \text{m s}^{-1})/(3.086 \times 10^{22}\ \text{m}) = 2.18 \times 10^{-18}\ \text{s}^{-1}$

2 1 mark: $\rho_c = 3(2.18 \times 10^{-18}\ \text{s}^{-1})^2/(8\pi \times 6.674 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}) = 8.5 \times 10^{-27}\ \text{kg m}^{-3}$

3 1 mark: $\rho_c/m_H = (8.53 \times 10^{-27}\ \text{kg m}^{-3})/(1.674 \times 10^{-27}\ \text{kg}) = 5.1$ hydrogen atoms per cubic metre

Mixing kilometres in the speed with metres in the megaparsec leaves H_0 a thousand times too small and ρ_c a million times too small.

(c) DERIVE · U2 **[3]**

For $K = 0$ and $\rho \propto a^{-3}$, derive $a(t)$ from the Friedmann equation, taking $a = 0$ at $t = 0$, and hence obtain the age t_0 of the universe in terms of H_0 and in gigayears.

1 1 mark: $\rho = \rho_c\, a^{-3} = (3H_0^2/8\pi G)a^{-3}$ turns the Friedmann equation into $\dot{a}^2 = H_0^2/a$, so a^(1/2) da = H_0 dt

2 1 mark: integrating from $a = 0$ at $t = 0$ gives (2/3)a^(3/2) = H_0t, so a(t) = (3H_0t/2)^(2/3) $\propto$ t^(2/3)

3 1 mark: $a = 1$ today gives $t_0 = 2/(3H_0) = 3.05 \times 10^{17}\ \text{s} = 9.7$ Gyr

Quoting $1/H_0 = 14.5$ Gyr as the age assumes the expansion rate has never changed. A decelerating universe was expanding faster in the past, and so is younger than $1/H_0$.

(d) EVALUATE · U4 **[2]**

The oldest globular-cluster stars in the Milky Way are dated at about 12.5 Gyr. Evaluate the flat, matter-only model in the light of this, including the value of H_0 the model would need.

1 1 mark: 9.7 Gyr is younger than stars that formed within the universe, so the model with the measured H_0 is excluded; an age of 12.5 Gyr would need $H_0 = 2/(3t_0) \approx 52$ km s^{-1} Mpc^{-1}, well below the measured value of about 67 km s^{-1} Mpc^{-1}

2 1 mark: the model must change, not H_0: a component that makes the expansion accelerate, such as a cosmological constant, means the expansion was slower in the past and the universe older for the same H_0 (about 13.8 Gyr for matter and dark-energy fractions of 0.315 and 0.685 with $H_0 = 67.4$ km s^{-1} Mpc^{-1}), and pressureless dust alone cannot supply it

An age younger than the oldest stars is a contradiction, not a tension. The discriminating answer names what must change, and checks that lowering H_0 to fit would conflict with its measurement.

(e) CALCULATE · U3 **[2]**

The cosmic microwave background was released at redshift $z = 1090$. Calculate the scale factor at that time and the factor by which the matter density then exceeded its present value.

1 1 mark: $a = 1/(1 + z) = 1/1091 = 9.17 \times 10^{-4}$

2 1 mark: $\rho/\rho_0 = a^{-3} = 1091^3 = 1.30 \times 10^9$

Using $a = 1/z$ is a negligible error at $z = 1090$ but a factor of two at $z = 1$. Redshift measures the stretch of wavelengths, $\lambda_{obs}/\lambda_{emit} = 1 + z = 1/a$.

University Physics V · Midterm Examination

Examines Units 1 to 10: states and Hermitian operators in Dirac notation, the experimental case for quanta, Born's rule and unitary evolution, the box, the finite well and tunnelling, the oscillator's ladder, commutators and the Robertson bound, and angular momentum and spin one-half.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**72**	**14**	**8**

SECTION — **UPV · Midterm · University Physics V — Midterm examination**

IN THE FORMAT OF — A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation

DEMAND — **Routine 5 · Demanding 7 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPV-01 · UPV-02 · UPV-03 · UPV-04 · UPV-05 · UPV-06 · UPV-07 · UPV-08 · UPV-09 · UPV-10

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Physics V · Midterm Examination

Answer all questions. 72 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-03.01 **[1]**

$|a\rangle$ and $|b\rangle$ are orthonormal kets, and a system is prepared in $|\psi\rangle = (|a\rangle + i|b\rangle)/\sqrt{2}$. Which of the following kets describes the same physical state as $|\psi\rangle$?

A $(|a\rangle - i|b\rangle)/\sqrt{2}$

B $(i|a\rangle + |b\rangle)/\sqrt{2}$

C $(i|a\rangle - |b\rangle)/\sqrt{2}$

D $(|a\rangle + |b\rangle)/\sqrt{2}$

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.03 **[1]**

In a semiclassical model of the interaction of light with matter, the electromagnetic field is a classical wave and only the electrons in the detector are treated quantum mechanically. Which observation can this model NOT account for?

A Light from a single trapped atom, re-excited by one short laser pulse at a time, falls on a 50:50 beam splitter with a detector at each output, and coincident clicks at the two detectors within one pulse are far rarer than any classical wave of the same mean intensity allows.

B The maximum kinetic energy of photoelectrons increases linearly with the frequency of the light, with gradient h.

C No photoelectrons are emitted below a threshold frequency, however intense the light.

D Photoelectrons are emitted with no measurable delay after very weak light is switched on.

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-03.02 · UPV-05.06 **[1]**

A particle is in the second excited state, $n = 3$, of an infinitely deep one-dimensional well occupying $0 < x < L$. What is the probability of finding it in the first quarter of the well, $0 < x < L/4$?

A 0.15

B 0.20

C 0.25

D 0.30

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-06.04 **[1]**

A steady beam of non-relativistic particles, each with kinetic energy E, travels in the +x direction and meets a potential step of height $V_0 = 3E/4$ at $x = 0$. What fraction of the incident probability current is transmitted past the step?

A 1/9

B 8/9

C 16/9

D 16/25

Answer ☐

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-07.02 · UPV-07.05 **[1]**

For a one-dimensional harmonic oscillator of mass m and angular frequency ω, the position operator is $x = \sqrt{(\hbar/2m\omega)}(a + a^\dagger)$, with $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$. What is the matrix element $\langle 3|x^2|1\rangle$?

A 0

B $\sqrt{2}\,\hbar/(2m\omega)$

C $\sqrt{6}\,\hbar/(2m\omega)$

D $3\hbar/(2m\omega)$

Answer ☐

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-08.07 **[1]**

Two Hermitian operators A and B on a finite-dimensional state space satisfy $[A, B] = 0$. Which statement must be true?

A Every eigenvector of A is also an eigenvector of B.

B $\Delta A\,\Delta B = 0$ in every state.

C A measurement of B made straight after a measurement of A leaves the state produced by the A measurement unchanged.

D There is an orthonormal basis made entirely of simultaneous eigenvectors of A and B.

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.04 **[1]**

X-ray photons of wavelength 60.0 pm scatter from free electrons initially at rest, and photons scattered through 120° are detected. Take $h/(m_e c) = 2.426$ pm and $hc = 1240$ eV nm. What is the kinetic energy of an electron that has scattered one of these detected photons?

A 0.41 keV

B 1.18 keV

C 19.5 keV

D 20.7 keV

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-10.06 · UPV-10.07 **[1]**

Beam P consists of spin-½ atoms each prepared in $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z. Beam Q is an equal statistical mixture of atoms in $|\uparrow\rangle$ and atoms in $|\downarrow\rangle$. Which measurement, repeated on many atoms from each beam, distinguishes the two beams?

A S_x: every atom from P gives $+\hbar/2$, while atoms from Q give $+\hbar/2$ and $-\hbar/2$ equally often.

B S_z: each atom from P gives either result with probability ½, whereas each atom in Q already has a definite value.

C S_y: the relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ in P can only be detected by a measurement along y.

D None: in both beams each atom is found up along z with probability ½, so no measurement can separate them.

Answer ☐

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9 CORE COURSE

STRUCTURED · ROUTINE · UPV-01.06 · UPV-01.07 · UPV-04.03 · UPV-04.04 **[10]**

An electron shared between two coupled quantum dots is described in the orthonormal basis $\{|1\rangle, |2\rangle\}$ of the two dot states by a Hamiltonian with matrix elements $H_{11} = 4.0$ meV, $H_{22} = -4.0$ meV, $H_{12} = 3.0i$ meV and $H_{21} = -3.0i$ meV. At $t = 0$ the electron is in $|1\rangle$. Take $\hbar = 6.582 \times 10^{-16}$ eV s.

(a) SHOW THAT **[2]**

Show that the energy eigenvalues are $E_+ = +5.0$ meV and $E_- = -5.0$ meV.

(b) DETERMINE **[2]**

Determine the normalised eigenvector $|E_+\rangle$ as a combination of $|1\rangle$ and $|2\rangle$.

(c) CALCULATE **[1]**

Calculate the probability that a measurement of the energy at $t = 0$ gives $+5.0$ meV.

(d) DERIVE **[3]**

Starting from the time-dependent Schrödinger equation, derive an expression for the probability $P_2(t)$ that the electron is found in $|2\rangle$ at time t.

(e) CALCULATE **[2]**

Calculate the earliest time after $t = 0$ at which P_2 is greatest, and state that greatest value.

10 CORE COURSE

STRUCTURED · DEMANDING · UPV-05.03 · UPV-06.01 · UPV-06.02 **[10]**

An electron of mass $m_e = 9.109 \times 10^{-31}$ kg is bound in a symmetric one-dimensional square well occupying $-a < x < a$, with $a = 0.500$ nm. Its potential energy is zero inside the well and $V_0 = 1.20$ eV outside it. Measure the energy E from the bottom of the well and write $k = \sqrt{(2m_eE)}/\hbar$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_eV_0)}$. The even bound states satisfy $z \tan z = \sqrt{(z_0^2 - z^2)}$ and the odd bound states satisfy $-z \cot z = \sqrt{(z_0^2 - z^2)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) DETERMINE **[3]**

Calculate z_0, and hence determine how many bound states the well holds.

(b) SHOW THAT **[2]**

Show that $z = 1.15$ satisfies the even-parity condition to within 1%.

(c) COMPARE **[3]**

Calculate the ground-state energy of the electron, and compare it with the ground-state energy the electron would have in an infinitely deep well of the same width $2a$.

(d) SKETCH **[2]**

On one set of axes of ψ against x, from $x = -2a$ to $x = +2a$, sketch the wavefunctions of the two bound states. Mark $x = -a$ and $x = +a$.

11 CORE COURSE

STRUCTURED · DEMANDING · UPV-10.04 · UPV-10.07 · UPV-10.08 **[10]**

Slow neutrons, which have spin ½ and gyromagnetic ratio $\gamma_n = -1.832 \times 10^8$ rad s^{-1} T^{-1}, leave a polariser in the state $|+z\rangle$. The spin operator is $S = (\hbar/2)\sigma$; in the $\{|+z\rangle, |-z\rangle\}$ basis σ_y has rows $(0, -i)$ and $(i, 0)$, and $\exp(-i\alpha\sigma_y) = \cos\alpha\, I - i \sin\alpha\, \sigma_y$ for real α. An analyser set along a unit vector $\hat{n}$ transmits only neutrons found in the $+\hbar/2$ eigenstate of $\hat{n}\cdot S$, and blocks the rest.

(a) CALCULATE **[2]**

The neutrons pass first through an analyser whose $\hat{n}$ lies in the xz-plane at 45° to the $+z$ axis, and then through an analyser set along $-z$, which transmits only $|-z\rangle$. Calculate the fraction of the neutrons leaving the polariser that emerge from the second analyser.

(b) EXPLAIN **[2]**

With the first analyser removed, no neutrons emerge from the $-z$ analyser. Explain why inserting an analyser that removes neutrons from the beam increases the number emerging, and identify which step in the sequence is not unitary.

(c) SHOW THAT **[3]**

The 45° analyser is removed. Instead, the $|+z\rangle$ neutrons cross a region of uniform magnetic field B along $+y$, where the Hamiltonian is $H = -\gamma_n B\, S_y$, taking a time t. Show that on leaving the field the spin state is $\cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$, where $\omega = |\gamma_n|B$.

(d) CALCULATE **[2]**

The neutrons travel at 2200 m s^{-1} through a field region 5.00 cm long. Calculate the field B that turns $|+z\rangle$ into $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$.

(e) PREDICT **[1]**

Predict the fraction of the neutrons that a $+z$ analyser placed after the field region would transmit if the same field pointed along $+z$ instead of $+y$, and give the reason.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UPV-07.07 · UPV-07.09 · UPV-08.05 **[10]**

A single $^{40}Ca^{+}$ ion of mass $m = 6.64 \times 10^{-26}$ kg is held in an ion trap whose potential along the trap axis is harmonic with angular frequency $\omega = 2\pi \times 1.20$ MHz $= 7.54 \times 10^{6}$ rad s^{-1}. Along that axis the position and momentum operators are $x = \sqrt{(\hbar/2m\omega)}(a + a^{\dagger})$ and $p = i\sqrt{(m\hbar\omega/2)}(a^{\dagger} - a)$, with $[a, a^{\dagger}] = 1$ and number states $|n\rangle$ of energy $(n + \tfrac{1}{2})\hbar\omega$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) ESTIMATE **[2]**

After laser cooling, the axial motion is in thermal equilibrium at $T = 0.50$ mK. Estimate the mean number of vibrational quanta $\bar{n}$, stating the approximation you make and why it is justified.

(b) SHOW THAT **[2]**

Further cooling and a laser pulse prepare the state $|\psi(0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Show that $\langle x\rangle(t) = \sqrt{(\hbar/2m\omega)}\cos\omega t$.

(c) DETERMINE **[4]**

Determine the product $\Delta x\,\Delta p$ at $t = 0$ and at $\omega t = \pi/4$, expressing each as a multiple of $\hbar$.

(d) EXPLAIN **[2]**

Explain why $\Delta x\,\Delta p$ exceeds $\hbar/2$ at every time for this state, although the ground state $|0\rangle$ reaches $\hbar/2$.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UPV-06.05 · UPV-06.09 **[10]**

A scanning tunnelling microscope tip is held above a clean metal surface at a fixed bias of 0.100 V. With the feedback loop switched off, the tip is withdrawn in steps, and the tunnelling current I is recorded against Δd, the increase in the tip–surface gap from its starting value. In the opaque-barrier limit $I = I_0 \exp(-2\kappa \Delta d)$, where $\kappa = \sqrt{(2 m_e \varphi)/\hbar}$ and φ is the apparent barrier height. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

Tunnelling current as the tip is withdrawn, at a bias of 0.100 V

Reading	Increase in gap Δd (pm)	Tunnelling current I (nA)
1	0	2.50
2	40	1.13
3	80	0.471
4	120	0.218
5	160	0.0924
6	200	0.0420

(a) STATE **[1]**

State the gradient of a graph of ln(I/nA) against Δd, in terms of κ.

(b) DETERMINE **[3]**

Determine κ from the data.

(c) CALCULATE **[2]**

Calculate the apparent barrier height φ in electronvolts.

(d) ESTIMATE **[2]**

In constant-current imaging the feedback loop moves the tip to hold I fixed. Estimate the change in gap that changes the current by 10%, stating the assumption you make.

(e) EVALUATE **[2]**

Photoemission gives the work function of this surface as 4.60 eV. A student concludes that either the STM data or the tunnelling model must be wrong. Evaluate this conclusion.

14 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-08.03 · UPV-09.03 · UPV-09.04 **[14]**

For Hermitian operators A and B and a normalised state $|\psi\rangle$, write $\langle A\rangle = \langle\psi|A|\psi\rangle$ and $(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$. The Cauchy–Schwarz inequality states that $\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$ for any kets $|f\rangle$ and $|g\rangle$. The components of an angular momentum obey $[J_x, J_y] = i\hbar J_z$ and its cyclic permutations, $J_\pm = J_x \pm iJ_y$, and the simultaneous eigenstates $|j, m\rangle$ of J^2 and J_z satisfy $J^2|j, m\rangle = \hbar^2 j(j + 1)|j, m\rangle$, $J_z|j, m\rangle = m\hbar|j, m\rangle$ and $J_\pm|j, m\rangle = \hbar\sqrt{[j(j + 1) - m(m \pm 1)]}\,|j, m \pm 1\rangle$.

(a) DERIVE **[4]**

Starting from the Cauchy–Schwarz inequality, derive the Robertson uncertainty relation, a lower bound on $\Delta A\,\Delta B$ in terms of the commutator $[A, B]$.

(b) SHOW THAT **[4]**

Show that in the state $|j, m\rangle$, $\langle J_x\rangle = \langle J_y\rangle = 0$ and $\langle J_x^2\rangle = \langle J_y^2\rangle = \tfrac{1}{2}\hbar^2[j(j + 1) - m^2]$.

(c) DETERMINE **[3]**

Determine the states $|j, m\rangle$ in which $\Delta J_x\,\Delta J_y$ is exactly equal to the Robertson bound for J_x and J_y.

(d) EVALUATE **[3]**

A student argues: 'In the state $|j = 1, m = 0\rangle$ the Robertson bound for J_x and J_y is zero, so a careful enough experiment could find both J_x and J_y with no spread in that state.' Evaluate this argument.

MARK SCHEME

University Physics V · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-03.01 **[1]**

|a⟩ and |b⟩ are orthonormal kets, and a system is prepared in |ψ⟩ = (|a⟩ + i|b⟩)/√2. Which of the following kets describes the same physical state as |ψ⟩?

KEY **C** **(i|a⟩ − |b⟩)/√2**

C is i|ψ⟩: multiplying both components by the same unit-modulus factor changes no probability in any basis, so it is the same ray in Hilbert space. A reverses the sign of the relative phase, and B is simply i times A, so A and B are the same state as each other but not the state |ψ⟩; the pair is a reminder that a phase can only be factored out if it multiplies both components. D is tempting because it has the same moduli and so gives the same probabilities, one half each, in the {|a⟩, |b⟩} basis. But a measurement in the basis (|a⟩ ± |b⟩)/√2 gives the '+' outcome with certainty for D and with probability one half for |ψ⟩, so a relative phase is observable even though a global phase is not.

U1

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.03 **[1]**

In a semiclassical model of the interaction of light with matter, the electromagnetic field is a classical wave and only the electrons in the detector are treated quantum mechanically. Which observation can this model NOT account for?

KEY **A** **Light from a single trapped atom, re-excited by one short laser pulse at a time, falls on a 50:50 beam splitter with a detector at each output, and coincident clicks at the two detectors within one pulse are far rarer than any classical wave of the same mean intensity allows.**

B, C and D are the three features usually quoted as proof that light is quantised, which is exactly why they tempt. Treat the light as a classical wave of frequency $f = \omega/2\pi$ perturbing a quantised electron: first-order perturbation theory drives transitions only to states lying $\hbar\omega = hf$ above the initial one, giving $K_{max} = hf - \varphi$, a straight line of gradient h against f (B), and a threshold at $hf = \varphi$ whatever the amplitude (C). The same theory gives a transition rate proportional to the intensity from the moment the light is switched on, so the first electrons can leave at once (D); the long delay was a prediction for a classical electron slowly soaking up energy, not for a quantum electron in a classical field. A is different in kind. For any classical field divided equally by a beam splitter, $\langle I_1 I_2 \rangle \geq \langle I_1 \rangle \langle I_2 \rangle$, so the coincidence ratio $g^{(2)}(0)$ cannot fall below 1. A single atom gives up its energy one quantum per excitation, and the observed suppression of coincidences requires a quantised field: it is the experiment that closes the gap the photoelectric effect leaves open.

U4

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-03.02 · UPV-05.06 **[1]**

A particle is in the second excited state, n = 3, of an infinitely deep one-dimensional well occupying 0 < x < L. What is the probability of finding it in the first quarter of the well, 0 < x < L/4?

KEY **D** **0.30**

With $\psi_3(x) = \sqrt{(2/L)}\sin(3\pi x/L)$, the integral of $|\psi_3|^2$ from 0 to L/4 is $1/4 + 1/(6\pi) = 0.30$. C is the classical value, a quarter of the probability in a quarter of the well; the quantum result can differ from it by up to $1/(2n\pi)$, which is appreciable only at small n. For n = 3 the quantum value is larger because the interval ends three-quarters of the way through the first lobe, which spans 0 < x < L/3: it keeps that lobe's maximum at x = L/6 and leaves out only the low-density region next to the node at x = L/3. B is $1/4 - 1/(6\pi) = 0.20$, which follows from writing the antiderivative of $\sin^2 u$ as $u/2 + (\sin 2u)/4$, the antiderivative of $\cos^2 u$; it is in fact the probability for the next quarter, L/4 < x < L/2, since $|\psi_3|^2$ is symmetric about x = L/2 and the two quarters must share one half. A is 0.30 halved, the result of dropping the normalisation factor and using $\sin(3\pi x/L)/\sqrt{L}$ as the wavefunction.

U3

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-06.04 **[1]**

A steady beam of non-relativistic particles, each with kinetic energy E, travels in the +x direction and meets a potential step of height $V_0 = 3E/4$ at x = 0. What fraction of the incident probability current is transmitted past the step?

KEY **B** **8/9**

With $k_1 \propto \sqrt{E}$ and $k_2 \propto \sqrt{(E - V_0)}$, $k_2 = k_1/2$. The transmitted amplitude is $t = 2k_1/(k_1 + k_2) = 4/3$, and the flux ratio is $T = (k_2/k_1)|t|^2 = 8/9$, so that $R + T = 1/9 + 8/9 = 1$. C is $|t|^2$ on its own: an amplitude ratio may exceed 1 because the wave slows beyond the step, and leaving out the velocity factor k_2/k_1 produces a 'probability' greater than one. A is the reflection coefficient $R = [(k_1 - k_2)/(k_1 + k_2)]^2$, the right calculation answering the wrong question. D follows from taking $k_2/k_1 = 1/4$, the ratio of kinetic energies, instead of its square root.

U3

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-07.02 · UPV-07.05 **[1]**

For a one-dimensional harmonic oscillator of mass m and angular frequency ω, the position operator is $x = \sqrt{(\hbar/2m\omega)}(a + a\dagger)$, with $a|n\rangle = \sqrt{n}\,|n - 1\rangle$ and $a\dagger|n\rangle = \sqrt{(n + 1)}\,|n + 1\rangle$. What is the matrix element $\langle 3|x^2|1\rangle$?

KEY **C** **$\sqrt{6}\,\hbar/(2m\omega)$**

Expanding $(a + a\dagger)^2$ gives a^2, $a\dagger^2$, $aa\dagger$ and $a\dagger a$; only $a\dagger^2$ can take $|1\rangle$ to $|3\rangle$, and $a\dagger a\dagger|1\rangle = \sqrt{2}\,\sqrt{3}\,|3\rangle$, so the element is $\sqrt{6}\,\hbar/(2m\omega)$. A carries over the selection rule for x itself, which connects only states whose n differ by one; x^2 connects states whose n differ by zero or two, and parity allows this element because $|1\rangle$ and $|3\rangle$ are both odd. B uses $a\dagger|n\rangle = \sqrt{n}\,|n + 1\rangle$, putting the square root on the wrong index. D is the diagonal element $\langle 1|x^2|1\rangle = (2n + 1)\hbar/(2m\omega)$ with n = 1, which answers a different question.

U2

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-08.07 **[1]**

Two Hermitian operators A and B on a finite-dimensional state space satisfy [A, B] = 0. Which statement must be true?

KEY **D** **There is an orthonormal basis made entirely of simultaneous eigenvectors of A and B.**

D is the theorem: B maps each eigenspace of A into itself, and diagonalising B within each eigenspace produces a common orthonormal eigenbasis. A is guaranteed only when A is non-degenerate, because within a degenerate eigenspace of A nothing forces B to be a multiple of the identity. Take A = L^2 and B = L_z on the l = 0 and l = 1 states, a four-dimensional space: (|1, 1⟩ + |1, 0⟩)/√2 is an eigenvector of L^2 but not of L_z. C fails for the same reason: if the L^2 measurement leaves that superposition, a measurement of L_z projects it onto |1, 1⟩ or |1, 0⟩, changing the state although the value of L^2 survives, so compatibility protects the eigenvalue, not the state. B mistakes a lower bound for a value: the Robertson bound ½|⟨[A, B]⟩| is zero, but (|0, 0⟩ + |1, 1⟩)/√2 has $\Delta L^2 = \hbar^2$ and $\Delta L_z = \hbar/2$, so $\Delta A\,\Delta B = \hbar^3/2$ in that state.

U1

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.04 **[1]**

X-ray photons of wavelength 60.0 pm scatter from free electrons initially at rest, and photons scattered through 120° are detected. Take $h/(m_e c)$ = 2.426 pm and hc = 1240 eV nm. What is the kinetic energy of an electron that has scattered one of these detected photons?

KEY **B** **1.18 keV**

The Compton shift is $\Delta\lambda = (h/m_e c)(1 - \cos 120°) = 2.426$ pm × 1.5 = 3.64 pm, so $\lambda' = 63.64$ pm, and the electron carries away the difference in photon energy: 20.67 keV − 19.48 keV = 1.18 keV. A comes from writing 1 − cos 120° = 0.5, losing the sign of cos 120° = −0.5; the shift is then a third of its true size, and the recoil energy falls to about a third as well. D is the energy of the incident photon and C that of the scattered photon. Each is a correct number for a different question, and both tempt because the recoil energy is small: the wavelength changes by only about 6%.

U3

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-10.06 · UPV-10.07 **[1]**

Beam P consists of spin-½ atoms each prepared in (|↑⟩ + |↓⟩)/√2, where |↑⟩ and |↓⟩ are the eigenstates of S_z. Beam Q is an equal statistical mixture of atoms in |↑⟩ and atoms in |↓⟩. Which measurement, repeated on many atoms from each beam, distinguishes the two beams?

KEY **A** **S_x: every atom from P gives $+\hbar/2$, while atoms from Q give $+\hbar/2$ and $-\hbar/2$ equally often.**

P is the pure state |+x⟩, with its Bloch vector along +x, while Q has density matrix I/2 and sits at the centre of the Bloch sphere, so an S_x measurement separates them completely. B describes a genuine difference in how the beams were prepared, but the S_z statistics are one half each for both beams, so this measurement cannot see the difference. C is right that the relative phase is what distinguishes a superposition from a mixture, but a real, positive relative phase puts the Bloch vector along x, not y: S_y gives one half each for P as well as for Q, and would only be the right choice for (|↑⟩ + i|↓⟩)/√2. D generalises from one basis to every basis, which is precisely the error that the interference term in P exposes.

U1

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9 CORE COURSE

STRUCTURED · ROUTINE · UPV-01.06 · UPV-01.07 · UPV-04.03 · UPV-04.04 **[10]**

An electron shared between two coupled quantum dots is described in the orthonormal basis $\{|1\rangle, |2\rangle\}$ of the two dot states by a Hamiltonian with matrix elements $H_{11} = 4.0$ meV, $H_{22} = -4.0$ meV, $H_{12} = 3.0i$ meV and $H_{21} = -3.0i$ meV. At $t = 0$ the electron is in $|1\rangle$. Take $\hbar = 6.582 \times 10^{-16}$ eV s.

(a) SHOW THAT · U2 **[2]**

Show that the energy eigenvalues are $E_+ = +5.0$ meV and $E_- = -5.0$ meV.

1. Sets $\det(H - EI) = 0$ with the off-diagonal product written out: $(4.0 - E)(-4.0 - E) - (3.0i)(-3.0i) = 0$, with E in meV
2. $(3.0i)(-3.0i) = +9.0$, so $E^2 - 16 - 9 = 0$, giving $E^2 = 25$ and $E\pm = \pm 5.0$ meV

(b) DETERMINE · U2 **[2]**

Determine the normalised eigenvector $|E_+\rangle$ as a combination of $|1\rangle$ and $|2\rangle$.

1. From the first row of $(H - 5.0\ \text{meV}\ I)c = 0$: $(4.0 - 5.0)c_1 + 3.0i\ c_2 = 0$, so $c_2 = -ic_1/3$
2. Normalising with $|c_1|^2 + |c_2|^2 = 1$ gives $|E_+\rangle = (3|1\rangle - i|2\rangle)/\sqrt{10}$, to within an overall phase

   Normalising with $c_1^2 + c_2^2$ instead of $|c_1|^2 + |c_2|^2$ gives $1/\sqrt{8}$ and a vector that is not normalised. Substituting the result into the second row, $-3.0i\ c_1 - 9.0\ c_2 = 0$, is a ten-second check that also catches a sign error in c_2.

(c) CALCULATE · U3 **[1]**

Calculate the probability that a measurement of the energy at $t = 0$ gives $+5.0$ meV.

1. $P(+5.0\ \text{meV}) = |\langle E_+|1\rangle|^2 = |3/\sqrt{10}|^2 = 0.90$

(d) DERIVE · U2 **[3]**

Starting from the time-dependent Schrödinger equation, derive an expression for the probability $P_2(t)$ that the electron is found in $|2\rangle$ at time t.

1. Because H is time independent, $i\hbar\, d|\psi\rangle/dt = H|\psi\rangle$ gives $|\psi(t)\rangle = \Sigma\, e^{(-iE\pm t/\hbar)}\, |E\pm\rangle\langle E\pm|1\rangle$, with $\langle E_+|1\rangle = 3/\sqrt{10}$, $\langle E_-|1\rangle = 1/\sqrt{10}$ and $|E_-\rangle = (|1\rangle + 3i|2\rangle)/\sqrt{10}$ (or, since $H^2 = (5.0\ \text{meV})^2\, I$, uses $e^{(-iHt/\hbar)} = \cos(Et/\hbar)\, I - i \sin(Et/\hbar)\, H/E$)
2. $\langle 2|\psi(t)\rangle = (3/\sqrt{10})(-i/\sqrt{10})e^{(-iEt/\hbar)} + (1/\sqrt{10})(3i/\sqrt{10})e^{(iEt/\hbar)} = (3i/10)(e^{(iEt/\hbar)} - e^{(-iEt/\hbar)}) = -(3/5)\sin(Et/\hbar)$, where $E = 5.0$ meV
3. $P_2(t) = (9/25)\sin^2(Et/\hbar) = 0.36 \sin^2[(5.0\ \text{meV})t/\hbar]$

   Building $|E_-\rangle$ by orthogonality to $|E_+\rangle$ is where a forgotten complex conjugate costs marks: dropping it gives $(|1\rangle - 3i|2\rangle)/\sqrt{10}$, which is not an eigenvector of H and leads to $P_2 = 0.36 \cos^2(Et/\hbar)$, non-zero at $t = 0$ when the electron is certainly in $|1\rangle$. The other common error is to attach separate phases to the dot-basis amplitudes of $|1\rangle$; only energy eigenstates evolve by a phase alone.

(e) CALCULATE · U3 **[2]**

Calculate the earliest time after $t = 0$ at which P_2 is greatest, and state that greatest value.

1. P_2 is greatest, at 0.36, when $(5.0\ \text{meV})t/\hbar = \pi/2$
2. $t = \pi\hbar/(2 \times 5.0\ \text{meV}) = \pi \times 6.582 \times 10^{-16}\ \text{eV s} / (1.0 \times 10^{-2}\ \text{eV}) = 2.1 \times 10^{-13}$ s

   A candidate who expects the electron to transfer completely has overlooked the unequal diagonal elements: the detuning caps the transfer at $|H_{12}|^2/E_+^2 = 9/25$.

10

CORE COURSE

STRUCTURED · DEMANDING · UPV-05.03 · UPV-06.01 · UPV-06.02 **[10]**

An electron of mass $m_e = 9.109 \times 10^{-31}$ kg is bound in a symmetric one-dimensional square well occupying $-a < x < a$, with $a = 0.500$ nm. Its potential energy is zero inside the well and $V_0 = 1.20$ eV outside it. Measure the energy E from the bottom of the well and write $k = \sqrt{(2m_eE)}/\hbar$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_eV_0)}$. The even bound states satisfy $z \tan z = \sqrt{(z_0^2 - z^2)}$ and the odd bound states satisfy $-z \cot z = \sqrt{(z_0^2 - z^2)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a)

DETERMINE · U3 **[3]**

Calculate z_0, and hence determine how many bound states the well holds.

1. $z_0 = (0.500 \times 10^{-9}\ \text{m} / 1.055 \times 10^{-34}\ \text{J s}) \times \sqrt{(2 \times 9.109 \times 10^{-31}\ \text{kg} \times 1.20 \times 1.602 \times 10^{-19}\ \text{J})}$, with V_0 converted to joules
2. $z_0 = 2.80$ (dimensionless)
3. An even root always exists; an odd root exists because $z_0 > \pi/2 = 1.57$; a second even root would need $z_0 > \pi = 3.14$. So the well holds 2 bound states, equivalently $\text{floor}(2z_0/\pi) + 1 = \text{floor}(1.79) + 1 = 2$

   Leaving V_0 in electronvolts inside the square root gives $z_0 \approx 7 \times 10^9$ and a well with billions of bound states. A dimensionless z_0 of order one is the quickest unit check in the question.

(b)

SHOW THAT · U3 **[2]**

Show that $z = 1.15$ satisfies the even-parity condition to within 1%.

1. Left side: $1.15 \times \tan(1.15\ \text{rad}) = 1.15 \times 2.234 = 2.57$
2. Right side: $\sqrt{(2.80^2 - 1.15^2)} = 2.55$ (2.56 with $z_0 = 2.805$), which differs from 2.57 by less than 1% (0.45% to 0.66%, depending on the z_0 used)

   Evaluating tan 1.15 in degrees gives a left side of 0.023 and apparent failure. Every trigonometric argument in a matching condition is in radians.

(c)

COMPARE · U3 **[3]**

Calculate the ground-state energy of the electron, and compare it with the ground-state energy the electron would have in an infinitely deep well of the same width 2a.

1. $E_1 = \hbar^2z^2/(2m_ea^2) = V_0(z/z_0)^2 = 1.20\ \text{eV} \times (1.15/2.80)^2 = 0.20$ eV
2. Infinite well of width 2a = 1.00 nm: $E_{1,\infty} = \pi^2\hbar^2/[2m_e(2a)^2] = 6.03 \times 10^{-20}$ J $= 0.376$ eV
3. The finite-well value is about 0.54 of the infinite-well value because the wavefunction extends into the classically forbidden region: spread over a greater effective width, it has less curvature and less kinetic energy. Both are non-zero, as confinement to a finite region requires

   Using a instead of 2a in the infinite-well formula quadruples $E_{1,\infty}$ to 1.51 eV, which lies above V_0 and should itself signal the error.

(d)

SKETCH · U1 **[2]**

On one set of axes of ψ against x, from $x = -2a$ to $x = +2a$, sketch the wavefunctions of the two bound states. Mark $x = -a$ and $x = +a$.

1. Ground state: even with no node, cosine-like inside the well, joining smoothly (ψ and dψ/dx continuous) onto exponential decay outside, and non-zero at $x = \pm a$
2. First excited state: odd with a single node at $x = 0$, and with exponential tails that decay more slowly than the ground state's, because it lies nearer the top of the well ($\kappa a \approx 1.7$ against 2.6)

   The mark most often lost is for drawing wavefunctions that reach zero at $x = \pm a$, as in the infinite well. The non-zero value at the walls is the feature that lowered the energy in (c).

11

CORE COURSE

STRUCTURED · DEMANDING · UPV-10.04 · UPV-10.07 · UPV-10.08 **[10]**

Slow neutrons, which have spin ½ and gyromagnetic ratio $\gamma_n = -1.832 \times 10^8$ rad s^{-1} T^{-1}, leave a polariser in the state $|+z\rangle$. The spin operator is $S = (\hbar/2)\sigma$; in the $\{|+z\rangle, |-z\rangle\}$ basis σ_y has rows (0, −i) and (i, 0), and $\exp(-i\alpha\sigma_y) = \cos\alpha\, I - i \sin\alpha\, \sigma_y$ for real α. An analyser set along a unit vector $\hat{n}$ transmits only neutrons found in the $+\hbar/2$ eigenstate of $\hat{n}\cdot S$, and blocks the rest.

(a) CALCULATE · U3 **[2]**

The neutrons pass first through an analyser whose n̂ lies in the xz-plane at 45° to the +z axis, and then through an analyser set along −z, which transmits only $|-z\rangle$. Calculate the fraction of the neutrons leaving the polariser that emerge from the second analyser.

1. $|+\hat{n}\rangle = \cos 22.5° \, |+z\rangle + \sin 22.5° \, |-z\rangle$, so the first analyser transmits $|\langle +\hat{n}|+z\rangle|^2 = \cos^2 22.5° = 0.854$
2. The second analyser transmits $|\langle -z|+\hat{n}\rangle|^2 = \sin^2 22.5° = 0.146$ of these, so the fraction is $0.854 \times 0.146 = 0.125$ (exactly 1/8)

   Using the angle between the axes instead of half of it, $\cos^2 45° \times \sin^2 45°$, gives 0.25. The half-angle is the signature of spin one-half, for which opposite directions in space are orthogonal states.

(b) EXPLAIN · U1 **[2]**

With the first analyser removed, no neutrons emerge from the −z analyser. Explain why inserting an analyser that removes neutrons from the beam increases the number emerging, and identify which step in the sequence is not unitary.

1. $|+z\rangle$ and $|-z\rangle$ are orthogonal, so with no intermediate analyser the amplitude to pass the −z analyser is zero; the 45° analyser replaces the state by $|+\hat{n}\rangle$, which has amplitude $\sin 22.5°$ on $|-z\rangle$
2. Each analyser performs a projective measurement: projecting with $|+\hat{n}\rangle\langle +\hat{n}|$ and renormalising what survives is not unitary, since it discards part of the beam and cannot be reversed, whereas free flight between analysers is unitary

   Saying that the first analyser 'rotates the spin' earns nothing. A rotation is unitary and loses no neutrons, whereas this analyser blocks about 15% of the beam; the increase comes from a projection that destroys the orthogonality between the prepared state and the final filter.

(c) SHOW THAT · U2 **[3]**

The 45° analyser is removed. Instead, the $|+z\rangle$ neutrons cross a region of uniform magnetic field B along +y, where the Hamiltonian is $H = -\gamma_n B \, S_y$, taking a time t. Show that on leaving the field the spin state is $\cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$, where $\omega = |\gamma_n|B$.

1. Because $\gamma_n < 0$, $H = |\gamma_n| B \, S_y = (\hbar\omega/2)\sigma_y$ with $\omega = |\gamma_n|B$; H is time independent, so the propagator is $U = \exp(-iHt/\hbar) = \exp(-i(\omega t/2)\sigma_y)$
2. By the given identity, $U = \cos(\omega t/2)\, I - i \sin(\omega t/2)\, \sigma_y$
3. $\sigma_y|+z\rangle = i|-z\rangle$, so $U|+z\rangle = \cos(\omega t/2)|+z\rangle - i \sin(\omega t/2)(i|-z\rangle) = \cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$

   Dropping the sign of γ_n gives $\cos(\omega t/2)|+z\rangle - \sin(\omega t/2)|-z\rangle$, a spin turning towards −x instead of +x. The z probabilities agree, so only a careful candidate notices, but the sense of precession is physical and measurable along x.

(d) CALCULATE · U3 **[2]**

The neutrons travel at 2200 m s^{-1} through a field region 5.00 cm long. Calculate the field B that turns $|+z\rangle$ into $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$.

1. $|+x\rangle$ requires $\omega t/2 = \pi/4$, so $\omega t = \pi/2$, with $t = L/v = 0.0500 \text{ m} / 2200 \text{ m s}^{-1} = 2.27 \times 10^{-5}$ s
2. $B = \pi/(2|\gamma_n|t) = \pi / (2 \times 1.832 \times 10^8 \text{ rad s}^{-1} \text{ T}^{-1} \times 2.27 \times 10^{-5} \text{ s}) = 3.77 \times 10^{-4}$ T

   Setting $\omega t/2 = \pi/2$ gives 7.55×10^{-4} T, the field that turns the spin all the way to −z. The spin vector turns through ωt, twice the angle that appears in the spinor components.

(e) PREDICT · U1 **[1]**

Predict the fraction of the neutrons that a +z analyser placed after the field region would transmit if the same field pointed along +z instead of +y, and give the reason.

1. All of them, a fraction of 1: with B along z, H is proportional to S_z, so $|+z\rangle$ is a stationary state that acquires only a global phase, and the z populations do not change

   Candidates who reason that 'the spin precesses, so some neutrons are lost' have not asked about the axis of precession: a spin along the field has nothing to precess about it.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UPV-07.07 · UPV-07.09 · UPV-08.05 **[10]**

A single $^{40}Ca^{+}$ ion of mass $m = 6.64 \times 10^{-26}$ kg is held in an ion trap whose potential along the trap axis is harmonic with angular frequency $\omega = 2\pi \times 1.20$ MHz $= 7.54 \times 10^{6}$ rad s^{-1}. Along that axis the position and momentum operators are $x = \sqrt{(\hbar/2m\omega)}(a + a\dagger)$ and $p = i\sqrt{(m\hbar\omega/2)}(a\dagger - a)$, with $[a, a\dagger] = 1$ and number states $|n\rangle$ of energy $(n + \tfrac{1}{2})\hbar\omega$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) ESTIMATE · U4 **[2]**

After laser cooling, the axial motion is in thermal equilibrium at $T = 0.50$ mK. Estimate the mean number of vibrational quanta $\bar{n}$, stating the approximation you make and why it is justified.

1. $\hbar\omega/(k_BT) = (1.055 \times 10^{-34}\ \text{J s} \times 7.54 \times 10^{6}\ \text{rad s}^{-1})/(1.381 \times 10^{-23}\ \text{J K}^{-1} \times 5.0 \times 10^{-4}\ \text{K}) = 0.115$, so $k_BT \gg \hbar\omega$ and the high-temperature (equipartition) limit $\bar{n} \approx k_BT/\hbar\omega$ applies
2. $\bar{n} \approx 1/0.115 \approx 9$ (8.7); the Bose–Einstein value $1/(e^{0.115} - 1) = 8.2$ is also accepted, as is any value from 8 to 9

The regime is decided by comparing k_BT with $\hbar\omega$, not by how small T looks. Candidates who call 0.5 mK 'ultracold' and place the ion in its ground state miss about eight quanta of occupation.

(b) SHOW THAT · U2 **[2]**

Further cooling and a laser pulse prepare the state $|\psi(0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Show that $\langle x\rangle(t) = \sqrt{(\hbar/2m\omega)}\cos\omega t$.

1. Each number state evolves as $e^{-i(n + \frac{1}{2})\omega t}|n\rangle$, so up to a global phase $|\psi(t)\rangle = (|0\rangle + e^{-i\omega t}|1\rangle)/\sqrt{2}$
2. Only $\langle 0|a|1\rangle = 1$ and $\langle 1|a\dagger|0\rangle = 1$ contribute, giving $\langle a\rangle = \tfrac{1}{2}e^{-i\omega t}$ and $\langle a\dagger\rangle = \tfrac{1}{2}e^{i\omega t}$, so $\langle x\rangle = \sqrt{(\hbar/2m\omega)}(\langle a\rangle + \langle a\dagger\rangle) = \sqrt{(\hbar/2m\omega)}\cos\omega t$

Losing the relative phase $e^{-i\omega t}$ gives a constant $\langle x\rangle = \sqrt{(\hbar/2m\omega)}$. A stationary expectation value from a superposition of two different energies should prompt a check.

(c) DETERMINE · U2 **[4]**

Determine the product $\Delta x\,\Delta p$ at $t = 0$ and at $\omega t = \pi/4$, expressing each as a multiple of $\hbar$.

1. $(a + a\dagger)^2 = a^2 + a\dagger^2 + 2N + 1$, and a^2 and $a\dagger^2$ cannot connect $|0\rangle$ with $|1\rangle$, so $\langle x^2\rangle = (\hbar/2m\omega)(2\langle N\rangle + 1) = \hbar/(m\omega)$ with $\langle N\rangle = \tfrac{1}{2}$; likewise $-(a\dagger - a)^2 = 2N + 1 - a^2 - a\dagger^2$ gives $\langle p^2\rangle = m\hbar\omega$
2. $\langle p\rangle = i\sqrt{(m\hbar\omega/2)}(\langle a\dagger\rangle - \langle a\rangle) = -\sqrt{(m\hbar\omega/2)}\sin\omega t$, which equals $m\,d\langle x\rangle/dt$ as Ehrenfest's theorem requires
3. $(\Delta x)^2 = (\hbar/2m\omega)(2 - \cos^2\omega t)$ and $(\Delta p)^2 = (m\hbar\omega/2)(2 - \sin^2\omega t)$, so $\Delta x\,\Delta p = (\hbar/2)\sqrt{[(2 - \cos^2\omega t)(2 - \sin^2\omega t)]}$
4. At $t = 0$, $\Delta x\,\Delta p = (\hbar/2)\sqrt{2} = 0.71\hbar$; at $\omega t = \pi/4$, $\Delta x\,\Delta p = (\hbar/2)(3/2) = 0.75\hbar$; both exceed $\hbar/2$

The usual error is to take $(\Delta x)^2 = \langle x^2\rangle$, which holds only when $\langle x\rangle = 0$, as in a stationary state but not here. That route gives $\Delta x\,\Delta p = \hbar$ at every time and misses the oscillation of the product entirely.

(d) EXPLAIN · U1 **[2]**

Explain why $\Delta x\,\Delta p$ exceeds $\hbar/2$ at every time for this state, although the ground state $|0\rangle$ reaches $\hbar/2$.

1. Equality in $\Delta x\,\Delta p \geq \hbar/2$ requires both inequalities of the derivation to be saturated, Cauchy–Schwarz and the dropping of $\mathrm{Re}\langle f|g\rangle$, which together demand $(p - \langle p\rangle)|\psi\rangle = i\lambda(x - \langle x\rangle)|\psi\rangle$ with λ real, and normalisability needs $\lambda > 0$; the only solutions are Gaussian wavefunctions
2. In position space this state is proportional to $(1 + c(t)x)\, e^{-m\omega x^2/2\hbar}$ with $c(t) = e^{-i\omega t}\sqrt{2m\omega/\hbar}$: a Gaussian multiplied by a polynomial, which is not a Gaussian at any t, so the bound is never reached, whereas $|0\rangle$ is a Gaussian and reaches it

   'Superpositions carry more uncertainty' earns nothing: a coherent state is a superposition of every $|n\rangle$ and still reaches $\hbar/2$. The deciding property is the Gaussian shape, which follows from the condition for equality.

13

CORE COURSE

DATA ANALYSIS · DEMANDING · UPV-06.05 · UPV-06.09 **[10]**

A scanning tunnelling microscope tip is held above a clean metal surface at a fixed bias of 0.100 V. With the feedback loop switched off, the tip is withdrawn in steps, and the tunnelling current I is recorded against Δd, the increase in the tip–surface gap from its starting value. In the opaque-barrier limit $I = I_0 \exp(-2\kappa\Delta d)$, where $\kappa = \sqrt{2m_e\varphi}/\hbar$ and φ is the apparent barrier height. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) STATE · U2 **[1]**

State the gradient of a graph of $\ln(I/\mathrm{nA})$ against Δd, in terms of κ.

1. Gradient $= -2\kappa$, since $\ln(I/\mathrm{nA}) = \ln(I_0/\mathrm{nA}) - 2\kappa\Delta d$

   Candidates who write $+2\kappa$, or κ, carry a sign or a factor of two into every later part; the 2 is there because the current follows the probability, the squared modulus of an amplitude decaying as $e^{-\kappa\Delta d}$.

(b) DETERMINE · U4 **[3]**

Determine κ from the data.

1. $\ln(I/\mathrm{nA})$ for the six readings: 0.916, 0.122, −0.753, −1.523, −2.382, −3.170
2. The points lie on a straight line (no residual larger than about 0.03) of gradient -2.05×10^{-2} pm^{-1} (least squares -0.0205 pm^{-1}; accept -0.0200 to -0.0210 pm^{-1})
3. $\kappa = 0.0205$ pm^{-1} / 2 $= 1.03 \times 10^{-2}$ pm^{-1} $= 1.03 \times 10^{10}$ m^{-1} (accept 1.00×10^{10} to 1.05×10^{10} m^{-1})

   Two errors dominate: using $\log_{10}$ instead of ln, which scales the gradient by 0.434 and leads to a barrier height about five times too small, and failing to convert the gradient from pm^{-1} to m^{-1}.

(c) CALCULATE · U3 **[2]**

Calculate the apparent barrier height φ in electronvolts.

1. $\varphi = \hbar^2\kappa^2/(2m_e) = (1.055 \times 10^{-34}\ \mathrm{J\,s})^2 \times (1.03 \times 10^{10}\ \mathrm{m^{-1}})^2 / (2 \times 9.109 \times 10^{-31}\ \mathrm{kg}) = 6.4 \times 10^{-19}$ J
2. $\varphi = 4.0$ eV (accept 3.8 to 4.2 eV, following from the κ found in (b))

   Because φ goes as κ^2, a 2% error in the gradient becomes a 4% error in φ. Rounding κ to 1×10^{10} m^{-1} before squaring gives 3.8 eV.

(d) ESTIMATE · U4 **[2]**

In constant-current imaging the feedback loop moves the tip to hold I fixed. Estimate the change in gap that changes the current by 10%, stating the assumption you make.

1. Assumes the exponential law holds over a few picometres with the fitted κ and an unchanged prefactor I_0, so that the change in gap is $\ln(1.10)/(2\kappa)$ for a 10% rise in current, or $\ln(1/0.90)/(2\kappa)$ for a 10% fall
2. $\Delta d = 0.0953 / 0.0205\ \text{pm}^{-1} \approx 5$ pm (4.6 pm for a rise, 5.1 pm for a fall, and 4.9 pm from the small-change form $0.10/(2\kappa)$ are all accepted), so holding the current to 10% fixes the gap to a few picometres, about 2% of an atomic diameter

   The estimate rests on κ being the same for a displacement of a few picometres as over the 200 pm range fitted, and an answer that does not say so loses the first mark whatever its number.

(e) EVALUATE · U4 **[2]**

Photoemission gives the work function of this surface as 4.60 eV. A student concludes that either the STM data or the tunnelling model must be wrong. Evaluate this conclusion.

1. The fitted $\varphi \approx 4.0$ eV is about 13% below 4.60 eV, but the apparent barrier height is not the work function: the image-charge potential rounds and lowers the barrier between tip and surface, and the tip's own work function also enters, so a lower apparent height is expected
2. The ln I values lie on a straight line to within about 0.03 over the whole 200 pm, which supports the exponential tunnelling model; the conclusion is not justified, because the data fit the model and the gap from 4.60 eV is explained by what φ represents

   Strong answers separate the two things the student has run together: whether the model fits, which the straightness of the line decides, and what the fitted parameter means, which is an image-lowered average barrier rather than the work function.

14

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-08.03 · UPV-09.03 · UPV-09.04 **[14]**

For Hermitian operators A and B and a normalised state $|\psi\rangle$, write $\langle A\rangle = \langle\psi|A|\psi\rangle$ and $(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$. The Cauchy–Schwarz inequality states that $\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$ for any kets $|f\rangle$ and $|g\rangle$. The components of an angular momentum obey $[J_x, J_y] = i\hbar J_z$ and its cyclic permutations, $J_\pm = J_x \pm iJ_y$, and the simultaneous eigenstates $|j, m\rangle$ of J^2 and J_z satisfy $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$, $J_z|j, m\rangle = m\hbar|j, m\rangle$ and $J_\pm|j, m\rangle = \hbar\sqrt{[j(j+1) - m(m \pm 1)]}\,|j, m \pm 1\rangle$.

(a) DERIVE · U2 **[4]**

Starting from the Cauchy–Schwarz inequality, derive the Robertson uncertainty relation, a lower bound on $\Delta A\,\Delta B$ in terms of the commutator $[A, B]$.

1. Defines $|f\rangle = (A - \langle A\rangle)|\psi\rangle$ and $|g\rangle = (B - \langle B\rangle)|\psi\rangle$; because the shifted operators are Hermitian, $\langle f|f\rangle = (\Delta A)^2$ and $\langle g|g\rangle = (\Delta B)^2$
2. Cauchy–Schwarz gives $(\Delta A)^2(\Delta B)^2 \geq |\langle f|g\rangle|^2 \geq (\text{Im}\langle f|g\rangle)^2 = [(\langle f|g\rangle - \langle g|f\rangle)/2i]^2$
3. $\langle f|g\rangle - \langle g|f\rangle = \langle\psi|[A - \langle A\rangle, B - \langle B\rangle]|\psi\rangle = \langle[A, B]\rangle$, because the numbers $\langle A\rangle$ and $\langle B\rangle$ commute with every operator
4. Hence $\Delta A\,\Delta B \geq \tfrac{1}{2}|\langle[A, B]\rangle|$, a real bound because $[A, B]$ is anti-Hermitian and its expectation value is purely imaginary

   Applying Cauchy–Schwarz to $A|\psi\rangle$ and $B|\psi\rangle$ without subtracting the means bounds $\langle A^2\rangle\langle B^2\rangle$, not the variances. The mean shift is the step that turns an inequality about vectors into an uncertainty relation.

(b) SHOW THAT · U2 **[4]**

Show that in the state $|j, m\rangle$, $\langle J_x\rangle = \langle J_y\rangle = 0$ and $\langle J_x^2\rangle = \langle J_y^2\rangle = \tfrac{1}{2}\hbar^2[j(j+1) - m^2]$.

1. $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/2i$; $J_\pm$ changes m by ± 1 and $\langle j, m|j, m \pm 1\rangle = 0$, so $\langle J_x\rangle = \langle J_y\rangle = 0$
2. $J_x^2 = \tfrac{1}{4}(J_+^2 + J_-^2 + J_+J_- + J_-J_+)$ and $J_y^2 = -\tfrac{1}{4}(J_+^2 + J_-^2 - J_+J_- - J_-J_+)$; the $J_\pm^2$ terms change m by ± 2 and have zero expectation value, so $\langle J_x^2\rangle = \langle J_y^2\rangle = \tfrac{1}{4}\langle J_+J_- + J_-J_+\rangle$
3. $J_+J_- + J_-J_+ = 2(J_x^2 + J_y^2) = 2(J^2 - J_z^2)$; equivalently $\langle J_+J_-\rangle + \langle J_-J_+\rangle = \hbar^2[j(j+1) - m(m-1)] + \hbar^2[j(j+1) - m(m+1)]$ from the given ladder elements
4. $\langle J_x^2\rangle = \langle J_y^2\rangle = \tfrac{1}{2}\langle J^2 - J_z^2\rangle = \tfrac{1}{2}\hbar^2[j(j+1) - m^2]$

   The result is given, so every mark is for the ladder algebra. Asserting $\langle J_x^2\rangle = \langle J_y^2\rangle$ 'by symmetry' earns nothing towards the value; the candidate must still show that the $J_\pm^2$ cross terms drop out.

(c) DETERMINE · U2 **[3]**

Determine the states $|j, m\rangle$ in which $\Delta J_x\,\Delta J_y$ is exactly equal to the Robertson bound for J_x and J_y.

1. With both means zero, (b) gives $\Delta J_x\,\Delta J_y = \tfrac{1}{2}\hbar^2[j(j+1) - m^2]$
2. The Robertson bound is $\tfrac{1}{2}|\langle[J_x, J_y]\rangle| = \tfrac{1}{2}\hbar|\langle J_z\rangle| = \tfrac{1}{2}\hbar^2|m|$
3. Equality requires $j(j+1) - m^2 = |m|$, that is $(j - |m|)(j + |m| + 1) = 0$, and since $j + |m| + 1 > 0$ this means $|m| = j$: only the extreme states $|j, \pm j\rangle$ reach the bound, which for spin ½ includes both S_z eigenstates

   Solving the quadratic in $|m|$ gives the roots j and $-(j+1)$. Rejecting the second, because $|m|$ cannot be negative, is the step most often left unstated, and without it the answer is not determined.

(d) EVALUATE · U1 **[3]**

A student argues: 'In the state $|j = 1, m = 0\rangle$ the Robertson bound for J_x and J_y is zero, so a careful enough experiment could find both J_x and J_y with no spread in that state.' Evaluate this argument.

1. In $|1, 0\rangle$, (b) gives $\Delta J_x = \Delta J_y = \hbar$, so $\Delta J_x\,\Delta J_y = \hbar^2$: the spreads belong to the prepared state, no experimental care reduces them, and a lower bound of zero is not a value the state attains
2. Zero spread in both would need a common eigenvector of J_x and J_y; then $[J_x, J_y]|\psi\rangle = 0$ forces $J_z|\psi\rangle = 0$, the cyclic relations then force $J_x|\psi\rangle = J_y|\psi\rangle = 0$, so $J^2|\psi\rangle = 0$ and $j = 0$
3. Judgement: the argument fails. A vanishing bound is necessary but not sufficient for joint sharpness, and no state with $j \geq \tfrac{1}{2}$, $|1, 0\rangle$ included, has both components sharp

   This part separates candidates who treat the Robertson relation as the whole content of uncertainty from those who see it as one inequality with a state-dependent right-hand side. The strongest answers give the commutator argument, which settles every state at once, rather than checking $|1, 0\rangle$ alone.

UNIVERSITY · GIOPHYSICS UNIVERSITY PHYSICS · PHYSICS I–V, QUANTUM MECHANICS I–II, ADVANCED QM AND QFT I

University Physics V · Final Examination

A cumulative final on University Physics V, weighted towards its second half: the hydrogen atom, multi-electron atoms and spectroscopy, nuclear structure and decay, fission and fusion, particle classification, the Standard Model, conservation-law audits and quantum information, with the state-space, spin and oscillator methods of the first half sampled throughout.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**120**	**19**	**10**

SECTION — **UPV · Final · University Physics V — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 2 · Routine 7 · Demanding 8 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UPV-01 · UPV-02 · UPV-03 · UPV-04 · UPV-05 · UPV-06 · UPV-07 · UPV-08 · UPV-09 · UPV-10 · UPV-11 · UPV-12 · UPV-13 · UPV-14 · UPV-15 · UPV-16 · UPV-17 · UPV-18 · UPV-19 · UPV-20

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Physics V · Final Examination

Answer all questions. 120 marks in 150 minutes.

1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-02.08 · UPV-03.07 **[1]**

A single photon on an integrated-optics chip is divided equally among three waveguides, leaving it in $|\psi\rangle = (|1\rangle + |2\rangle + |3\rangle)/\sqrt{3}$, where $|k\rangle$ is the state in which the photon travels in waveguide k. A heater on waveguide 3 then multiplies the amplitude in that waveguide by e^(iπ). One of two measurements is made afterwards: M_1 asks whether the photon is in waveguide 3, with projector $|3\rangle\langle 3|$, and M_2 asks whether it is in the original state $|\psi\rangle$, with projector $|\psi\rangle\langle\psi|$. How does the probability of the outcome 'yes' in each measurement compare with its value when the heater is off?

A Neither changes: M_1 stays at 1/3 and M_2 stays at 1, because multiplying an amplitude by a phase factor cannot change a probability.

B M_1 stays at 1/3, and M_2 falls from 1 to 1/9.

C M_1 stays at 1/3, and M_2 falls from 1 to 1/3.

D M_1 stays at 1/3, and M_2 falls from 1 to 0.

Answer

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-06.04 **[1]**

Conduction electrons travelling in the +x direction along a single-channel nanowire meet a thin impurity layer at x = 0, modelled as the potential $V(x) = \alpha\delta(x)$ with $\alpha > 0$. The potential is zero on both sides of the layer, so the wavenumber k is the same on both sides. Matching ψ at x = 0, together with the jump in dψ/dx that the delta function requires, gives a reflected amplitude $r = -i\beta/(1 + i\beta)$ and a transmitted amplitude $t = 1/(1 + i\beta)$ for unit incident amplitude, where $\beta = m\alpha/(\hbar^2 k)$ and m is the effective mass of the conduction electrons. For electrons with $\beta = 2$, what is the reflection coefficient R, the ratio of the reflected to the incident probability current?

A 1/5

B 4/9

C 4/5

D 2/√5

Answer

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-07.05 · UPV-07.07 **[1]**

A mode of a superconducting microwave resonator behaves as a harmonic oscillator of angular frequency ω, with Hamiltonian $H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$ and number states $|n\rangle$ that satisfy $a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle$ and $a^\dagger a|n\rangle = n|n\rangle$. A sequence of control pulses prepares the mode in $|\psi\rangle = \eta(1 + a^\dagger)^2|0\rangle$, where 1 is the identity operator and η is a positive constant that normalises the state. What is the energy uncertainty ΔE in this state?

A $(2\sqrt{5}/7)\hbar\omega$

B $\hbar\omega/\sqrt{3}$

C $(2\sqrt{5}/7 + 1/2)\hbar\omega$

D $\sqrt{(12/7)}\,\hbar\omega$

Answer

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4 CORE COURSE

MULTIPLE CHOICE · RECALL · UPV-08.07 · UPV-09.03 **[1]**

A particle is in a simultaneous eigenstate of L^2 and L_z with quantum numbers $l = 2$ and $m_l = 2$. The component L_x is then measured. Which list gives every result that measurement can return?

A $+2\hbar$ only, because the state already has a definite component of angular momentum

B 0 only, because $\langle L_x\rangle = 0$ in this state

C $-\hbar$, 0 and $+\hbar$

D $-2\hbar$, $-\hbar$, 0, $+\hbar$ and $+2\hbar$

Answer

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UPV-11.05 · UPV-11.06 **[1]**

In the non-relativistic Coulomb model of hydrogen, with the electron's spin carried as a tensor factor but no spin-dependent term in the Hamiltonian, how many orthonormal states share the energy $E_4 = -0.850$ eV?

A 7

B 32

C 16

D 8

Answer

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-12.07 **[1]**

Treat the ground configuration of nickel as [Ar] $3d^8\,4s^2$, with the $4s^2$ subshell closed. Using antisymmetry and Hund's rules, what is the ground term?

A $^1G_4$

B $^3F_2$

C $^3G_5$

D $^3F_4$

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-14.04 · UPV-14.09 **[1]**

A nucleus with A = 236 and binding energy per nucleon 7.59 MeV splits into two fragments: one with A = 138 and 8.39 MeV per nucleon, and one with A = 98 and 8.60 MeV per nucleon. Ignoring any emitted neutrons, what energy is released?

A 209 MeV

B −209 MeV

C 0.887 MeV

D 1.79×10^3 MeV

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-15.02 **[1]**

A nuclide decays by α emission with branching fraction 0.25 and by β^- emission with branching fraction 0.75. Its half-life, found by following the number of undecayed nuclei, is 48.0 min. A detector that records only the α particles is used instead. With what half-life does the α count rate fall?

A 12.0 min

B 192 min

C 48.0 min

D 64.0 min

Answer ☐

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-17.03 · UPV-19.06 · UPV-19.09 **[1]**

Which of these proposed processes is forbidden by a conservation law of the Standard Model with massless neutrinos?

A $\mu^- \to e^- + \gamma$

B $K^- + p \to \Omega^- + K^+ + K^0$

C $\bar{\nu}_e + p \to n + e^+$

D $\pi^+ \to \mu^+ + \nu_\mu$

Answer ☐

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10 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-20.04 **[1]**

Which of these normalised two-qubit states, each written in the computational basis $|q_1 q_2\rangle$, is entangled?

A $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$

B $(|00\rangle - |01\rangle + |10\rangle - |11\rangle)/2$

C $(|00\rangle + i|01\rangle + i|10\rangle - |11\rangle)/2$

D $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$

Answer

11 CORE COURSE

STRUCTURED · ROUTINE · UPV-04.04 · UPV-05.07 · UPV-05.08 **[10]**

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.0670\ m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, with eigenstates $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$ for $0 \le x \le L$ and energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ it is prepared in $\Psi(x, 0) = [3\psi_1(x) + 4\psi_2(x)]/5$. Use $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.626 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C. You may use $\int_0^L \psi_1(x)\,\psi_2(x)\,dx = -16L/(9\pi^2)$.

(a) CALCULATE **[2]**

Calculate E_1 and E_2, in meV.

(b) SHOW THAT **[2]**

Show that the expectation value of the energy is $73E_1/25$ at every time t.

(c) DETERMINE **[3]**

Determine $\langle x \rangle$ as a function of time, giving the centre and the amplitude of its oscillation in nm.

(d) CALCULATE **[1]**

Calculate the period of the oscillation of $\langle x \rangle$.

(e) PREDICT **[2]**

The electron is instead prepared in $[3\psi_1 + 4\psi_3]/5$. Predict how $\langle x \rangle$ behaves in time, and give the reason.

12 CORE COURSE

STRUCTURED · ROUTINE · UPV-11.05 · UPV-11.07 · UPV-11.08 · UPV-11.09 **[11]**

X-ray spectra of the hot gas in galaxy clusters contain lines from O^{7+}, an oxygen nucleus ($Z = 8$) with a single bound electron. Treat the ion with the non-relativistic Coulomb Hamiltonian and an infinitely heavy nucleus, for which $E_n = -(13.6\ \text{eV})Z^2/n^2$. The ground-state radial probability density is $P(r) = (4Z^3/a_0^3)\, r^2\, e^{(-2Zr/a_0)}$, with $a_0 = 52.9$ pm. Use $hc = 1239.8$ eV nm.

(a) CALCULATE **[3]**

Calculate the ionisation energy of ground-state O^{7+}, and the photon energy and wavelength of its $2p \to 1s$ (Lyman-α) line.

(b) SHOW THAT **[3]**

Show that the most probable distance of the electron from the nucleus in the ground state is a_0/Z, and calculate that distance for O^{7+}.

(c) JUSTIFY **[3]**

The electron is excited into one of the $n = 3$ states. Justify which of 3s, 3p and 3d can decay to 1s by emitting a single electric-dipole photon, and state the route to 1s for each of the others.

(d) EXPLAIN **[2]**

In this model the 3s, 3p and 3d levels are exactly degenerate. Explain why this degeneracy is special to the Coulomb potential, and name the effect that splits these levels in a real O^{7+} ion.

13 CORE COURSE

STRUCTURED · DEMANDING · UPV-04.03 · UPV-08.03 · UPV-10.04 · UPV-10.08 **[12]**

A water sample sits in the uniform static field $B_0 = 9.40$ T of an NMR magnet, directed along +z. Consider the spin of a single hydrogen nucleus. The proton's magnetic moment is $\mu = (g_p \mu_N/\hbar)S$, with $g_p = 5.586$ and $\mu_N = 5.051 \times 10^{-27}$ J T^{-1}, so the Hamiltonian is $H = -\mu\cdot B = -\omega_0 S_z$ with $\omega_0 = g_p \mu_N B_0/\hbar$. Take $\hbar = 1.055 \times 10^{-34}$ J s. A radio-frequency pulse, treated as instantaneous, leaves the spin at $t = 0$ in the $+\hbar/2$ eigenstate of $\hat{n}\cdot S$, where $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the unit vector with polar angle θ and azimuth φ. The spin operators are $S_i = (\hbar/2)\sigma_i$, and in the basis of the S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ the Pauli matrices are $\sigma_x = (0, 1; 1, 0)$, $\sigma_y = (0, -i; i, 0)$ and $\sigma_z = (1, 0; 0, -1)$, with rows separated by semicolons.

(a) DETERMINE **[2]**

Determine the matrix of $\hat{n}\cdot S$ in the S_z basis, and its normalised eigenvector with eigenvalue $+\hbar/2$ written as a combination of $|\uparrow\rangle$ and $|\downarrow\rangle$.

(b) SHOW THAT **[3]**

Using the propagator $U(t) = \exp(-iHt/\hbar)$, show that at time t the spin is in the $+\hbar/2$ eigenstate of $\hat{n}(t)\cdot S$, where $\hat{n}(t)$ has the same polar angle θ and the azimuth $\varphi - \omega_0 t$.

(c) CALCULATE **[2]**

Calculate ω_0, and, for $0 < \theta < \pi$, the shortest time after $t = 0$ at which the spin is again in the same physical state as at $t = 0$.

(d) DESCRIBE **[2]**

A receiver coil with its axis along x picks up a voltage proportional to $d\langle S_x\rangle/dt$. Describe the signal it records, and how the amplitude of that signal depends on θ.

(e) SHOW THAT **[3]**

For operators A and B the Robertson relation reads $\Delta A\,\Delta B \geq \tfrac{1}{2}|\langle[A, B]\rangle|$. Apply it to H and S_x, and show that at every time t this spin obeys $\Delta E\,\Delta S_x \geq (\hbar/2)|d\langle S_x\rangle/dt|$, where ΔE is the uncertainty in its energy.

14 CORE COURSE

STRUCTURED · DEMANDING · UPV-12.07 · UPV-13.04 · UPV-13.07 **[12]**

The magnesium line at $\lambda = 517.3$ nm is the transition from the upper level $3s4s\ {}^3S_1$ to the lower level $3s3p\ {}^3P_1$. The source is placed in a uniform magnetic field $B = 1.20$ T, weak enough that LS coupling holds and each level shifts by $\Delta E = g_J \mu_B B M_J$, with $g_J = 1 + [J(J+1) + S(S+1) - L(L+1)]/[2J(J+1)]$. Use $\mu_B = 5.788 \times 10^{-5}$ eV T^{-1} and $hc = 1239.8$ eV nm. In the lower term, the ${}^3P_1$ and ${}^3P_2$ levels are 5.05 meV apart.

(a) CALCULATE **[2]**

Calculate the Landé g factors of the upper and lower levels.

(b) JUSTIFY **[2]**

Justify that this line is an allowed electric-dipole transition.

(c) DETERMINE **[4]**

Determine how many Zeeman components the line splits into, and the shift of each from the zero-field photon energy in units of $\mu_B B$.

(d) CALCULATE **[2]**

Calculate the wavelength separation between the two outermost components.

(e) ESTIMATE **[2]**

Estimate the field above which this anomalous Zeeman pattern gives way to the Paschen–Back regime, stating the criterion you use.

15 CORE COURSE

STRUCTURED · DEMANDING · UPV-17.08 · UPV-19.02 · UPV-19.03 · UPV-19.09 **[12]**

A π^- beam strikes a liquid-hydrogen target, and the reaction $\pi^- + p \to K^0 + \Lambda^0$ is studied. The Λ^0 then decays as $\Lambda^0 \to p + \pi^-$ with mean lifetime 2.63×10^{-10} s. Rest energies: $m_\pi c^2 = 139.57$ MeV, $m_p c^2 = 938.27$ MeV, $m_K c^2 = 497.61$ MeV and $m_\Lambda c^2 = 1115.68$ MeV. The quark contents are $K^0 = d\bar{s}$ and $\Lambda^0 = uds$, and a strange quark carries strangeness $S = -1$.

(a) STATE **[2]**

State the total charge, baryon number and strangeness on each side of the production reaction, and state which interaction can mediate it.

(b) DERIVE **[3]**

Derive, from four-momentum conservation, an expression for the minimum laboratory energy E_π of the pion, with the target proton at rest, for which the reaction can occur.

(c) CALCULATE **[2]**

Calculate the threshold kinetic energy and momentum of the pion.

(d) COMPARE **[2]**

The Σ^0 baryon, also uds, decays as $\Sigma^0 \to \Lambda^0 + \gamma$ with mean lifetime 7.4×10^{-20} s. Compare its lifetime with that of the Λ^0, relating the difference to the interaction responsible for each decay.

(e) CALCULATE **[3]**

Calculate the momentum of the proton in the rest frame of the Λ^0 for the decay $\Lambda^0 \to p + \pi^-$.

16 CORE COURSE

STRUCTURED · DEMANDING · UPV-20.03 · UPV-20.05 **[10]**

A source sends one qubit of each pair to Alice and the other to Bob, in the state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Alice measures the observable $A(a) = \cos a\, Z + \sin a\, X$ on her qubit and Bob measures $B(b) = \cos b\, Z + \sin b\, X$ on his, each with outcomes ± 1, where X and Z are Pauli matrices. The correlation is $E(a, b) = \langle\Phi^+|A(a) \otimes B(b)|\Phi^+\rangle$, and the CHSH combination is $S = E(a, b) + E(a, b') + E(a', b) - E(a', b')$. Every local hidden-variable model obeys $|S| \le 2$.

(a) SHOW THAT **[3]**

Show that $E(a, b) = \cos(a - b)$.

(b) CALCULATE **[2]**

Calculate S for the settings $a = 0$, $a' = \pi/2$, $b = \pi/4$ and $b' = -\pi/4$.

(c) DETERMINE **[3]**

Noise in the source replaces $|\Phi^+\rangle\langle\Phi^+|$ by the density operator $\rho = p|\Phi^+\rangle\langle\Phi^+| + (1 - p)I/4$, with $0 \le p \le 1$. Determine S for the same settings as a function of p, and the range of p for which these settings show a violation.

(d) EVALUATE **[2]**

An experiment using these settings reports $S = 2.38 \pm 0.05$. Evaluate what this result establishes and what it does not.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UPV-15.02 · UPV-15.07 · UPV-15.09 **[12]**

A radiopharmacy confirms the identity of a batch of fluorine-18 by measuring its half-life. A sample is placed in a well counter and the counts in 100 s are recorded at intervals; t is the time at the middle of each 100 s count, measured from the middle of the first. With no sample present, the counter records 1200 counts in 1000 s. The site's quality-control rule accepts the batch if the measured half-life lies between 105 min and 115 min. The accepted half-life of fluorine-18 is 109.8 min.

Gross counts recorded in 100 s intervals

t / min	Gross counts in 100 s
0	48252
60	32822
120	22668
180	15665
240	10629
300	7284
360	5123

(a) EXPLAIN **[2]**

Explain why the background must be subtracted from each reading before logarithms are taken, and why the uncertainty in the background itself can be neglected here.

(b) DETERMINE **[3]**

Determine the decay constant λ from the data, using either a straight-line fit of ln(net counts) against t or two well-separated readings.

(c) CALCULATE **[2]**

Calculate the half-life and the mean lifetime.

(d) ESTIMATE **[3]**

Estimate the uncertainty in λ that arises from counting statistics, assuming each reading is Poisson-distributed and using the two readings from part (b).

(e) EVALUATE **[2]**

Evaluate whether the batch passes the quality-control rule, and whether the data show any sign of a longer-lived radionuclidic impurity.

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18 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPV-14.04 · UPV-16.06 **[15]**

In a deuterium–tritium plasma the reaction $^2H + {}^3H \rightarrow {}^4He + n$ releases the energy that fusion devices aim to harvest. Atomic masses: $m(^2H) = 2.014102$ u, $m(^3H) = 3.016049$ u and $m(^4He) = 4.002603$ u; the neutron mass is 1.008665 u; 1 u = 931.494 MeV/c². For a Maxwellian plasma at temperature T, the non-resonant reaction rate per pair of nuclei is $\langle\sigma v\rangle \propto \int_0^\infty S(E)\exp[-E/(kT) - \sqrt{(E_G/E)}]\,dE$, where E is the centre-of-mass energy, S(E) varies slowly with E, $e^{(-E/kT)}$ is the Maxwell–Boltzmann factor and $e^{(-\sqrt{(E_G/E)})}$ is the probability of tunnelling through the Coulomb barrier. For D–T the Gamow energy is $E_G = 2\mu c^2(\pi\alpha Z_1 Z_2)^2 = 1182$ keV.

(a) CALCULATE **[2]**

Calculate the Q value of the reaction.

(b) SKETCH **[2]**

On one set of axes against E, sketch the Maxwell–Boltzmann factor, the tunnelling factor and their product, marking the energy E_0 at which the product peaks.

(c) DERIVE **[3]**

Starting from the rate integral given in the stem, derive an expression for E_0 in terms of E_G and kT by maximising the exponent of its integrand.

(d) SHOW THAT **[3]**

By expanding the exponent to second order about E_0, show that the integrand is approximately proportional to $\exp[-(E - E_0)^2/(\Delta/2)^2]$ with $\Delta = 4\sqrt{(E_0 kT/3)}$.

(e) CALCULATE **[3]**

Calculate E_0 and Δ for kT = 10.0 keV, and comment on how sharply the Gaussian form localises the reacting energies at this temperature.

(f) EVALUATE **[2]**

The Coulomb barrier between a deuteron and a triton at nuclear contact is about 0.44 MeV. A student argues that since kT = 10.0 keV is some 44 times smaller, fusion in this plasma must be negligible. Evaluate this argument.

19

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-18.04 · UPV-18.05 **[16]**

In the electroweak model the Higgs field is a complex doublet φ with hypercharge ½ and potential $V(\varphi) = -\mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$, with $\mu^2 > 0$ and $\lambda > 0$. Its covariant derivative is $D_\mu = \partial_\mu - ig(\sigma^a/2)W^a_\mu - i(g'/2)B_\mu$, summed over $a = 1, 2, 3$, where σ^a are the Pauli matrices, and the Lagrangian contains $(D_\mu\varphi)^\dagger(D^\mu\varphi) - V(\varphi)$. In unitary gauge the vacuum is $\varphi_0 = (0, v/\sqrt{2})^{\mathsf{T}}$. A real vector field V_μ of mass M contributes $\tfrac{1}{2}M^2V_\mu V^\mu$ to the Lagrangian, and the charged fields $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$ of mass M_W contribute $M_W^2W^+_\mu W^{-\mu}$. Take $g = 0.652$, $g' = 0.357$, $v = 246$ GeV and $\lambda = 0.129$.

(a) DERIVE **[3]**

Derive the vacuum value v in terms of μ and λ. Then, writing $\varphi = (0, (v + h)/\sqrt{2})^{\mathsf{T}}$, derive the mass M_H of the physical scalar h.

(b) STATE **[2]**

State how many real field components the doublet has, how many generators of $SU(2)_L \times U(1)_Y$ the vacuum breaks, and what becomes of the corresponding field components.

(c) SHOW THAT **[4]**

Show that the kinetic term evaluated on φ_0 gives $M_W = gv/2$, and that the neutral fields acquire the mass-squared matrix $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$ in the basis (W^3, B), with rows separated by semicolons.

(d) DETERMINE **[3]**

Determine the eigenvalues and normalised eigenvectors of M^2, identify the photon and the Z boson, and express the Weinberg angle θ_W and the ratio M_W/M_Z in terms of g and g'.

(e) CALCULATE **[2]**

Calculate M_W, M_Z, $\sin^2\theta_W$ and M_H.

(f) EVALUATE **[2]**

The measured values are $M_W = 80.37$ GeV, $M_Z = 91.19$ GeV and $M_H = 125.1$ GeV. Evaluate which of your results are genuine predictions of the model and which only return the numbers put in.

MARK SCHEME

University Physics V · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-02.08 · UPV-03.07 **[1]**

A single photon on an integrated-optics chip is divided equally among three waveguides, leaving it in |ψ⟩ = (|1⟩ + |2⟩ + |3⟩)/√3, where |k⟩ is the state in which the photon travels in waveguide k. A heater on waveguide 3 then multiplies the amplitude in that waveguide by e^(iπ). One of two measurements is made afterwards: M_1 asks whether the photon is in waveguide 3, with projector |3⟩⟨3|, and M_2 asks whether it is in the original state |ψ⟩, with projector |ψ⟩⟨ψ|. How does the probability of the outcome 'yes' in each measurement compare with its value when the heater is off?

KEY **B** **M_1 stays at 1/3, and M_2 falls from 1 to 1/9.**

With the heater on, the state is |ψ′⟩ = (|1⟩ + |2⟩ − |3⟩)/√3. A is the misreading this item is built to catch: a phase factor is unobservable only when it multiplies the whole state, whereas e^(iπ) = −1 on one component is a relative phase, and any measurement whose projector mixes the waveguide states can detect it. C finds the overlap correctly, ⟨ψ|ψ′⟩ = (1 + 1 − 1)/3 = 1/3, but stops at the amplitude, when the Born rule takes its squared modulus. D carries over the two-path interferometer, where a phase of π on one arm makes the two amplitudes cancel; with three equal amplitudes, reversing one leaves 1 + 1 − 1 = 1, not 0. M_1 is unchanged in every option because |⟨3|ψ′⟩|² = |−1/√3|² = 1/3: a phase on one basis component leaves that component's modulus alone. Only B pairs this with $P(M_2)$ = |1/3|² = 1/9.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-06.04 **[1]**

Conduction electrons travelling in the +x direction along a single-channel nanowire meet a thin impurity layer at x = 0, modelled as the potential V(x) = αδ(x) with α > 0. The potential is zero on both sides of the layer, so the wavenumber k is the same on both sides. Matching ψ at x = 0, together with the jump in dψ/dx that the delta function requires, gives a reflected amplitude r = −iβ/(1 + iβ) and a transmitted amplitude t = 1/(1 + iβ) for unit incident amplitude, where β = mα/(ħ²k) and m is the effective mass of the conduction electrons. For electrons with β = 2, what is the reflection coefficient R, the ratio of the reflected to the incident probability current?

KEY **C** **4/5**

The reflected wave has the same wavenumber as the incident wave, so no velocity factor enters and R = |r|² = β²/(1 + β²) = 4/5; T = |t|² = 1/(1 + β²) = 1/5 confirms R + T = 1. A is that transmission coefficient T, the right calculation answering the wrong question. B comes from dropping the i in the denominator and treating |1 + iβ| as 1 + β = 3, which gives (2/3)² = 4/9; the modulus is √(1 + β²) = √5, and with this error R + T comes to 5/9 instead of 1. D is |r| = 2/√5 ≈ 0.894 left unsquared, but a probability current is proportional to the squared modulus of an amplitude, not to the amplitude itself. The check R + T = 1 exposes both B and D.

U3

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-07.05 · UPV-07.07 **[1]**

A mode of a superconducting microwave resonator behaves as a harmonic oscillator of angular frequency ω, with Hamiltonian $H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$ and number states $|n\rangle$ that satisfy $a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle$ and $a^\dagger a|n\rangle = n|n\rangle$. A sequence of control pulses prepares the mode in $|\psi\rangle = \eta(1 + a^\dagger)^2|0\rangle$, where 1 is the identity operator and η is a positive constant that normalises the state. What is the energy uncertainty ΔE in this state?

KEY **A** **$(2\sqrt{5/7})\hbar\omega$**

Expanding the operator gives $(1 + a^\dagger)^2|0\rangle = |0\rangle + 2a^\dagger|0\rangle + a^\dagger a^\dagger|0\rangle = |0\rangle + 2|1\rangle + \sqrt{2}|2\rangle$, so $\eta^2 = 1/7$ and the probabilities of n = 0, 1 and 2 are 1/7, 4/7 and 2/7. Then $\langle n\rangle = 8/7$, $\langle n^2\rangle = 12/7$ and $(\Delta n)^2 = 12/7 - 64/49 = 20/49$; since $H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$, $\Delta E = \hbar\omega\,\Delta n = (2\sqrt{5/7})\hbar\omega$. B writes $a^\dagger a^\dagger|0\rangle = |2\rangle$, missing the factor $\sqrt{1}\,\sqrt{2} = \sqrt{2}$ from the two raising steps; the state $|0\rangle + 2|1\rangle + |2\rangle$ has $\Delta n = 1/\sqrt{3}$. C adds the zero-point energy to the spread, but $\tfrac{1}{2}\hbar\omega$ shifts every level by the same amount, so it moves $\langle E\rangle$ to $(23/14)\hbar\omega$ and leaves ΔE alone. D is $\sqrt{\langle n^2\rangle}\,\hbar\omega$, a root-mean-square value with $\langle n\rangle^2$ never subtracted, which mistakes how far the energy sits above the zero point for how widely it is spread.

U2

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4 CORE COURSE

MULTIPLE CHOICE · RECALL · UPV-08.07 · UPV-09.03 **[1]**

A particle is in a simultaneous eigenstate of L^2 and L_z with quantum numbers l = 2 and $m_l = 2$. The component L_x is then measured. Which list gives every result that measurement can return?

KEY **D** **$-2\hbar$, $-\hbar$, 0, $+\hbar$ and $+2\hbar$**

A treats a definite L_z as a definite angular-momentum vector. L_x does not commute with L_z, so an L_z eigenstate is not an L_x eigenstate and the result of measuring L_x is not fixed. B confuses a mean with an outcome: $\langle L_x\rangle = 0$ here, but a single measurement returns an eigenvalue, and the zero mean is built from several different ones. C is the spectrum for l = 1. Within the l = 2 subspace L_x has the same eigenvalues as L_z, because a rotation carries one into the other, and the stretched state $|2, 2\rangle$ has a non-zero overlap with every L_x eigenstate, with probabilities 1/16, 1/4, 3/8, 1/4 and 1/16.

U1

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UPV-11.05 · UPV-11.06 **[1]**

In the non-relativistic Coulomb model of hydrogen, with the electron's spin carried as a tensor factor but no spin-dependent term in the Hamiltonian, how many orthonormal states share the energy $E_4 = -0.850$ eV?

KEY **B** **32**

C is $n^2 = 16$, the sum of $2l + 1$ over l = 0 to 3. It is the orbital count alone, correct only before the spin factor is carried, and the stem says it is carried. A is $2l + 1$ for the largest value, l = 3, which counts one subshell and forgets that every l from 0 to $n - 1$ has the same energy in a pure Coulomb field. D counts four l values with two spin states each but drops the $2l + 1$ values of m_l. The full label set n, l, m_l, m_s gives $2n^2 = 32$.

U1

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-12.07 **[1]**

Treat the ground configuration of nickel as [Ar] $3d^8\,4s^2$, with the $4s^2$ subshell closed. Using antisymmetry and Hund's rules, what is the ground term?

KEY **D** **$^3F_4$**

A d^8 subshell behaves as two holes. C maximises S and L independently: ^3G would need two holes of the same spin both in $m_l = 2$, which antisymmetry forbids, so a spin-aligned pair reaches at most $M_L = 2 + 1 = 3$, an F term. A is the ^1G term, which antisymmetry does allow, but it takes the largest L before the largest S and so reverses the order of Hund's first two rules. B has the right term, ^3F, but applies the less-than-half-filled rule $J = |L - S|$. The subshell is more than half full, so $J = L + S = 4$ and the ground term is $^3F_4$.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-14.04 · UPV-14.09 **[1]**

A nucleus with A = 236 and binding energy per nucleon 7.59 MeV splits into two fragments: one with A = 138 and 8.39 MeV per nucleon, and one with A = 98 and 8.60 MeV per nucleon. Ignoring any emitted neutrons, what energy is released?

KEY **A** **209 MeV**

B has the right size and the wrong sign. The fragments are more tightly bound, so their total mass is smaller and energy is released: Q = B(fragments) − B(parent) is positive. C is the gain in binding energy per nucleon averaged over the fragments, 0.887 MeV, which still has to be multiplied by the 236 nucleons that share it. D is the parent's total binding energy, 236 × 7.59 = 1791 MeV, the energy needed to take the nucleus apart completely rather than the energy released by rearranging it. Q = 138 × 8.39 + 98 × 8.60 − 236 × 7.59 = 1157.8 + 842.8 − 1791.2 = 209 MeV.

U3

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-15.02 **[1]**

A nuclide decays by α emission with branching fraction 0.25 and by β^- emission with branching fraction 0.75. Its half-life, found by following the number of undecayed nuclei, is 48.0 min. A detector that records only the α particles is used instead. With what half-life does the α count rate fall?

KEY **C** **48.0 min**

B is the partial half-life of the α channel, 48.0/0.25 = 192 min, and it is the trap: a partial half-life says what share of the decays go by that route, not how fast that route's rate falls. Both channels draw on the same population of undecayed nuclei, so the α activity is λ_α N(t) with N(t) = N₀e^(−λt), and it falls with the total decay constant λ = λ_α + λ_β. D is the partial half-life of the β^- channel, 48.0/0.75 = 64.0 min. A multiplies the half-life by the branching fraction instead of dividing. The α rate is one quarter of the total activity at every instant, so it falls with the same 48.0 min half-life.

U1

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UPV-17.03 · UPV-19.06 · UPV-19.09 **[1]**

Which of these proposed processes is forbidden by a conservation law of the Standard Model with massless neutrinos?

KEY **A** **$\mu^- \to e^- + \gamma$**

B is tempting because it makes a baryon of three strange quarks, but its ledger balances: strangeness −1 on the left and −3 + 1 + 1 = −1 on the right, charge 0 on each side, baryon number 1 on each side, so the strong interaction can mediate it above threshold. C looks as if an antineutrino turns into a positron, but both carry $L_e = -1$, charge is +1 on each side and baryon number is 1 on each side: it is inverse beta decay, the reaction used to detect reactor antineutrinos. D balances muon number as 0 = −1 + 1. A conserves charge, energy and total lepton number, but changes L_μ from 1 to 0 and L_e from 0 to 1. With massless neutrinos each lepton-family number is conserved, so A is forbidden; the family-number violation that neutrino oscillation implies would make this decay far rarer than any search can reach.

U1

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10 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UPV-20.04 **[1]**

Which of these normalised two-qubit states, each written in the computational basis $|q_1 q_2\rangle$, is entangled?

KEY **D** **$(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$**

Arrange the coefficients c_{ij} of $\Sigma\, c_{ij}|ij\rangle$ as a 2×2 matrix: the state is a product, with Schmidt rank 1, exactly when that matrix has zero determinant. A is $|+\rangle\otimes|+\rangle$, with determinant (1·1 − 1·1)/4 = 0. B carries minus signs that look like correlations, but it is $|+\rangle\otimes|-\rangle$ and its determinant is (−1 + 1)/4 = 0. C is the strongest distractor, with complex phases and a minus sign, yet it is $(|0\rangle + i|1\rangle)\otimes(|0\rangle + i|1\rangle)/2$ and its determinant is $(-1 - i^2)/4 = 0$. D differs from A by a single sign, and that sign makes the determinant (−1 − 1)/4 = −1/2, so no product state can reproduce it; it is the state a controlled-Z gate makes from $|+\rangle\otimes|+\rangle$.

U2

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11 CORE COURSE

STRUCTURED · ROUTINE · UPV-04.04 · UPV-05.07 · UPV-05.08 **[10]**

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.0670\ m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, with eigenstates $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$ for $0 \le x \le L$ and energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ it is prepared in $\Psi(x, 0) = [3\psi_1(x) + 4\psi_2(x)]/5$. Use $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.626 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C. You may use $\int_0^L \psi_1(x)\, x\, \psi_2(x)\, dx = -16L/(9\pi^2)$.

(a) CALCULATE · U3 **[2]**

Calculate E_1 and E_2, in meV.

1. $E_1 = \pi^2\hbar^2/(2m^*L^2) = \pi^2(1.055 \times 10^{-34}\ \text{J s})^2/(2 \times 0.0670 \times 9.109 \times 10^{-31}\ \text{kg} \times (12.0 \times 10^{-9}\ \text{m})^2) = 6.25 \times 10^{-21}$ J
2. $E_1 = 39.0$ meV and $E_2 = 4E_1 = 156$ meV

Using the free-electron mass instead of the effective mass gives $E_1 = 2.61$ meV, fifteen times too small; the small effective mass is why quantum-well levels reach tens of meV.

(b) SHOW THAT · U2 **[2]**

Show that the expectation value of the energy is $73E_1/25$ at every time t.

1. $\langle E\rangle = |c_1|^2E_1 + |c_2|^2E_2 = (9/25)E_1 + (16/25)(4E_1) = 73E_1/25$ (= 114 meV)
2. time evolution multiplies each c_n by $e^{-iE_n t/\hbar}$, which leaves every $|c_n|^2$ unchanged, so $\langle E\rangle$ is the same at all t (equivalently, the propagator commutes with H)

Writing $\langle E\rangle = (3E_1 + 4E_2)/5$ weights by amplitudes instead of probabilities; the Born rule squares the coefficients.

(c) DETERMINE · U2 **[3]**

Determine $\langle x\rangle$ as a function of time, giving the centre and the amplitude of its oscillation in nm.

1. $\Psi(x, t) = [3\psi_1 e^{-iE_1 t/\hbar} + 4\psi_2 e^{-iE_2 t/\hbar}]/5$, with $\langle\psi_1|x|\psi_1\rangle = \langle\psi_2|x|\psi_2\rangle = L/2$
2. $\langle x\rangle = L/2 + 2(3/5)(4/5)x_{12}\cos(\omega t)$, where $\omega = (E_2 - E_1)/\hbar$ and $x_{12} = -16L/(9\pi^2)$
3. $\langle x\rangle = L/2 - [384/(225\pi^2)]L\cos(\omega t) = 6.00\ \text{nm} - (2.08\ \text{nm})\cos(\omega t)$

The oscillation lives entirely in the cross term. A candidate who adds the two diagonal contributions and stops has computed $\langle x\rangle$ for a mixture, which sits at $L/2$ for ever.

(d) CALCULATE · U3 **[1]**

Calculate the period of the oscillation of $\langle x\rangle$.

1. $T = 2\pi\hbar/(E_2 - E_1) = h/(3E_1) = 6.626 \times 10^{-34}\ \text{J s}/(3 \times 6.25 \times 10^{-21}\ \text{J}) = 3.53 \times 10^{-14}$ s (35.3 fs); $2\pi\hbar/(3E_1)$ with the given $\hbar$ gives 3.54×10^{-14} s, equally acceptable

The period depends on the difference $E_2 - E_1$ alone; using E_2 or E_1 by itself treats an unobservable global phase as if it set a frequency.

(e) PREDICT · U1 **[2]**

The electron is instead prepared in $[3\psi_1 + 4\psi_3]/5$. Predict how $\langle x\rangle$ behaves in time, and give the reason.

1. $\langle x\rangle = L/2$ at all times: it does not oscillate
2. ψ_1 and ψ_3 are both even about $x = L/2$, so $(x - L/2)\psi_1\psi_3$ is odd about the centre and integrates to zero, which with $\langle\psi_1|\psi_3\rangle = 0$ gives $x_{13} = \int\psi_1\, x\, \psi_3\, dx = 0$; the cross term vanishes even though $|\Psi|^2$ still changes shape symmetrically in time

Many candidates expect every two-state superposition to slosh from side to side. The box selection rule for x, non-zero only when $m - n$ is odd, decides whether it does.

12 CORE COURSE

STRUCTURED · ROUTINE · UPV-11.05 · UPV-11.07 · UPV-11.08 · UPV-11.09 **[11]**

X-ray spectra of the hot gas in galaxy clusters contain lines from O^{7+}, an oxygen nucleus ($Z = 8$) with a single bound electron. Treat the ion with the non-relativistic Coulomb Hamiltonian and an infinitely heavy nucleus, for which $E_n = -(13.6\ \text{eV})Z^2/n^2$. The ground-state radial probability density is $P(r) = (4Z^3/a_0^3)\, r^2\, e^{(-2Zr/a_0)}$, with $a_0 = 52.9$ pm. Use $hc = 1239.8$ eV nm.

(a) CALCULATE · U3 **[3]**

Calculate the ionisation energy of ground-state O^{7+}, and the photon energy and wavelength of its $2p \to 1s$ (Lyman-α) line.

1. ionisation energy $= (13.6\ \text{eV}) \times 8^2 = 870$ eV
2. photon energy for $2p \to 1s = 870.4\ \text{eV} \times (1 - 1/4) = 653$ eV
3. $\lambda = hc/E = 1239.8\ \text{eV nm}/652.8\ \text{eV} = 1.90$ nm

Forgetting the factor Z^2 gives 10.2 eV, the hydrogen Lyman-α line in the ultraviolet, rather than a line in the X-ray band where this one is observed.

(b) SHOW THAT · U2 **[3]**

Show that the most probable distance of the electron from the nucleus in the ground state is a_0/Z, and calculate that distance for O^{7+}.

1. $dP/dr \propto (2r - 2Zr^2/a_0)e^{(-2Zr/a_0)}$
2. $dP/dr = 0$ for $r > 0$ gives $r = a_0/Z$, a maximum because $P = 0$ at $r = 0$ and $P \to 0$ as $r \to \infty$
3. $r = 52.9\ \text{pm}/8 = 6.61$ pm

Maximising $|\psi|^2$ instead of $r^2|\psi|^2$ returns $r = 0$: the probability per unit volume peaks at the nucleus, but the probability per unit radius carries the shell area $4\pi r^2$.

(c) JUSTIFY · U1 **[3]**

The electron is excited into one of the $n = 3$ states. Justify which of 3s, 3p and 3d can decay to 1s by emitting a single electric-dipole photon, and state the route to 1s for each of the others.

1. the dipole operator is a rank-1 vector operator of odd parity, so the Wigner–Eckart triangle rule and parity together require $\Delta l = \pm 1$ (with $\Delta m_l = 0, \pm 1$)
2. only $3p \to 1s$ (Lyman-β) is allowed directly; $3s \to 1s$ has $\Delta l = 0$ and $3d \to 1s$ has $\Delta l = 2$, so both are forbidden
3. 3s and 3d each decay first to 2p ($\Delta l = -1$ from 3d, $\Delta l = +1$ from 3s) and then $2p \to 1s$

Candidates often allow $3d \to 1s$ because Δn is unrestricted; the rule constrains l, not n. Sending 3s to 2s is the other common slip, since that step has $\Delta l = 0$.

(d) EXPLAIN · U1 **[2]**

In this model the 3s, 3p and 3d levels are exactly degenerate. Explain why this degeneracy is special to the Coulomb potential, and name the effect that splits these levels in a real O^{7+} ion.

1. for a $1/r$ potential the Runge–Lenz vector commutes with H and, with L, closes an so(4) algebra whose multiplets contain every l from 0 to $n - 1$; any other central potential keeps only the $(2l + 1)$-fold rotational degeneracy in m_l
2. relativistic fine structure (the kinetic-energy correction, spin–orbit coupling and the Darwin term), which scales as $Z^4\alpha^2$ and is about $8^4 \approx 4 \times 10^3$ times larger than in hydrogen, splits the levels by j; levels of equal n and j are then separated by the much smaller Lamb shift

Rotational symmetry alone explains only the m_l degeneracy, so attributing the l degeneracy to spherical symmetry is the commonest error. Screening is also wrong here: O^{7+} has only one electron.

13

CORE COURSE

STRUCTURED · DEMANDING · UPV-04.03 · UPV-08.03 · UPV-10.04 · UPV-10.08 **[12]**

A water sample sits in the uniform static field $B_0 = 9.40$ T of an NMR magnet, directed along +z. Consider the spin of a single hydrogen nucleus. The proton's magnetic moment is $\mu = (g_p \mu_N/\hbar)S$, with $g_p = 5.586$ and $\mu_N = 5.051 \times 10^{-27}$ J T^{-1}, so the Hamiltonian is $H = -\mu \cdot B = -\omega_0 S_z$ with $\omega_0 = g_p \mu_N B_0/\hbar$. Take $\hbar = 1.055 \times 10^{-34}$ J s. A radio-frequency pulse, treated as instantaneous, leaves the spin at $t = 0$ in the $+\hbar/2$ eigenstate of $\hat{n}\cdot S$, where $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the unit vector with polar angle θ and azimuth φ. The spin operators are $S_i = (\hbar/2)\sigma_i$, and in the basis of the S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ the Pauli matrices are $\sigma_x = (0, 1; 1, 0)$, $\sigma_y = (0, -i; i, 0)$ and $\sigma_z = (1, 0; 0, -1)$, with rows separated by semicolons.

(a) DETERMINE · U2 **[2]**

Determine the matrix of $\hat{n}\cdot S$ in the S_z basis, and its normalised eigenvector with eigenvalue $+\hbar/2$ written as a combination of $|\uparrow\rangle$ and $|\downarrow\rangle$.

1. $\hat{n}\cdot S = (\hbar/2)(\cos\theta, \sin\theta\, e^{-i\varphi}; \sin\theta\, e^{i\varphi}, -\cos\theta)$
2. $|\hat{n}+\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\varphi}\sin(\theta/2)|\downarrow\rangle$, normalised and checked against the first row: $\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2) = \cos(\theta/2)$; any overall phase is accepted

Writing full angles, $\cos\theta|\uparrow\rangle + e^{i\varphi}\sin\theta|\downarrow\rangle$, gives a normalised ket that is not an eigenvector. Testing the answer at $\theta = \pi$, where the eigenvector must be $|\downarrow\rangle$, catches the error at once: the full-angle form returns $-|\uparrow\rangle$.

(b) SHOW THAT · U2 **[3]**

Using the propagator $U(t) = \exp(-iHt/\hbar)$, show that at time t the spin is in the $+\hbar/2$ eigenstate of $\hat{n}(t)\cdot S$, where $\hat{n}(t)$ has the same polar angle θ and the azimuth $\varphi - \omega_0 t$.

1. $H = -\omega_0 S_z$ is diagonal in the S_z basis, with energy $-\hbar\omega_0/2$ for $|\uparrow\rangle$ and $+\hbar\omega_0/2$ for $|\downarrow\rangle$, so $U(t)$ multiplies $|\uparrow\rangle$ by $e^{i\omega_0 t/2}$ and $|\downarrow\rangle$ by $e^{-i\omega_0 t/2}$
2. $U(t)|\hat{n}+\rangle = e^{i\omega_0 t/2}\cos(\theta/2)|\uparrow\rangle + e^{i\varphi}\, e^{-i\omega_0 t/2}\sin(\theta/2)|\downarrow\rangle$
3. factoring out the global phase $e^{i\omega_0 t/2}$, which changes no prediction, leaves $\cos(\theta/2)|\uparrow\rangle + e^{i(\varphi - \omega_0 t)}\sin(\theta/2)|\downarrow\rangle$, the eigenvector of (a) with θ unchanged and φ replaced by $\varphi - \omega_0 t$

Losing the minus sign in $H = -\omega_0 S_z$ gives the azimuth $\varphi + \omega_0 t$, a precession in the wrong sense. The result is given, so the marks are for the route, and the sense is physical: a proton, whose moment is parallel to its spin, precesses clockwise as seen from +z.

(c) CALCULATE · U3 **[2]**

Calculate ω_0, and, for $0 < \theta < \pi$, the shortest time after $t = 0$ at which the spin is again in the same physical state as at $t = 0$.

1. $\omega_0 = g_p \mu_N B_0/\hbar = 5.586 \times 5.051 \times 10^{-27}\ \text{J T}^{-1} \times 9.40\ \text{T}/(1.055 \times 10^{-34}\ \text{J s}) = 2.51 \times 10^9\ \text{rad s}^{-1}$, a Larmor frequency $f_0 = \omega_0/2\pi = 400$ MHz
2. the azimuth first returns to φ when $\omega_0 t = 2\pi$, so $t = 2\pi/\omega_0 = 2.50 \times 10^{-9}$ s; the ket is then $-|\hat{n}+\rangle$, which is the same ray

A candidate who recalls that a spinor needs a rotation through 4π to return to itself gives 5.00×10^{-9} s. After 2π the ket has only changed sign, a global phase, so the physical state has already returned. Using h in place of $\hbar$ gives f_0 where ω_0 was asked for.

(d) DESCRIBE · U1 **[2]**

A receiver coil with its axis along x picks up a voltage proportional to $d\langle S_x\rangle/dt$. Describe the signal it records, and how the amplitude of that signal depends on θ.

1. the spin stays in the $+\hbar/2$ eigenstate of $\hat{n}(t)\cdot S$, so $\langle S\rangle = (\hbar/2)\hat{n}(t)$ keeps its length and its angle θ to the field while precessing about +z; $\langle S_x\rangle = (\hbar/2)\sin\theta\cos(\varphi - \omega_0 t)$, so the coil records a sinusoidal voltage at $f_0 = 400$ MHz
2. the amplitude is proportional to $\sin\theta$: greatest for $\theta = \pi/2$, a spin tipped into the xy-plane, and zero for $\theta = 0$ or π, when the state is an eigenstate of H and nothing precesses

Candidates who expect a signal whatever the pulse has done have not asked what precesses. When $\theta = 0$ or π every expectation value is constant, and only the part of $\langle S\rangle$ transverse to the field induces a voltage.

(e) SHOW THAT · U2 **[3]**

For operators A and B the Robertson relation reads $\Delta A\,\Delta B \geq \tfrac{1}{2}|\langle[A, B]\rangle|$. Apply it to H and S_x, and show that at every time t this spin obeys $\Delta E\,\Delta S_x \geq (\hbar/2)|d\langle S_x\rangle/dt|$, where ΔE is the uncertainty in its energy.

1. $[H, S_x] = -\omega_0[S_z, S_x] = -i\hbar\omega_0 S_y$, so the Robertson relation gives $\Delta E\,\Delta S_x \geq (\hbar\omega_0/2)|\langle S_y\rangle|$
2. $\langle S_x\rangle = (\hbar/2)\sin\theta\cos(\varphi - \omega_0 t)$ and $\langle S_y\rangle = (\hbar/2)\sin\theta\sin(\varphi - \omega_0 t)$, so $d\langle S_x\rangle/dt = (\hbar\omega_0/2)\sin\theta\sin(\varphi - \omega_0 t) = \omega_0\langle S_y\rangle$
3. substituting $\omega_0|\langle S_y\rangle| = |d\langle S_x\rangle/dt|$ gives $\Delta E\,\Delta S_x \geq (\hbar/2)|d\langle S_x\rangle/dt|$: the energy spread limits how fast $\langle S_x\rangle$ can change

Candidates who try to treat time as an operator have nothing to put into the Robertson relation. The route is through [H, S_x]: by Ehrenfest's theorem $\langle[H, S_x]\rangle = -i\hbar\, d\langle S_x\rangle/dt$, so a bound set by a commutator becomes a bound on a rate.

14

CORE COURSE

STRUCTURED · DEMANDING · UPV-12.07 · UPV-13.04 · UPV-13.07 **[12]**

The magnesium line at $\lambda = 517.3$ nm is the transition from the upper level $3s4s\ {}^3S_1$ to the lower level $3s3p\ {}^3P_1$. The source is placed in a uniform magnetic field $B = 1.20$ T, weak enough that LS coupling holds and each level shifts by $\Delta E = g_J \mu_B B M_J$, with $g_J = 1 + [J(J+1) + S(S+1) - L(L+1)]/[2J(J+1)]$. Use $\mu_B = 5.788 \times 10^{-5}$ eV T^{-1} and $hc = 1239.8$ eV nm. In the lower term, the ${}^3P_1$ and ${}^3P_2$ levels are 5.05 meV apart.

(a) CALCULATE · U3 **[2]**

Calculate the Landé g factors of the upper and lower levels.

1. ${}^3S_1$ has $L = 0$, $S = 1$, $J = 1$: $g = 1 + (2 + 2 - 0)/4 = 2$
2. ${}^3P_1$ has $L = 1$, $S = 1$, $J = 1$: $g = 1 + (2 + 2 - 2)/4 = 3/2$

Reading the superscript 3 as $S = 3$, rather than $2S + 1 = 3$, gives meaningless g factors; so does using one electron's l in place of the term's total L.

(b) JUSTIFY · U1 **[2]**

Justify that this line is an allowed electric-dipole transition.

1. a single electron jumps 4s → 3p with $\Delta l = 1$, so the configuration parity changes from even to odd, as the odd-parity dipole operator requires
2. $\Delta S = 0$, $\Delta L = 1$ and $\Delta J = 0$ between two $J = 1$ levels are all allowed; the only forbidden $\Delta J = 0$ case is $J = 0 \to J = 0$, which this is not

Candidates who quote $\Delta J = \pm 1$ alone would forbid this bright line; $\Delta J = 0$ is allowed except between two $J = 0$ levels.

(c) DETERMINE · U2 **[4]**

Determine how many Zeeman components the line splits into, and the shift of each from the zero-field photon energy in units of $\mu_B B$.

1. photon energy shift $= \mu_B B(g_u M_u - g_l M_l) = \mu_B B(2M_u - 1.5M_l)$, with $\Delta M = M_u - M_l = 0$ or ± 1, giving seven (M_u, M_l) pairs
2. the $M_u = 0 \to M_l = 0$ component vanishes because $\Delta J = 0$: its Clebsch–Gordan coefficient $\langle 1\ 0;\ 1\ 0|1\ 0\rangle$ is zero (the 3j symbol with all projections zero vanishes when $1 + 1 + 1$ is odd)
3. six components remain
4. the π components ($\Delta M = 0$) are shifted by $\pm 0.5\mu_B B$, and the σ components ($\Delta M = \pm 1$) by $\pm 1.5\mu_B B$ and $\pm 2.0\mu_B B$

Most candidates count seven components. The missing central π component is the discriminating point: it follows from Wigner–Eckart geometry, not from an extra rule to memorise.

(d) CALCULATE · U3 **[2]**

Calculate the wavelength separation between the two outermost components.

1. the outermost components are $4.0\mu_B B$ apart: $4 \times 5.788 \times 10^{-5}$ eV T^{-1} $\times$ 1.20 T = 2.78×10^{-4} eV
2. $\Delta\lambda = \lambda^2\Delta E/(hc) = (517.3\text{ nm})^2 \times 2.78 \times 10^{-4}$ eV$/(1239.8$ eV nm$) = 0.0600$ nm (60.0 pm)

   Using the shift of one outer component, $2.0\mu_B B$, in place of the separation of the pair halves the answer.

(e) ESTIMATE · U4 **[2]**

Estimate the field above which this anomalous Zeeman pattern gives way to the Paschen–Back regime, stating the criterion you use.

1. criterion: the magnetic energy $\mu_B B$ becomes comparable with the fine-structure interval of the ^{3}P term, 5.05 meV, so that the field decouples L from S
2. $B \approx 5.05 \times 10^{-3}$ eV$/5.788 \times 10^{-5}$ eV T^{-1} $\approx$ 90 T (of order 10^2 T), far above 1.20 T, which confirms that the weak-field treatment used here is valid

   An estimate needs its criterion; a bare number earns nothing. Comparing $\mu_B B$ with the photon energy instead of the fine-structure interval gives a field about 500 times too large.

15

CORE COURSE

STRUCTURED · DEMANDING · UPV-17.08 · UPV-19.02 · UPV-19.03 · UPV-19.09 **[12]**

A π^- beam strikes a liquid-hydrogen target, and the reaction $\pi^- + p \to K^0 + \Lambda^0$ is studied. The Λ^0 then decays as $\Lambda^0 \to p + \pi^-$ with mean lifetime 2.63×10^{-10} s. Rest energies: $m_\pi c^2 = 139.57$ MeV, $m_p c^2 = 938.27$ MeV, $m_K c^2 = 497.61$ MeV and $m_\Lambda c^2 = 1115.68$ MeV. The quark contents are $K^0 = d\bar{s}$ and $\Lambda^0 =$ uds, and a strange quark carries strangeness $S = -1$.

(a) STATE · U1 **[2]**

State the total charge, baryon number and strangeness on each side of the production reaction, and state which interaction can mediate it.

1. charge: $-1 + 1 = 0$ and $0 + 0 = 0$; baryon number: $0 + 1 = 1$ and $0 + 1 = 1$; strangeness: $0 + 0 = 0$ and $(+1) + (-1) = 0$
2. all three balance, strangeness included, so the strong interaction can mediate the production

   Giving K^0 strangeness -1 because it contains strangeness unbalances the ledger; its $\bar{s}$ antiquark carries $S = +1$.

(b) DERIVE · U2 **[3]**

Derive, from four-momentum conservation, an expression for the minimum laboratory energy E_π of the pion, with the target proton at rest, for which the reaction can occur.

1. the invariant $s = (P_\pi + P_p)\cdot(P_\pi + P_p)$ is the same in every frame, and in the laboratory $s = m_\pi^2c^4 + m_p^2c^4 + 2E_\pi m_p c^2$
2. at threshold the products are at rest in the centre-of-momentum frame, so $s = (m_K + m_\Lambda)^2c^4$
3. equating the two: $E_\pi = [(m_K + m_\Lambda)^2 - m_\pi^2 - m_p^2]c^2/(2m_p)$

   Equating the pion's kinetic energy to the mass deficit, as if the products could be left at rest in the laboratory, ignores momentum conservation and underestimates the threshold kinetic energy by about 30 per cent.

(c) CALCULATE · U3 **[2]**

Calculate the threshold kinetic energy and momentum of the pion.

1. $E_\pi = (1613.29^2 - 139.57^2 - 938.27^2)/(2 \times 938.27)$ MeV $= 907.5$ MeV, so the kinetic energy is $E_\pi - m_\pi c^2 = 767.9$ MeV
2. $p_\pi c = \sqrt{(E_\pi^2 - m_\pi^2c^4)} = 896.7$ MeV, so $p_\pi = 897$ MeV/c

   Quoting E_π as the kinetic energy is the usual slip; the rest energy has to be removed.

(d) COMPARE · U1 **[2]**

The Σ^0 baryon, also uds, decays as $\Sigma^0 \to \Lambda^0 + \gamma$ with mean lifetime 7.4×10^{-20} s. Compare its lifetime with that of the Λ^0, relating the difference to the interaction responsible for each decay.

1. $\Lambda^0 \to p + \pi^-$ changes strangeness from -1 to 0, which neither the strong nor the electromagnetic interaction allows, so it is a weak decay; $\Sigma^0 \to \Lambda^0 + \gamma$ conserves strangeness and emits a photon, so it is electromagnetic
2. the lifetimes differ by a factor 2.63×10^{-10} s$/7.4 \times 10^{-20}$ s $\approx 4 \times 10^{9}$; by the golden rule the width $\Gamma = \hbar/\tau$ scales as the square of the transition amplitude, so the ratio reflects how much weaker the weak amplitude is (phase space differs as well, but cannot account for nine orders of magnitude)

   Saying only that weak decays are slow earns nothing; the mark is for the strangeness change that closes the faster routes.

(e) CALCULATE · U3 **[3]**

Calculate the momentum of the proton in the rest frame of the Λ^0 for the decay $\Lambda^0 \to p + \pi^-$.

1. momentum conservation makes the two momenta equal and opposite, of magnitude p^*, and energy conservation gives $\sqrt{(p^{*2}c^2 + m_p^2c^4)} + \sqrt{(p^{*2}c^2 + m_\pi^2c^4)} = m_\Lambda c^2$
2. solving: $p^*c = \sqrt{([m_\Lambda^2 - (m_p + m_\pi)^2][m_\Lambda^2 - (m_p - m_\pi)^2])}\, c^2/(2m_\Lambda)$
3. $p^* = 100.6$ MeV/c; the proton's kinetic energy is then 5.4 MeV and the pion's 32.5 MeV, sharing $Q = 37.8$ MeV

   Treating the proton non-relativistically is harmless, but the pion's kinetic energy is almost a quarter of its rest energy: an all-non-relativistic solution gives 96 MeV/c, wrong in the second significant figure.

16

CORE COURSE

STRUCTURED · DEMANDING · UPV-20.03 · UPV-20.05 **[10]**

A source sends one qubit of each pair to Alice and the other to Bob, in the state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Alice measures the observable $A(a) = \cos a\, Z + \sin a\, X$ on her qubit and Bob measures $B(b) = \cos b\, Z + \sin b\, X$ on his, each with outcomes ± 1, where X and Z are Pauli matrices. The correlation is $E(a, b) = \langle \Phi^+ | A(a) \otimes B(b) | \Phi^+ \rangle$, and the CHSH combination is $S = E(a, b) + E(a, b') + E(a', b) - E(a', b')$. Every local hidden-variable model obeys $|S| \le 2$.

(a) SHOW THAT · U2 **[3]**

Show that $E(a, b) = \cos(a - b)$.

1. expands $A \otimes B = \cos a \cos b\, Z\otimes Z + \cos a \sin b\, Z\otimes X + \sin a \cos b\, X\otimes Z + \sin a \sin b\, X\otimes X$
2. $Z\otimes Z$ and $X\otimes X$ leave $|\Phi^+\rangle$ unchanged, so $\langle Z\otimes Z\rangle = \langle X\otimes X\rangle = 1$, while $Z\otimes X$ and $X\otimes Z$ map $|\Phi^+\rangle$ to $\pm(|01\rangle - |10\rangle)/\sqrt{2}$, which is orthogonal to it, so those expectation values are 0
3. $E(a, b) = \cos a \cos b + \sin a \sin b = \cos(a - b)$

   Each qubit on its own is maximally mixed, so $\langle A\rangle = \langle B\rangle = 0$. Candidates who factorise E as $\langle A\rangle \langle B\rangle$ get zero and have assumed away the correlation the question is about.

(b) CALCULATE · U3 **[2]**

Calculate S for the settings $a = 0$, $a' = \pi/2$, $b = \pi/4$ and $b' = -\pi/4$.

1. $E(0, \pi/4) = E(0, -\pi/4) = E(\pi/2, \pi/4) = \cos(\pi/4) = 0.707$ and $E(\pi/2, -\pi/4) = \cos(3\pi/4) = -0.707$
2. $S = 3 \times 0.707 + 0.707 = 2\sqrt{2} = 2.83$, above the local bound of 2

   The minus sign on the fourth term is where the violation comes from; dropping it gives $S = 1.41$.

(c) DETERMINE · U2 **[3]**

Noise in the source replaces $|\Phi^+\rangle\langle\Phi^+|$ by the density operator $\rho = p|\Phi^+\rangle\langle\Phi^+| + (1 - p)I/4$, with $0 \le p \le 1$. Determine S for the same settings as a function of p, and the range of p for which these settings show a violation.

1. $E_\rho(a, b) = \mathrm{Tr}[\rho\, A \otimes B] = p \cos(a - b) + (1 - p)\mathrm{Tr}(A)\mathrm{Tr}(B)/4$
2. A and B are traceless, so the noise term vanishes and $S = 2\sqrt{2}\, p$
3. a violation needs $2\sqrt{2}\, p > 2$, so $p > 1/\sqrt{2} = 0.707$

   Candidates who give the white-noise part a correlation of its own, often $\pm 1/4$, have forgotten that every Pauli combination here is traceless.

(d) EVALUATE · U4 **[2]**

An experiment using these settings reports $S = 2.38 \pm 0.05$. Evaluate what this result establishes and what it does not.

1. S exceeds 2 by $(2.38 - 2)/0.05 = 7.6$ standard uncertainties, so the correlations are incompatible with every local hidden-variable model under the test's assumptions; in the noise model it implies $p = 2.38/2.83 = 0.84$, above the threshold of 0.707
2. it excludes local hidden variables only if the loopholes are closed (settings chosen at spacelike separation, and detection efficient enough that the recorded pairs are not a biased sample), and it neither proves that exactly $|\Phi^+\rangle$ was prepared nor allows any signal to pass between Alice and Bob

   A strong answer states the size of the violation in standard uncertainties and names the assumptions it rests on; the bare claim that quantum mechanics is proved earns neither mark.

17

CORE COURSE

DATA ANALYSIS · DEMANDING · UPV-15.02 · UPV-15.07 · UPV-15.09 **[12]**

A radiopharmacy confirms the identity of a batch of fluorine-18 by measuring its half-life. A sample is placed in a well counter and the counts in 100 s are recorded at intervals; t is the time at the middle of each 100 s count, measured from the middle of the first. With no sample present, the counter records 1200 counts in 1000 s. The site's quality-control rule accepts the batch if the measured half-life lies between 105 min and 115 min. The accepted half-life of fluorine-18 is 109.8 min.

(a) EXPLAIN · U4 **[2]**

Explain why the background must be subtracted from each reading before logarithms are taken, and why the uncertainty in the background itself can be neglected here.

1. the background adds a constant 120 counts per 100 s to an exponential, so ln(gross counts) is not linear in t: the late readings sit above the straight line and a fit would give too small a λ, that is, too long a half-life
2. the background uncertainty is √1200/10 ≈ 3.5 counts per 100 s, which shifts ln(net counts) by at most 3.5/5003 ≈ 7 × 10^{-4} at t = 360 min, far below that reading's own Poisson uncertainty of about 0.014

Omitting the background, or subtracting it only after fitting ln(gross counts), is the commonest procedural error in decay counting; it bends the late-time data upward and mimics a long-lived impurity.

(b) DETERMINE · U4 **[3]**

Determine the decay constant λ from the data, using either a straight-line fit of ln(net counts) against t or two well-separated readings.

1. subtracts 120 from each reading, for example net counts 48132 at t = 0 and 5003 at t = 360 min, giving ln(net counts) = 10.782 and 8.518
2. gradient = (8.518 − 10.782)/(360 min) = −6.29 × 10^{-3} min^{-1} (a least-squares fit to all seven points gives −6.30 × 10^{-3} min^{-1})
3. λ = 6.3 × 10^{-3} min^{-1} = 1.05 × 10^{-4} s^{-1}; accept values from 6.27 × 10^{-3} to 6.33 × 10^{-3} min^{-1}

Using $\log_{10}$ and reading its gradient as λ gives a value 2.30 times too small; only the natural logarithm has gradient −λ.

(c) CALCULATE · U3 **[2]**

Calculate the half-life and the mean lifetime.

1. $T_{½}$ = ln 2/λ = 0.6931/(6.29 × 10^{-3} min^{-1}) = 110 min (110.2 min from the two end readings, 109.9 min from the least-squares fit)
2. τ = 1/λ = 159 min

τ and $T_{½}$ differ by the factor ln 2. Quoting 1/λ = 159 min as the half-life would reject a batch that is in fact fluorine-18.

(d) ESTIMATE · U4 **[3]**

Estimate the uncertainty in λ that arises from counting statistics, assuming each reading is Poisson-distributed and using the two readings from part (b).

1. σ[ln(net counts)] = √(gross counts)/(net counts): √48252/48132 = 0.0046 at t = 0 and √5123/5003 = 0.0143 at t = 360 min
2. δλ ≈ √(0.0046² + 0.0143²)/(360 min) = 4.2 × 10^{-5} min^{-1}, a fractional uncertainty of 0.7 per cent
3. $\delta T_{½} = T_{½}\,\delta\lambda/\lambda$ ≈ 0.7 min, so $T_{½}$ = 110.2 ± 0.7 min (a weighted fit to all seven readings narrows this to about ±0.5 min)

Taking the uncertainty in ln N to be √N rather than √N/N makes λ look unmeasurable; using √(net) in place of √(gross) is only a minor slip here.

(e) EVALUATE · U4 **[2]**

Evaluate whether the batch passes the quality-control rule, and whether the data show any sign of a longer-lived radionuclidic impurity.

1. $T_{½}$ = 110.2 ± 0.7 min lies inside 105 min to 115 min and agrees with the accepted 109.8 min to within about 0.6 standard uncertainties, so the batch passes
2. the intermediate readings lie on both sides of a least-squares line, each within about one Poisson standard uncertainty (within two of the line through the end readings), with no systematic upward curvature at late times; a longer-lived impurity would lift the late readings above the line, so none is detectable above a few per cent of the count rate at 360 min

   Concluding that the data fit a straight line, without saying what an impurity would do to that line, is description rather than evaluation.

18

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPV-14.04 · UPV-16.06 **[15]**

In a deuterium–tritium plasma the reaction $^{2}H + {}^{3}H \rightarrow {}^{4}He + n$ releases the energy that fusion devices aim to harvest. Atomic masses: $m(^{2}H) = 2.014102$ u, $m(^{3}H) = 3.016049$ u and $m(^{4}He) = 4.002603$ u; the neutron mass is 1.008665 u; 1 u = 931.494 MeV/c². For a Maxwellian plasma at temperature T, the non-resonant reaction rate per pair of nuclei is $\langle\sigma v\rangle \propto \int_0^\infty S(E)\exp[-E/(kT) - \sqrt{(E_G/E)}]\,dE$, where E is the centre-of-mass energy, S(E) varies slowly with E, $e^{-E/kT}$ is the Maxwell–Boltzmann factor and $e^{-\sqrt{(E_G/E)}}$ is the probability of tunnelling through the Coulomb barrier. For D–T the Gamow energy is $E_G = 2\mu c^2(\pi\alpha Z_1 Z_2)^2 = 1182$ keV.

(a) CALCULATE · U3 **[2]**

Calculate the Q value of the reaction.

1. $\Delta m = 2.014102\text{ u} + 3.016049\text{ u} - 4.002603\text{ u} - 1.008665\text{ u} = 0.018883$ u, where atomic masses may be used because the two electrons on the left are matched by the two in ^{4}He
2. $Q = 0.018883\text{ u} \times 931.494\text{ MeV/u} = 17.59$ MeV

   Mixing atomic and nuclear masses, for instance subtracting $2m_e$ from ^{4}He alone, leaves two electron masses unbalanced, an error of 1.022 MeV, about 6 per cent of Q and far larger than the data's precision.

(b) SKETCH · U1 **[2]**

On one set of axes against E, sketch the Maxwell–Boltzmann factor, the tunnelling factor and their product, marking the energy E_0 at which the product peaks.

1. the Maxwell–Boltzmann factor falls monotonically from 1 at E = 0, and the tunnelling factor rises monotonically from 0 at E = 0, very flat near the origin, towards 1 at large E
2. their product, drawn on an expanded vertical scale and labelled as such, is a single peak at E_0, several times kT, and is smaller than both factors at every energy

   Placing the peak near kT, where the Maxwellian itself is largest (kT/2 for the energy distribution, kT for its flux-weighted form), misses the whole construction: the tunnelling factor pulls the reacting nuclei far into the tail.

(c) DERIVE · U2 **[3]**

Starting from the rate integral given in the stem, derive an expression for E_0 in terms of E_G and kT by maximising the exponent of its integrand.

1. writes $f(E) = -E/(kT) - E_G^{1/2}E^{-1/2}$, since S(E) varies slowly and the exponential fixes where the integrand peaks
2. $df/dE = -1/(kT) + ½E_G^{1/2}E^{-3/2} = 0$
3. $E_0^{3/2} = ½E_G^{1/2}kT$, so $E_0 = [E_G(kT)^2/4]^{1/3}$

   Differentiating $E^{-1/2}$ as $-½E^{1/2}$, raising the power instead of lowering it, gives $E_0 = 4/[E_G(kT)^2]$, which falls as T rises and is not even an energy; checking that E_0 grows with temperature and has the dimensions of energy catches it.

(d) SHOW THAT · U2 **[3]**

By expanding the exponent to second order about E_0, show that the integrand is approximately proportional to $\exp[-(E-E_0)^2/(\Delta/2)^2]$ with $\Delta = 4\sqrt{(E_0 kT/3)}$.

1. $f''(E) = -\tfrac{3}{4}E_G\hat{}(1/2)E\hat{}(-5/2)$, and the stationary condition gives $E_G\hat{}(1/2) = 2E_0\hat{}(3/2)/(kT)$, so $f''(E_0) = -3/(2E_0 kT)$
2. $f(E) \approx f(E_0) + \tfrac{1}{2}f''(E_0)(E-E_0)^2 = f(E_0) - 3(E-E_0)^2/(4E_0 kT)$, the first-order term vanishing at the maximum
3. matching $3/(4E_0 kT) = 1/(\Delta/2)^2$ gives $(\Delta/2)^2 = 4E_0 kT/3$, so $\Delta = 4\sqrt{(E_0 kT/3)}$

Candidates who evaluate f'' without eliminating E_G through the stationary condition usually reach a correct but unrecognisable expression and cannot finish.

(e) CALCULATE · U3 **[3]**

Calculate E_0 and Δ for $kT = 10.0$ keV, and comment on how sharply the Gaussian form localises the reacting energies at this temperature.

1. $E_0 = [1182 \text{ keV} \times (10.0 \text{ keV})^2/4]\hat{}(1/3) = 30.9$ keV, about $3.1kT$
2. $\Delta = 4\sqrt{(30.9 \text{ keV} \times 10.0 \text{ keV}/3)} = 40.6$ keV
3. $\Delta \approx 1.3E_0$, so the window is broad rather than sharp and noticeably asymmetric, extending further above E_0 than below it: the Gaussian gives the position and width of the window but not its exact shape (a numerical integration shows the Gaussian estimate of the integral is about 4 per cent low)

Laplace's method is exact only in the limit of a peak narrow compared with its position; at D–T temperatures Δ is comparable with E_0, and candidates who never compare the two have not checked the approximation they have just made.

(f) EVALUATE · U1 **[2]**

The Coulomb barrier between a deuteron and a triton at nuclear contact is about 0.44 MeV. A student argues that since $kT = 10.0$ keV is some 44 times smaller, fusion in this plasma must be negligible. Evaluate this argument.

1. the barrier height is not a threshold: nuclei tunnel through it, and at E_0 the tunnelling factor is $e\hat{}(-\sqrt{(1182/30.9)}) = e\hat{}(-6.18) \approx 2 \times 10^{-3}$, small but far from zero
2. the rate is set where this factor overlaps the Maxwellian tail (about 10 per cent of nuclei have $E > E_0 \approx 3kT$), so the reaction proceeds at an appreciable rate; the student's argument holds only classically, where no pair below 0.44 MeV could react

This is the argument Gamow's factor answers. A candidate who agrees with the student has treated the barrier classically straight after deriving a formula that contains the tunnelling probability.

19

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-18.04 · UPV-18.05 **[16]**

In the electroweak model the Higgs field is a complex doublet φ with hypercharge $\tfrac{1}{2}$ and potential $V(\varphi) = -\mu^2\varphi\dagger\varphi + \lambda(\varphi\dagger\varphi)^2$, with $\mu^2 > 0$ and $\lambda > 0$. Its covariant derivative is $D_\mu = \partial_\mu - ig(\sigma\hat{}a/2)W\hat{}a_\mu - i(g'/2)B_\mu$, summed over $a = 1, 2, 3$, where $\sigma\hat{}a$ are the Pauli matrices, and the Lagrangian contains $(D_\mu\varphi)\dagger(D\hat{}\mu\varphi) - V(\varphi)$. In unitary gauge the vacuum is $\varphi_0 = (0, v/\sqrt{2})^{\mathrm{T}}$. A real vector field V_μ of mass M contributes $\tfrac{1}{2}M^2V_\mu V\hat{}\mu$ to the Lagrangian, and the charged fields $W\hat{}\pm = (W^1 \mp iW^2)/\sqrt{2}$ of mass M_W contribute $M_W^2W^+_\mu W\hat{}-\mu$. Take $g = 0.652$, $g' = 0.357$, $v = 246$ GeV and $\lambda = 0.129$.

(a) DERIVE · U2 **[3]**

Derive the vacuum value v in terms of μ and λ. Then, writing $\varphi = (0, (v + h)/\sqrt{2})^{\mathrm{T}}$, derive the mass M_H of the physical scalar h.

1. with $\rho = \varphi\dagger\varphi$, $dV/d\rho = -\mu^2 + 2\lambda\rho = 0$ gives $\rho = \mu^2/(2\lambda) = v^2/2$, so $v = \mu/\sqrt{\lambda}$
2. substituting $\varphi\dagger\varphi = (v + h)^2/2$ gives $V = -\tfrac{1}{2}\mu^2(v + h)^2 + \tfrac{1}{4}\lambda(v + h)^4$, whose h^2 coefficient is $-\tfrac{1}{2}\mu^2 + (3/2)\lambda v^2 = \lambda v^2$
3. matching λv^2h^2 to $\tfrac{1}{2}M_H^2h^2$ gives $M_H = \sqrt{(2\lambda)}\, v = \sqrt{2}\,\mu$

Reading the mass from the h^2 coefficient without the $\tfrac{1}{2}$ in $\tfrac{1}{2}M_H^2h^2$ gives $M_H = \sqrt{\lambda}\, v$, too small by $\sqrt{2}$.

(b) STATE · U1 **[2]**

State how many real field components the doublet has, how many generators of SU(2)_L × U(1)_Y the vacuum breaks, and what becomes of the corresponding field components.

1. four real components; the vacuum is left invariant only by the U(1) generated by the charge $Q = T_3 + Y$, so $4 - 1 = 3$ generators are broken
2. the three would-be Goldstone modes become the longitudinal polarisations of W^+, W^- and Z, leaving one physical scalar, the Higgs boson h

Candidates who count four Goldstone bosons miss that one generator combination annihilates the vacuum, which is exactly why the photon stays massless.

(c) SHOW THAT · U2 **[4]**

Show that the kinetic term evaluated on φ_0 gives $M_W = gv/2$, and that the neutral fields acquire the mass-squared matrix $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$ in the basis (W^3, B), with rows separated by semicolons.

1. the gauge-field matrix is $\tfrac{1}{2}(gW^3 + g'B, g(W^1 - iW^2); g(W^1 + iW^2), -gW^3 + g'B)$, so $D_\mu\varphi_0 = -(i/2)(v/\sqrt{2})(g(W^1 - iW^2), -gW^3 + g'B)^T$
2. $(D_\mu\varphi_0)\dagger(D\hat{}\mu\varphi_0) = (v^2/8)[g^2(W^1W^1 + W^2W^2) + (gW^3 - g'B)^2]$
3. $W^1W^1 + W^2W^2 = 2W^+W^-$, so the charged term is $(g^2v^2/4)W^+W^-$ and $M_W = gv/2$
4. $(v^2/8)(gW^3 - g'B)^2 = \tfrac{1}{2}(W^3, B)M^2(W^3, B)^T$ with $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$

The factor ½ that separates a real field's mass term from a complex field's is where most attempts go wrong; tracking it separately for W^± and for the neutral pair is what earns the last two marks.

(d) DETERMINE · U2 **[3]**

Determine the eigenvalues and normalised eigenvectors of M^2, identify the photon and the Z boson, and express the Weinberg angle θ_W and the ratio M_W/M_Z in terms of g and g′.

1. $\det M^2 = (v^2/4)^2(g^2g'^2 - g^2g'^2) = 0$, so one eigenvalue is zero; its normalised eigenvector is the massless photon $A = (g'W^3 + gB)/\sqrt{(g^2 + g'^2)}$
2. the other eigenvalue equals the trace, $M_Z^2 = (v^2/4)(g^2 + g'^2)$, with normalised eigenvector the Z boson, $Z = (gW^3 - g'B)/\sqrt{(g^2 + g'^2)}$
3. writing $Z = \cos\theta_W W^3 - \sin\theta_W B$ gives $\tan\theta_W = g'/g$, and $M_W/M_Z = g/\sqrt{(g^2 + g'^2)} = \cos\theta_W$

Noticing the zero determinant finishes the diagonalisation in two lines; brute-force diagonalisation takes longer and often loses the normalisation of the eigenvectors.

(e) CALCULATE · U3 **[2]**

Calculate M_W, M_Z, $\sin^2\theta_W$ and M_H.

1. $M_W = 0.652 \times 246\ \text{GeV}/2 = 80.2\ \text{GeV}$ and $M_Z = (246\ \text{GeV}/2)\sqrt{(0.652^2 + 0.357^2)} = 91.4\ \text{GeV}$
2. $\sin^2\theta_W = g'^2/(g^2 + g'^2) = 0.231$ and $M_H = \sqrt{(2 \times 0.129)} \times 246\ \text{GeV} = 125\ \text{GeV}$

Taking $\sin\theta_W = g'/g$ confuses the sine with the tangent and gives $\sin^2\theta_W = 0.30$ instead of 0.231.

(f) EVALUATE · U1 **[2]**

The measured values are $M_W = 80.37$ GeV, $M_Z = 91.19$ GeV and $M_H = 125.1$ GeV. Evaluate which of your results are genuine predictions of the model and which only return the numbers put in.

1. g, g', v and λ are inputs fitted to data, so the individual values of M_W, M_Z and M_H are not independent predictions; M_H in particular only restates λ, which the model leaves free
2. the genuine tree-level predictions are structural: a massless photon, and $M_W = M_Z \cos\theta_W$, which follows from the doublet's quantum numbers and tests the model once θ_W is measured independently of the masses, in neutral-current couplings; the measured ratio $80.37/91.19 = 0.881$ agrees with $\cos\theta_W = 0.877$ at the half-per-cent level that loop corrections account for

Declaring that the model predicts $M_W = 80.2$ GeV is exactly the confusion this part is set to expose: with four fitted inputs, only relations among the outputs test the theory.

University Quantum Mechanics I · Midterm Examination

The first eight units on one paper: the experiments that forced a wavefunction, the Schrödinger equation and its probability current, operators, measurement and uncertainty, the free particle, the infinite and finite square wells, and tunnelling at steps and barriers. Every probability is an integral of $|\psi|^2$, and every coefficient is matched region by region before it is quoted.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**74**	**14**	**8**

SECTION	**UQMI · Midterm · University Quantum Mechanics I — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Recall 1 · Routine 7 · Demanding 5 · Discriminating 1** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UQMI-01 · UQMI-02 · UQMI-03 · UQMI-04 · UQMI-05 · UQMI-06 · UQMI-07 · UQMI-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Quantum Mechanics I · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.04 **[1]**

The Compton shift for a photon scattered through an angle θ by a free electron at rest is $\Delta\lambda = (h/m_e c)(1 - \cos\theta)$, with $h/m_e c = 2.43$ pm. Photons of wavelength 71.0 pm and photons of wavelength 500 nm are each scattered through 90°. Which statement best explains why the effect was established with X-rays rather than with visible light?

A The shift is 2.43 pm in both cases, which is 3.4% of 71.0 pm but only 4.9×10^{-6} of 500 nm, a fractional change about 7000 times smaller and far harder to detect.

B A visible photon carries no momentum, because its energy is far below the electron's rest energy, so it cannot make the electron recoil.

C The shift is proportional to the incident wavelength, so for 500 nm light it is so large that the scattered light leaves the visible band and goes undetected.

D The shift is 2.43 pm for X-rays but smaller by the factor 71.0 pm/500 nm for visible light, about 0.35 fm, which is too small to resolve.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**

Each function in the options is finite and continuous for all x, and takes the form stated at large |x|, where C and k are nonzero real constants. Which one could be normalised, and so represent the state of a particle?

A ψ(x) ≈ C/√|x| at large |x|

B ψ(x) ≈ C e^(ikx) at large |x|

C ψ(x) ≈ C/|x| at large |x|

D ψ(x) ≈ C cos(kx)/|x|^(1/4) at large |x|

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.05 **[1]**

A particle in a time-independent potential is in the state $\Psi(x,t) = [\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}]/\sqrt{2}$, where ψ_1 and ψ_2 are real, normalised, mutually orthogonal eigenfunctions of $\hat{H}$ with eigenvalues $E_1 \neq E_2$. Which quantity is independent of time whatever the forms of ψ_1 and ψ_2?

A $|\Psi(x,t)|^2$ at a point x where ψ_1 and ψ_2 are both nonzero

B $\langle x \rangle$

C $|\int \Psi^*(x,0)\Psi(x,t)\,dx|^2$, the probability of finding the particle still in its initial state

D $\langle H \rangle$

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-03.05 **[1]**

Acting on square-integrable functions on $-\infty < x < \infty$ that vanish, together with their derivatives, as $|x| \to \infty$, which one of these operators is Hermitian?

A d/dx

B $x\, d/dx$

C $i\, d^2/dx^2$

D $i\, d/dx$

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.05 **[1]**

A particle has Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, and $[\hat{x}, \hat{p}] = i\hbar$. What is the commutator $[\hat{x}, \hat{H}]$?

A $i\hbar\hat{p}/m$

B 0, because $\hat{x}$ commutes with $V(\hat{x})$

C $i\hbar\hat{p}/(2m)$

D $-i\hbar\hat{p}/m$

Answer

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6 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMI-04.09 **[1]**

Position is measured once on each of N electrons, all prepared in the same state ψ, with a detector whose resolution is far finer than the spread of the results. The readings have sample standard deviation s, and their mean has standard error $s/\sqrt{N}$. N is increased from 100 to 10 000. Which statement is correct?

A s and the standard error both fall by a factor of 10, because more data reduces every uncertainty.

B s stays close to σ_x of the state, while the standard error falls by a factor of 10.

C s and the standard error both stay the same, because the uncertainty principle fixes both.

D s grows, because each extra measurement disturbs the ensemble, while the standard error falls by a factor of 10.

Answer

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.09 · UQMI-05.01 · UQMI-05.04 **[1]**

A particle of mass m has V(x) = 0 everywhere on $-\infty < x < \infty$. Which statement about the solutions of its time-independent Schrödinger equation is correct?

A Only discrete energies are allowed, because ψ must be normalisable and only certain wavelengths fit.

B Every E > 0 is allowed, and each is twofold degenerate, with solutions e^(ikx) and e^(−ikx) where $k = \sqrt{(2mE)}/\hbar$.

C Every E > 0 is allowed, and each is non-degenerate, because states in one dimension are never degenerate.

D Negative energies are also allowed, with solutions e^(κx) and e^(−κx) where $\kappa = \sqrt{(-2mE)}/\hbar$.

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-06.03 **[1]**

A particle is in the ground state $\psi_1(x) = \sqrt{(2/L)}\sin(\pi x/L)$ of an infinite square well occupying $0 < x < L$. What is the probability that a measurement of position finds it in $0 < x < L/4$?

A 0.41

B 0.25

C 0.091

D 0.045

Answer

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9 CORE COURSE

STRUCTURED · ROUTINE · UQMI-01.07 · UQMI-01.08 · UQMI-01.09 · UQMI-02.08 **[9]**

A particle of mass m moving along the x-axis is in a stationary state of some potential V(x). Its wavefunction is ψ(x) = 0 for x < 0; ψ(x) = Ax/a for 0 ≤ x ≤ a; and ψ(x) = A e^(−(x−a)/a) for x > a, where a is a positive length and A is a positive real constant. ψ is continuous everywhere, but its slope jumps at x = 0 and at x = a.

(a) SHOW THAT **[3]**

Show that A = √(6/(5a)).

(b) CALCULATE **[2]**

Calculate the probability that a measurement of position finds the particle in 0 < x < a/2, and the probability that it finds the particle at x > a.

(c) EXPLAIN **[2]**

The slope of ψ must be continuous wherever V is finite. Explain why, and explain why the jump in slope at x = 0 is nevertheless admissible.

(d) DETERMINE **[2]**

The jump in slope at x = a requires V(x) to contain a delta-function term gδ(x − a). Determine g in terms of $\hbar$, m and a, and say whether the spike is attractive or repulsive.

10 CORE COURSE

STRUCTURED · DEMANDING · UQMI-04.08 · UQMI-05.02 · UQMI-05.05 · UQMI-05.07 · UQMI-05.08 · UQMI-05.09 **[9]**

An electron ($m_e = 9.109 \times 10^{-31}$ kg) moves freely along the x-axis, $V = 0$. At $t = 0$ it is prepared as a Gaussian wave packet centred on wavenumber k_0, with position standard deviation $\sigma_0 = 0.40$ nm and $\sigma_x\sigma_p = \hbar/2$. The kinetic energy corresponding to the central wavenumber is $E_0 = \hbar^2k_0^2/(2m_e) = 1.50$ eV. The width of the packet then evolves as $\sigma_x(t) = \sigma_0\sqrt{(1 + (\hbar t/(2m_e\sigma_0^2))^2)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) EXPLAIN **[2]**

Explain why no single plane wave e^(ik₀x) can describe the electron, and how a superposition of plane waves overcomes the difficulty.

(b) SHOW THAT **[2]**

Using the dispersion relation of a free particle, show that the centre of the packet moves at $v_g = \hbar k_0/m_e$, twice the phase velocity of its central component.

(c) CALCULATE **[3]**

Calculate the group velocity, the time at which σ_x has grown to $2\sigma_0$, and the distance the centre of the packet travels in that time.

(d) PREDICT **[2]**

The electron is instead prepared with $\sigma_0 = 0.80$ nm, still with $\sigma_x\sigma_p = \hbar/2$ at $t = 0$. Predict, with a reason, the new time for the width to double.

11 CORE COURSE

STRUCTURED · DEMANDING · UQMI-03.07 · UQMI-04.03 · UQMI-04.04 · UQMI-06.06 · UQMI-06.07 · UQMI-06.08 · UQMI-06.09 **[11]**

An electron ($m_e = 9.109 \times 10^{-31}$ kg) is confined along one axis of a nanostructure modelled as an infinite square well on $0 < x < L$, with $L = 1.20$ nm. Its energy eigenfunctions are $\psi_n(x) = \sqrt{(2/L)}\,\sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m_eL^2)$. At $t = 0$ its state is $\Psi(x,0) = \sqrt{(2/3)}\,\psi_1(x) + (i/\sqrt{3})\,\psi_2(x)$. You may use $\int_0^L \psi_n\, x\, \psi_n\, dx = L/2$ for every n, and $\int_0^L \psi_1\, x\, \psi_2\, dx = -16L/(9\pi^2)$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) CALCULATE **[2]**

Calculate the probabilities of obtaining E_1 and E_2 in a measurement of energy, and the expectation value $\langle H \rangle$ in electronvolts.

(b) SHOW THAT **[3]**

Write down $\Psi(x,t)$, and show that $\langle x \rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2))\,\sin(\omega t)$, where $\omega = (E_2 - E_1)/\hbar$.

(c) CALCULATE **[2]**

Calculate the period of the oscillation of $\langle x \rangle$, and its amplitude in nanometres.

(d) PREDICT **[2]**

Suppose instead that $\Psi(x,0) = \sqrt{(2/3)}\,\psi_1(x) + (1/\sqrt{3})\,\psi_2(x)$. Predict which of $P(E_1)$, $P(E_2)$, $\langle H \rangle$, and the period and amplitude of $\langle x \rangle(t)$ change, and what does change.

(e) DESCRIBE **[2]**

Return to the original state. At $t = 2.0$ fs a measurement of energy on the electron returns E_2. Describe the state immediately afterwards, and how $\langle x \rangle$ and the result of a second energy measurement behave from then on.

12 CORE COURSE

STRUCTURED · DEMANDING · UQMI-07.03 · UQMI-07.04 · UQMI-07.05 · UQMI-07.06 · UQMI-07.07 · UQMI-07.08 **[11]**

An electron is bound in a thin layer of material modelled as a symmetric finite square well: $V(x) = 0$ for $|x| < a$ and $V(x) = V_0$ for $|x| > a$, with well width $2a = 0.80$ nm and $V_0 = 2.00$ eV. Treat the electron as having the free-electron mass $m_e = 9.109 \times 10^{-31}$ kg. For a bound state with $0 < E < V_0$, write $k = \sqrt{(2m_eE)}/\hbar$, $\kappa = \sqrt{(2m_e(V_0 - E))}/\hbar$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_eV_0)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) JUSTIFY **[2]**

Justify looking for bound states that are either even or odd in x, rather than of general form.

(b) DERIVE **[4]**

Starting from the solutions of the Schrödinger equation in each region, derive the condition that the energies of the even bound states must satisfy, written as a single equation in z and z_0 alone.

(c) DETERMINE **[2]**

Calculate z_0, and hence determine how many bound states the well has and the parity of each.

(d) COMPARE **[3]**

The even condition has its lowest root at $z = 1.159$. Compare the ground-state energy with that of an infinite square well of the same width, 0.80 nm, and account for the difference.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UQMI-08.04 · UQMI-08.05 · UQMI-08.08 **[11]**

In a scanning tunnelling microscope a sharp tungsten tip is held above a clean metal surface with a small bias voltage between them, and the tunnelling current I is recorded as a piezoelectric actuator withdraws the tip in equal steps. Δd is the increase in the tip-to-surface gap from its starting value d_0, which is not known accurately but is estimated to be about 0.5 nm. The gap is modelled as a rectangular barrier of height φ above the energy of the tunnelling electrons. In the thick-barrier limit $I = I_0 \exp(-2\kappa d)$, with $\kappa = \sqrt{(2m_e\varphi)}/\hbar$, $d = d_0 + \Delta d$, and I_0 independent of d. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1\ \text{eV} = 1.602 \times 10^{-19}$ J.

Tunnelling current as the tip is withdrawn

Δd / pm	0	25	50	75	100	125
I / nA	3.20	1.85	1.13	0.644	0.388	0.222

(a)(i) EXPLAIN **[1]**

Explain why a plot of $\ln(I/\text{nA})$ against Δd should give a straight line even though d_0 is not known.

(a)(ii) STATE **[1]**

State what the gradient of that line represents.

(b) DETERMINE **[3]**

Determine κ from the data.

(c) CALCULATE **[2]**

Calculate the barrier height φ in electronvolts.

(d) EVALUATE **[2]**

Evaluate whether the thick-barrier form is adequate for analysing these readings.

(e) ESTIMATE **[2]**

Estimate, stating your assumptions, the change in gap that would alter the current by 5%.

14 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-02.06 · UQMI-02.07 · UQMI-08.01 · UQMI-08.02 **[15]**

A beam of electrons of mass m_e and energy E travels in the +x direction towards a potential step: $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, with V_0 real and positive. The electrons are described by stationary scattering states of the time-dependent Schrödinger equation $i\hbar\,\partial\Psi/\partial t = -(\hbar^2/2m_e)\partial^2\Psi/\partial x^2 + V(x)\Psi$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) DERIVE **[4]**

Derive, from the time-dependent Schrödinger equation and its complex conjugate, the continuity equation $\partial\rho/\partial t + \partial j/\partial x = 0$ for $\rho = |\Psi|^2$, obtaining an expression for the probability current j and stating where the reality of V is used.

(b) SHOW THAT **[4]**

For $E > V_0$, take $\psi(x) = e^{(ik_1x)} + r\,e^{(-ik_1x)}$ for $x < 0$ and $\psi(x) = t\,e^{(ik_2x)}$ for $x > 0$, where $k_1 = \sqrt{(2m_eE)}/\hbar$ and $k_2 = \sqrt{(2m_e(E - V_0))}/\hbar$. Explaining why there is no $e^{(-ik_2x)}$ term, show that the reflection and transmission probabilities are $R = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = 4k_1k_2/(k_1 + k_2)^2$.

(c) CALCULATE **[2]**

For electrons with E = 6.0 eV incident on a step of $V_0 = 4.5$ eV, calculate R, T and $|t|^2$, and confirm that $R + T = 1$.

(d) SKETCH **[3]**

The electrons now have E = 3.0 eV, below the step. Sketch $|\psi(x)|^2$ against x from about $x = -1$ nm to $x = +0.5$ nm, for an incident wave of unit amplitude, marking the spacing of the maxima for $x < 0$, the value of $|\psi|^2$ at $x = 0$, and the length over which $|\psi|^2$ falls by a factor of e for $x > 0$.

(e) EXPLAIN **[2]**

Explain why every electron is reflected when $E < V_0$, even though $|\psi|^2$ is not zero inside the step.

MARK SCHEME

University Quantum Mechanics I · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.04 **[1]**

The Compton shift for a photon scattered through an angle θ by a free electron at rest is $\Delta\lambda = (h/m_e c)(1 - \cos\theta)$, with $h/m_e c = 2.43$ pm. Photons of wavelength 71.0 pm and photons of wavelength 500 nm are each scattered through 90°. Which statement best explains why the effect was established with X-rays rather than with visible light?

KEY **A** **The shift is 2.43 pm in both cases, which is 3.4% of 71.0 pm but only 4.9 × 10^{-6} of 500 nm, a fractional change about 7000 times smaller and far harder to detect.**

D is the most tempting, because it reaches the right conclusion that visible light is a poor probe of the effect. It gets there by scaling the shift itself with wavelength, but the formula contains no λ at all: at 90° the absolute shift is 2.43 pm whatever the incident wavelength, and what differs is the fraction Δλ/λ. C makes the opposite scaling error, treating the shift as proportional to λ, which again contradicts the formula in the stem. B confuses energy with momentum: every photon carries p = h/λ, about 1.3 × 10^{-27} kg m s^{-1} at 500 nm, and the size of its energy compared with $m_e c^2$ decides how large the fractional shift is, not whether recoil happens. A is correct because 2.43/71.0 = 0.034, while 2.43 × 10^{-12} m / 500 × 10^{-9} m = 4.9 × 10^{-6}, about 7000 times smaller. A fractional change of that size is not beyond every optical instrument, since a good interferometer can resolve it; in practice the further obstacle is that a visible photon's few electronvolts are comparable with atomic binding energies, so the electrons do not recoil as free particles at rest.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**

Each function in the options is finite and continuous for all x, and takes the form stated at large |x|, where C and k are nonzero real constants. Which one could be normalised, and so represent the state of a particle?

KEY **C** **ψ(x) ≈ C/|x| at large |x|**

Normalisability needs $\int|\psi|^2\,dx$ to be finite, so the test is the tail of $|\psi|^2$, not the tail of ψ. A is the trap: ψ does fall to zero, but $|\psi|^2 = C^2/|x|$, and $\int dx/|x|$ grows as $\ln|x|$ without limit, so the integral diverges however slowly. D is tempting because the function both decays and oscillates, and oscillation looks as though it should cancel; but $|\psi|^2 = C^2\cos^2(kx)/\sqrt{|x|}$ is never negative and averages to $C^2/(2\sqrt{|x|})$, whose integral grows as $\sqrt{|x|}$. B is a plane wave: $|\psi|^2 = C^2$ everywhere, so the integral grows linearly and no choice of C normalises it. C is correct: $|\psi|^2 = C^2/x^2$, and $\int dx/x^2$ converges at large |x|.

U1

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.05 **[1]**

A particle in a time-independent potential is in the state $\Psi(x,t) = [\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}]/\sqrt{2}$, where ψ_1 and ψ_2 are real, normalised, mutually orthogonal eigenfunctions of $\hat{H}$ with eigenvalues $E_1 \neq E_2$. Which quantity is independent of time whatever the forms of ψ_1 and ψ_2?

KEY **D** **$\langle H \rangle$**

A is what a candidate expects if each phase factor is thought to cancel in the modulus squared, as it does for a single stationary state. With two terms the cross term $\psi_1\psi_2\cos((E_2 - E_1)t/\hbar)$ survives, so the density at any point where both functions are nonzero oscillates. B moves with the same cross term, weighted by $\int \psi_1 x \psi_2\, dx$; it is constant only in the special case where that integral vanishes, for example when ψ_1 and ψ_2 have the same parity, so it fails the words whatever the forms. C looks conserved because the state stays normalised, but normalisation is not overlap: up to an overall phase the integral is $[1 + e^{-i(E_2-E_1)t/\hbar)}]/2$, whose squared modulus is $\cos^2((E_2 - E_1)t/2\hbar)$. D is correct: $\langle H \rangle = (E_1 + E_2)/2$ at all times, because $\hat{H}$ returns each term multiplied by its own energy and orthogonality removes the cross terms.

U1

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4 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-03.05 **[1]**

Acting on square-integrable functions on $-\infty < x < \infty$ that vanish, together with their derivatives, as $|x| \to \infty$, which one of these operators is Hermitian?

KEY **D** **i d/dx**

Each option is tested by moving the operator across $\int f^*(\hat{O}g)\, dx$ with integration by parts, the boundary terms vanishing because the functions do. A gives $\int f^* g'\, dx = -\int (f)^* g\, dx$: one integration by parts flips the sign, so d/dx is anti-Hermitian. It is tempting because it is $\hat{p}$ with the constant $-i\hbar$ removed, and that factor of i is exactly what restores Hermiticity. B fails twice over: moving d/dx across gives a minus sign, and differentiating the x produces an extra term, so its adjoint is $-1 - x\, d/dx$. C needs two integrations by parts, which leave d^2/dx^2 itself Hermitian, but the i is complex-conjugated on the way across, so the adjoint is $-i\, d^2/dx^2$. D is correct: $\int f^*(ig')\, dx = -i\int (f)^* g\, dx = \int (if)^* g\, dx$, the sign from the integration by parts cancelled by the conjugation of i. It is $-\hat{p}/\hbar$.

U2

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.05 **[1]**

A particle has Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, and $[\hat{x}, \hat{p}] = i\hbar$. What is the commutator $[\hat{x}, \hat{H}]$?

KEY **A** **$i\hbar\hat{p}/m$**

B stops at the potential term. $V(\hat{x})$ does commute with $\hat{x}$, but the kinetic term does not, and it is the kinetic term that makes position and energy incompatible. C expands $[\hat{x}, \hat{p}^2]$ as $\hat{p}[\hat{x}, \hat{p}]$ alone and loses the second term; the product rule gives $[\hat{x}, \hat{p}^2] = \hat{p}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{p} = 2i\hbar\hat{p}$, and dividing by 2m leaves $i\hbar\hat{p}/m$. D is the sign error made by starting from $[\hat{p}, \hat{x}] = -i\hbar$ with the operators in the wrong order. A is correct, and it is consistent with $d\langle x \rangle/dt = (i/\hbar)\langle [\hat{H}, \hat{x}] \rangle = \langle p \rangle/m$.

U2

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6 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMI-04.09 **[1]**

Position is measured once on each of N electrons, all prepared in the same state ψ, with a detector whose resolution is far finer than the spread of the results. The readings have sample standard deviation s, and their mean has standard error $s/\sqrt{N}$. N is increased from 100 to 10 000. Which statement is correct?

KEY **B** **s stays close to σ_x of the state, while the standard error falls by a factor of 10.**

A treats the spread of the readings as a measurement error that averaging removes. It is not: s estimates σ_x, a property of the prepared state, and more readings only pin that estimate down more precisely. C is right about s but wrong about the mean, whose standard error does fall as $1/\sqrt{N}$, by $\sqrt{(10\,000/100)} = 10$, so $\langle x \rangle$ can be located as precisely as patience allows without any conflict with $\sigma_x \sigma_p \geq \hbar/2$. D pictures the measurements acting on one another, but each electron is measured once and then discarded, so every reading comes from a fresh copy of ψ. B is correct.

U4

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.09 · UQMI-05.01 · UQMI-05.04 **[1]**

A particle of mass m has V(x) = 0 everywhere on $-\infty < x < \infty$. Which statement about the solutions of its time-independent Schrödinger equation is correct?

KEY **B** **Every E > 0 is allowed, and each is twofold degenerate, with solutions e^(ikx) and e^(−ikx) where k = √(2mE)/ħ.**

A carries over the logic of the bound-state problems, where a condition at the walls or at infinity picks out discrete energies. With V = 0 there are no walls and no solution vanishes at infinity, so nothing selects particular values of k and the spectrum is continuous. C quotes a genuine theorem outside its scope: non-degeneracy in one dimension is proved for normalisable bound states, and plane waves are not normalisable, so e^(ikx) and e^(−ikx) share each energy. D writes down correct solutions of the differential equation, but every combination of real exponentials grows without bound in at least one direction, so none can be normalised even in the delta-function sense or superposed into a state, and E < 0 is excluded. B is correct.

U1

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-06.03 **[1]**

A particle is in the ground state $\psi_1(x) = \sqrt{(2/L)}\sin(\pi x/L)$ of an infinite square well occupying $0 < x < L$. What is the probability that a measurement of position finds it in $0 < x < L/4$?

KEY **C** **0.091**

B is the classical answer for a particle equally likely to be anywhere in the well, and also what comes from replacing $\sin^2$ by its average value of ½; but ψ_1 is small near a wall, so the quarter of the well beside the wall holds much less than a quarter of the probability. A comes from a sign slip in $\int\sin^2 u\,du = u/2 - (\sin 2u)/4$, which gives $¼ + 1/(2\pi)$. D uses 1/L instead of 2/L for $|\psi_1|^2$, forgetting what the normalisation constant is for. C is correct: $P = (2/L)\int_0^{(L/4)} \sin^2(\pi x/L)\,dx = ¼ - 1/(2\pi) = 0.091$.

U3

9

CORE COURSE

STRUCTURED · ROUTINE · UQMI-01.07 · UQMI-01.08 · UQMI-01.09 · UQMI-02.08 **[9]**

A particle of mass m moving along the x-axis is in a stationary state of some potential V(x). Its wavefunction is $\psi(x) = 0$ for $x < 0$; $\psi(x) = Ax/a$ for $0 \le x \le a$; and $\psi(x) = A\,e^{-(x-a)/a}$ for $x > a$, where a is a positive length and A is a positive real constant. ψ is continuous everywhere, but its slope jumps at $x = 0$ and at $x = a$.

(a) SHOW THAT · U2 **[3]**

Show that $A = \sqrt{(6/(5a))}$.

1. splits the normalisation integral at $x = a$ and evaluates the linear piece: $\int_0^a A^2x^2/a^2\,dx = A^2a/3$
2. evaluates the tail: $\int_a^\infty A^2e^{-2(x-a)/a}\,dx = A^2a/2$
3. sets $A^2(a/3 + a/2) = 5A^2a/6 = 1$, so $A = \sqrt{(6/(5a))}$

(b) CALCULATE · U3 **[2]**

Calculate the probability that a measurement of position finds the particle in $0 < x < a/2$, and the probability that it finds the particle at $x > a$.

1. $P(0 < x < a/2) = A^2\int_0^{a/2} x^2/a^2\,dx = A^2a/24 = (6/5)(1/24) = 0.050$
2. $P(x > a) = A^2a/2 = 3/5 = 0.60$

This part catches candidates who read a probability off $|\psi|^2$ at one point and multiply by the interval width, or who integrate ψ rather than $|\psi|^2$. The interval $0 < x < a/2$ is half of the linear region yet holds only a twentieth of the probability.

(c) EXPLAIN · U1 **[2]**

The slope of ψ must be continuous wherever V is finite. Explain why, and explain why the jump in slope at $x = 0$ is nevertheless admissible.

1. integrating $-(\hbar^2/2m)\psi'' + V\psi = E\psi$ from $x_0 - \varepsilon$ to $x_0 + \varepsilon$ gives $\psi'(x_0 + \varepsilon) - \psi'(x_0 - \varepsilon) = (2m/\hbar^2)\int(V - E)\psi\,dx$, which tends to zero as $\varepsilon \to 0$ when V is finite, so ψ' cannot jump at such a point
2. $\psi = 0$ for all $x < 0$ is the wavefunction beside an infinite wall ($V = \infty$ for $x < 0$); V is not finite there, so the slope may jump from 0 to A/a at $x = 0$ while ψ itself stays continuous

Candidates who state the rule without deriving it from the Schrödinger equation cannot say when it relaxes, and so often call the kink at the wall inadmissible too.

(d) DETERMINE · U2 **[2]**

The jump in slope at $x = a$ requires V(x) to contain a delta-function term $g\delta(x - a)$. Determine g in terms of $\hbar$, m and a, and say whether the spike is attractive or repulsive.

1. the slope changes from $+A/a$ to $-A/a$, so $\Delta\psi' = -2A/a$, while integrating the Schrödinger equation across $x = a$ gives $\Delta\psi' = (2m/\hbar^2)g\psi(a) = (2m/\hbar^2)gA$
2. $g = -\hbar^2/(ma)$: the spike is attractive, V(x) containing $-(\hbar^2/(ma))\delta(x - a)$

The sign is what separates answers. A cusp where the slope falls, at a point where ψ is positive, needs an attractive (negative) spike; candidates who remember only that a delta function makes a kink often give the right magnitude with the wrong sign.

10 CORE COURSE

STRUCTURED · DEMANDING · UQMI-04.08 · UQMI-05.02 · UQMI-05.05 · UQMI-05.07 · UQMI-05.08 · UQMI-05.09 **[9]**

An electron ($m_e = 9.109 \times 10^{-31}$ kg) moves freely along the x-axis, $V = 0$. At $t = 0$ it is prepared as a Gaussian wave packet centred on wavenumber k_0, with position standard deviation $\sigma_0 = 0.40$ nm and $\sigma_x\sigma_p = \hbar/2$. The kinetic energy corresponding to the central wavenumber is $E_0 = \hbar^2k_0^2/(2m_e) = 1.50$ eV. The width of the packet then evolves as $\sigma_x(t) = \sigma_0\sqrt{(1 + (\hbar t/(2m_e\sigma_0^2))^2)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) EXPLAIN · U1 **[2]**

Explain why no single plane wave e^(ik₀x) can describe the electron, and how a superposition of plane waves overcomes the difficulty.

1. $|e^{(ik_0x)}|^2 = 1$ at every x, so $\int|\psi|^2\,dx$ over the whole line diverges: the plane wave cannot be normalised and so is not a physical state
2. a Fourier superposition $\psi(x) = (1/\sqrt{(2\pi)})\int\varphi(k)e^{(ikx)}\,dk$ with square-integrable $\varphi(k)$ is normalisable, since $\int|\psi|^2\,dx = \int|\varphi|^2\,dk$ (Plancherel's theorem); the price is a spread of wavenumbers, and so of momenta, about k_0

The error this catches is saying only that a plane wave is not localised. The marks are for the divergent normalisation integral, and for seeing that the spread in k is what buys normalisability rather than a side effect of preparing the state.

(b) SHOW THAT · U2 **[2]**

Using the dispersion relation of a free particle, show that the centre of the packet moves at $v_g = \hbar k_0/m_e$, twice the phase velocity of its central component.

1. for $V = 0$, $\hbar\omega = \hbar^2k^2/(2m_e)$, so $\omega(k) = \hbar k^2/(2m_e)$ and the phase velocity of the central component is $v_p = \omega/k = \hbar k_0/(2m_e)$
2. the envelope moves at $v_g = d\omega/dk$ evaluated at k_0, which is $\hbar k_0/m_e = 2v_p$ and equals the classical speed p_0/m_e

Candidates who set $\hbar\omega$ equal to $\frac{1}{2}mv^2$ and then take $v = \omega/k$ obtain half the particle's speed. The part rewards knowing that a detector registers the envelope, and that the envelope moves at $d\omega/dk$.

(c) CALCULATE · U3 **[3]**

Calculate the group velocity, the time at which σ_x has grown to $2\sigma_0$, and the distance the centre of the packet travels in that time.

1. $v_g = \sqrt{(2E_0/m_e)} = \sqrt{(2 \times 1.50 \times 1.602 \times 10^{-19}\text{ J} / 9.109 \times 10^{-31}\text{ kg})} = 7.26 \times 10^5$ m s^{-1}
2. $\sigma_x = 2\sigma_0$ when $(\hbar t/(2m_e\sigma_0^2))^2 = 3$, so $t = 2\sqrt{3}\,m_e\sigma_0^2/\hbar = 2\sqrt{3} \times 9.109 \times 10^{-31}$ kg $\times$ $(0.40 \times 10^{-9}\text{ m})^2 / 1.055 \times 10^{-34}$ J s $= 4.79 \times 10^{-15}$ s
3. distance $= v_g\,t = 7.26 \times 10^5$ m s^{-1} $\times$ 4.79×10^{-15} s $= 3.5 \times 10^{-9}$ m (3.5 nm)

Setting $\hbar t/(2m_e\sigma_0^2) = 2$, instead of solving $1 + (\hbar t/(2m_e\sigma_0^2))^2 = 4$, gives 5.5×10^{-15} s and is the commonest slip. The width has doubled when the squared ratio equals 3, not when the ratio equals 2.

(d) PREDICT · U1 **[2]**

The electron is instead prepared with $\sigma_0 = 0.80$ nm, still with $\sigma_x\sigma_p = \hbar/2$ at $t = 0$. Predict, with a reason, the new time for the width to double.

1. the packet spreads because ω is quadratic in k, so its components travel with a spread of group velocities σ_p/m_e; with $\sigma_p = \hbar/(2\sigma_0)$ halved, they separate half as fast, while the width to be doubled is twice as large
2. the doubling time scales as σ_0^2, so it becomes 4 times longer: 1.91×10^{-14} s

The trap is the intuition that a wider packet is less confined and so should spread at least as fast. A narrower packet spreads faster because confining it in x broadens it in k, a property of the Fourier transform rather than a disturbance.

11 CORE COURSE

STRUCTURED · DEMANDING · UQMI-03.07 · UQMI-04.03 · UQMI-04.04 · UQMI-06.06 · UQMI-06.07 · UQMI-06.08 · UQMI-06.09 **[11]**

An electron ($m_e = 9.109 \times 10^{-31}$ kg) is confined along one axis of a nanostructure modelled as an infinite square well on $0 < x < L$, with $L = 1.20$ nm. Its energy eigenfunctions are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m_eL^2)$. At $t = 0$ its state is $\Psi(x,0) = \sqrt{(2/3)}\,\psi_1(x) + (i/\sqrt{3})\,\psi_2(x)$. You may use $\int_0^L \psi_n\, x\, \psi_n\, dx = L/2$ for every n, and $\int_0^L \psi_1\, x\, \psi_2\, dx = -16L/(9\pi^2)$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) CALCULATE · U3 **[2]**

Calculate the probabilities of obtaining E_1 and E_2 in a measurement of energy, and the expectation value $\langle H\rangle$ in electronvolts.

1. $P(E_1) = |\sqrt{(2/3)}|^2 = 2/3$ and $P(E_2) = |i/\sqrt{3}|^2 = 1/3$, which sum to 1, so Ψ is normalised
2. $E_1 = \pi^2\hbar^2/(2m_eL^2) = 4.19 \times 10^{-20}$ J $= 0.261$ eV, so $\langle H\rangle = (2/3)E_1 + (1/3)(4E_1) = 2E_1 = 0.523$ eV

   Squaring $i/\sqrt{3}$ to $-1/3$, which leaves probabilities summing to 1/3, is the error to catch: $|c|^2$ is c^*c, not c^2. A candidate who gives $\langle H\rangle$ as one of the eigenvalues has confused an average over identically prepared systems with the result of one measurement.

(b) SHOW THAT · U2 **[3]**

Write down $\Psi(x,t)$, and show that $\langle x\rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2))\sin(\omega t)$, where $\omega = (E_2 - E_1)/\hbar$.

1. $\Psi(x,t) = \sqrt{(2/3)}\psi_1 e^{(-iE_1t/\hbar)} + (i/\sqrt{3})\psi_2 e^{(-iE_2t/\hbar)}$, with each phase factor attached to its own term
2. $\langle x\rangle = (2/3)(L/2) + (1/3)(L/2) + 2\mathrm{Re}[c_1^*c_2 e^{(-i\omega t)}]\int\psi_1 x\psi_2\, dx = L/2 + 2\mathrm{Re}[(i\sqrt{2}/3)e^{(-i\omega t)}] \times (-16L/(9\pi^2))$
3. $\mathrm{Re}[i\, e^{(-i\omega t)}] = \sin(\omega t)$, so $\langle x\rangle = L/2 - (2\sqrt{2}/3)(16L/(9\pi^2))\sin(\omega t) = L/2 - (32\sqrt{2}L/(27\pi^2))\sin(\omega t)$

   Pulling a single factor $e^{(-iEt/\hbar)}$ out in front of the whole sum freezes $|\Psi|^2$ and removes the cross term from $\langle x\rangle(t)$. The other slip is dropping the i in c_2, which gives a cosine and the wrong value of $\langle x\rangle$ at $t = 0$.

(c) CALCULATE · U3 **[2]**

Calculate the period of the oscillation of $\langle x\rangle$, and its amplitude in nanometres.

1. $\omega = 3E_1/\hbar = 3 \times 4.19 \times 10^{-20}$ J $/ 1.055 \times 10^{-34}$ J s $= 1.19 \times 10^{15}$ rad s^{-1}, so the period is $2\pi/\omega = 5.28 \times 10^{-15}$ s
2. amplitude $= 32\sqrt{2}L/(27\pi^2) = 0.170L = 0.204$ nm

   Using E_2 alone, or $E_1 + E_2$, in place of the difference $E_2 - E_1$ is the error this part is set to catch: only the beat between the two phases is observable.

(d) PREDICT · U1 **[2]**

Suppose instead that $\Psi(x,0) = \sqrt{(2/3)}\,\psi_1(x) + (1/\sqrt{3})\,\psi_2(x)$. Predict which of $P(E_1)$, $P(E_2)$, $\langle H\rangle$, and the period and amplitude of $\langle x\rangle(t)$ change, and what does change.

1. $P(E_1)$, $P(E_2)$ and $\langle H\rangle$ are unchanged because they depend only on $|c_n|^2$; the period and amplitude are unchanged because they depend only on $E_2 - E_1$ and on $|c_1||c_2|$
2. only the phase of the oscillation changes: $\langle x\rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2))\cos(\omega t)$, so $\langle x\rangle$ starts at its minimum, $0.600 - 0.204 = 0.396$ nm, instead of at the centre of the well

   Candidates who expect a relative phase to leave everything untouched, because it cancels in each $|c_n|^2$, miss that it enters the interference term. The converse error, predicting that the energy probabilities change, is just as common.

(e) DESCRIBE · U1 **[2]**

Return to the original state. At t = 2.0 fs a measurement of energy on the electron returns E_2. Describe the state immediately afterwards, and how $\langle x \rangle$ and the result of a second energy measurement behave from then on.

1. the state collapses to ψ_2, multiplied by a phase factor of no physical consequence and normalised, with no ψ_1 component remaining
2. it is then stationary: $\langle x \rangle = L/2 = 0.600$ nm at all later times, and a repeated energy measurement returns $E_2 = 4E_1 = 1.05$ eV with probability 1

The error to catch is carrying on with the pre-measurement superposition, so that $\langle x \rangle$ keeps oscillating, or keeping a reduced ψ_1 component and renormalising. Collapse removes the other term entirely.

12

CORE COURSE

STRUCTURED · DEMANDING · UQMI-07.03 · UQMI-07.04 · UQMI-07.05 · UQMI-07.06 · UQMI-07.07 · UQMI-07.08 **[11]**

An electron is bound in a thin layer of material modelled as a symmetric finite square well: $V(x) = 0$ for $|x| < a$ and $V(x) = V_0$ for $|x| > a$, with well width $2a = 0.80$ nm and $V_0 = 2.00$ eV. Treat the electron as having the free-electron mass $m_e = 9.109 \times 10^{-31}$ kg. For a bound state with $0 < E < V_0$, write $k = \sqrt{(2m_eE)}/\hbar$, $\kappa = \sqrt{(2m_e(V_0 - E))}/\hbar$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_eV_0)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) JUSTIFY · U1 **[2]**

Justify looking for bound states that are either even or odd in x, rather than of general form.

1. $V(-x) = V(x)$, so if $\psi(x)$ solves the time-independent Schrödinger equation with energy E, then $\psi(-x)$ solves it with the same E
2. bound states in one dimension are non-degenerate, so $\psi(-x) = \lambda\psi(x)$; reflecting twice gives $\lambda^2 = 1$, so $\lambda = \pm 1$ and every bound state is even or odd as a matter of theorem, not merely of choice

Answers that say only that the well is symmetric, so the split is allowed, earn the first mark. The second needs non-degeneracy, which is what turns a convenient choice into a result.

(b) DERIVE · U2 **[4]**

Starting from the solutions of the Schrödinger equation in each region, derive the condition that the energies of the even bound states must satisfy, written as a single equation in z and z_0 alone.

1. inside the well $\psi = C\cos(kx)$; for $x > a$, $\psi = De^{-\kappa x}$, mirrored for $x < -a$, the growing solution $e^{+\kappa x}$ being discarded because ψ must be normalisable
2. continuity of ψ at $x = a$: $C\cos(ka) = De^{-\kappa a}$
3. continuity of ψ' at $x = a$, which holds because V is finite there: $-kC\sin(ka) = -\kappa De^{-\kappa a}$; dividing the two conditions eliminates C and D and gives $k\tan(ka) = \kappa$
4. multiplying by a and using $k^2 + \kappa^2 = 2m_eV_0/\hbar^2$, so that $(\kappa a)^2 = z_0^2 - z^2$, gives $z\tan z = \sqrt{(z_0^2 - z^2)}$

The order of the argument is what matters: normalisability removes $e^{+\kappa x}$ before any matching is done, and matching then fixes the amplitude ratio and the energy. Candidates who keep both exponentials are left with more unknowns than conditions, and candidates who stop at $k\tan(ka) = \kappa$ have not used $k^2 + \kappa^2 = 2m_eV_0/\hbar^2$ to remove κ.

(c) DETERMINE · U3 **[2]**

Calculate z_0, and hence determine how many bound states the well has and the parity of each.

1. $z_0 = (0.40 \times 10^{-9}\ \text{m} / 1.055 \times 10^{-34}\ \text{J s}) \times \sqrt{(2 \times 9.109 \times 10^{-31}\ \text{kg} \times 2.00 \times 1.602 \times 10^{-19}\ \text{J})} = 2.90$
2. an even state exists for each $n\pi < z_0$ and an odd state for each $(n + \tfrac{1}{2})\pi < z_0$, with $n = 0, 1, 2, \ldots$; since $\pi/2 < 2.90 < \pi$ there are two bound states, the even ground state and one odd state

Using the full width 0.80 nm in place of a doubles z_0 to 5.79 and predicts four states. Checking that $2z_0/\pi = 1.84$ rounds up to 2 is a quick guard against that slip.

(d) COMPARE · U3 **[3]**

The even condition has its lowest root at $z = 1.159$. Compare the ground-state energy with that of an infinite square well of the same width, 0.80 nm, and account for the difference.

1. $E = V_0(z/z_0)^2 = 2.00\ \text{eV} \times (1.159/2.897)^2 = 0.320$ eV (accept 0.319 eV from $z_0 = 2.90$)
2. infinite well of width 2a: $E = \pi^2\hbar^2/(2m_e(2a)^2) = 0.588$ eV, so the finite-well ground state is lower, about 0.54 of the infinite-well value
3. ψ leaks into the classically forbidden region over a decay length $1/\kappa = a/\sqrt{(z_0^2 - z^2)} = 0.15$ nm, so it fits a longer wavelength inside the well: $ka = 1.159$ rather than $\pi/2 = 1.571$, giving a smaller k and a lower energy

Candidates often argue that the finite-well level must be higher because the electron needs extra energy to enter the barrier. The comparison has to rest on the curvature of ψ inside the well, which the leakage reduces.

13

CORE COURSE

DATA ANALYSIS · DEMANDING · UQMI-08.04 · UQMI-08.05 · UQMI-08.08 **[11]**

In a scanning tunnelling microscope a sharp tungsten tip is held above a clean metal surface with a small bias voltage between them, and the tunnelling current I is recorded as a piezoelectric actuator withdraws the tip in equal steps. Δd is the increase in the tip-to-surface gap from its starting value d_0, which is not known accurately but is estimated to be about 0.5 nm. The gap is modelled as a rectangular barrier of height φ above the energy of the tunnelling electrons. In the thick-barrier limit $I = I_0 \exp(-2\kappa d)$, with $\kappa = \sqrt{(2m_e\varphi)}/\hbar$, $d = d_0 + \Delta d$, and I_0 independent of d. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) (i) EXPLAIN · U4 **[1]**

Explain why a plot of ln(I/nA) against Δd should give a straight line even though d_0 is not known.

1 $\ln I = (\ln I_0 - 2\kappa d_0) - 2\kappa\Delta d$, so ln I is linear in Δd, and the unknown d_0 only shifts the intercept

Candidates who think d_0 must be known before κ can be found have not separated the intercept from the gradient. The measurement is designed so that the gradient needs only changes in the gap.

(a) (ii) STATE · U4 **[1]**

State what the gradient of that line represents.

1 the gradient is -2κ

Giving the gradient as $-\kappa$, by dropping the 2 that comes from squaring the amplitude to get a probability, is the slip this catches; it carries through to a barrier height four times too large.

(b) DETERMINE · U4 **[3]**

Determine κ from the data.

1 computes ln(I/nA) = 1.163, 0.615, 0.122, −0.440, −0.947, −1.505

2 gradient of the best-fit straight line of ln(I/nA) against Δd = -0.0212 pm^{-1} (accept -0.0205 to -0.0220 pm^{-1}), from a least-squares fit or from widely separated points on a drawn line rather than from adjacent readings

3 $\kappa = 0.0212$ pm^{-1} / 2 = 1.06×10^{10} m^{-1} (accept 1.03×10^{10} to 1.10×10^{10} m^{-1})

Adjacent pairs of readings give gradients anywhere from -0.0197 to -0.0225 pm^{-1}, because the readings scatter by 1 to 2 per cent; the least-squares gradient, -0.02124 pm^{-1}, uses all six. Forgetting the factor of 2 doubles κ and quadruples the barrier height in (c).

(c) CALCULATE · U3 **[2]**

Calculate the barrier height φ in electronvolts.

1 $\varphi = \hbar^2\kappa^2/(2m_e) = (1.055 \times 10^{-34}\ \text{J s} \times 1.06 \times 10^{10}\ \text{m}^{-1})^2 / (2 \times 9.109 \times 10^{-31}\ \text{kg}) = 6.9 \times 10^{-19}$ J

2 φ = 4.3 eV (accept 4.0 to 4.6 eV when consistent with the value of κ from (b))

The squared κ makes this result sensitive: a 3% error in the gradient becomes a 6% error in φ. Candidates who leave φ in joules, or divide by e twice, are the ones this part catches.

(d) EVALUATE · U4 **[2]**

Evaluate whether the thick-barrier form is adequate for analysing these readings.

1 at the smallest gap $2\kappa d \approx 2 \times 1.06 \times 10^{10}\ \text{m}^{-1} \times 0.5 \times 10^{-9}\ \text{m} \approx 11 \gg 1$, so the exact rectangular-barrier transmission differs from the thick-barrier form only by corrections of relative order e^(−2κd), a few parts in 10^5, and the prefactor that form discards does not depend on d, so it cannot change the gradient

2 the six values of ln I lie within about 0.02 of a straight line, which supports the model over this range; judgement: adequate for finding the gradient, but φ is an apparent barrier height, because the real barrier is not rectangular (image-charge forces round and lower it) and the tip geometry is not one-dimensional

A strong answer weighs both the size of 2κd and the residuals of the fit, and then limits the claim: the analysis gives an effective φ for a model barrier, not the work function of the surface directly.

(e) ESTIMATE · U4 **[2]**

Estimate, stating your assumptions, the change in gap that would alter the current by 5%.

1. assumes I_0 stays fixed and the change is small enough to linearise the exponential, so that $|\Delta I|/I \approx 2\kappa|\Delta d|$
2. $|\Delta d| \approx 0.05 / (2.1 \times 10^{10}\ \mathrm{m^{-1}}) \approx 2 \times 10^{-12}$ m (2 pm), a small fraction of an atomic diameter

The error to catch is reading the change off the 25 pm spacing of the table instead of using the local slope. The estimate is what makes sub-atomic height resolution plausible: a change of a couple of picometres already gives a measurable change in current.

14

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-02.06 · UQMI-02.07 · UQMI-08.01 · UQMI-08.02 **[15]**

A beam of electrons of mass m_e and energy E travels in the +x direction towards a potential step: $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, with V_0 real and positive. The electrons are described by stationary scattering states of the time-dependent Schrödinger equation $i\hbar\,\partial\Psi/\partial t = -(\hbar^2/2m_e)\partial^2\Psi/\partial x^2 + V(x)\Psi$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) DERIVE · U2 **[4]**

Derive, from the time-dependent Schrödinger equation and its complex conjugate, the continuity equation $\partial\rho/\partial t + \partial j/\partial x = 0$ for $\rho = |\Psi|^2$, obtaining an expression for the probability current j and stating where the reality of V is used.

1. writes the conjugate equation $-i\hbar\,\partial\Psi^*/\partial t = -(\hbar^2/2m_e)\partial^2\Psi^*/\partial x^2 + V\Psi^*$, which uses $V^* = V$
2. $\partial\rho/\partial t = \Psi^*\partial\Psi/\partial t + \Psi\partial\Psi^*/\partial t$; on substituting both equations the potential terms $-(i/\hbar)V|\Psi|^2$ and $+(i/\hbar)V|\Psi|^2$ cancel, because V is real
3. $\partial\rho/\partial t = (i\hbar/2m_e)(\Psi^*\partial^2\Psi/\partial x^2 - \Psi\partial^2\Psi^*/\partial x^2) = (i\hbar/2m_e)\partial/\partial x(\Psi^*\partial\Psi/\partial x - \Psi\partial\Psi^*/\partial x)$, the terms in $(\partial\Psi^*/\partial x)(\partial\Psi/\partial x)$ cancelling
4. so $\partial\rho/\partial t + \partial j/\partial x = 0$ with $j = (\hbar/2im_e)(\Psi^*\partial\Psi/\partial x - \Psi\partial\Psi^*/\partial x) = (\hbar/m_e)\mathrm{Im}(\Psi^*\partial\Psi/\partial x)$

The step candidates miss is recognising $\Psi^*\partial^2\Psi/\partial x^2 - \Psi\partial^2\Psi^*/\partial x^2$ as an exact derivative; without it no current can be identified. Candidates who never write $V^* = V$ have not shown where conservation of probability could fail, and a complex potential is exactly how absorption is modelled.

(b) SHOW THAT · U2 **[4]**

For $E > V_0$, take $\psi(x) = e^{(ik_1x)} + r\,e^{(-ik_1x)}$ for $x < 0$ and $\psi(x) = t\,e^{(ik_2x)}$ for $x > 0$, where $k_1 = \sqrt{(2m_eE)/\hbar}$ and $k_2 = \sqrt{(2m_e(E - V_0))/\hbar}$. Explaining why there is no $e^{(-ik_2x)}$ term, show that the reflection and transmission probabilities are $R = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = 4k_1k_2/(k_1 + k_2)^2$.

1. there is no $e^{(-ik_2x)}$ term because no wave comes in from $x = +\infty$: electrons are sent in from the left only
2. continuity of ψ at $x = 0$ gives $1 + r = t$, and continuity of ψ', which holds because V is finite, gives $ik_1(1 - r) = ik_2t$
3. solving: $r = (k_1 - k_2)/(k_1 + k_2)$ and $t = 2k_1/(k_1 + k_2)$
4. using the current from (a): $j = (\hbar k_1/m_e)(1 - |r|^2)$ for $x < 0$, with no interference term, and $j = (\hbar k_2/m_e)|t|^2$ for $x > 0$, so $R = |r|^2 = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = (k_2/k_1)|t|^2 = 4k_1k_2/(k_1 + k_2)^2$

Writing $T = |t|^2$ is the error this part is built to catch. The current, not the density, counts particles crossing a point per unit time, so the speed ratio k_2/k_1 has to multiply $|t|^2$; because T is printed in the prompt, a candidate who writes $T = |t|^2$ and then forces agreement earns nothing for the last line.

(c) CALCULATE · U3 **[2]**

For electrons with E = 6.0 eV incident on a step of V_0 = 4.5 eV, calculate R, T and $|t|^2$, and confirm that R + T = 1.

1. $k_2/k_1 = \sqrt{((E - V_0)/E)} = \sqrt{(1.5/6.0)} = 0.50$, so $R = (0.5/1.5)^2 = 1/9 = 0.11$ and $T = 4(0.5)/(1.5)^2 = 8/9 = 0.89$, giving R + T = 1
2. $t = 2/(1 + 0.5) = 4/3$, so $|t|^2 = 16/9 = 1.78$, and $T = (k_2/k_1)|t|^2 = 0.89$ is less than 1

A value of $|t|^2$ greater than 1 is not an error: the transmitted electrons move at half the speed, so they bunch to a higher density while carrying less flux. Candidates who conclude that a probability has exceeded one have mistaken a density ratio for a flux ratio.

(d) SKETCH · U1 **[3]**

The electrons now have E = 3.0 eV, below the step. Sketch $|\psi(x)|^2$ against x from about x = −1 nm to x = +0.5 nm, for an incident wave of unit amplitude, marking the spacing of the maxima for x < 0, the value of $|\psi|^2$ at x = 0, and the length over which $|\psi|^2$ falls by a factor of e for x > 0.

1. for x < 0: a standing-wave pattern oscillating between 0 and 4, with true zeros because |r| = 1, and maxima spaced $\pi/k_1 = 0.35$ nm apart ($k_1 = 8.87 \times 10^9$ m^{-1})
2. at x = 0: matching gives $t = 1 + r = 2k_1/(k_1 + i\kappa)$, so $|\psi(0)|^2 = 4k_1^2/(k_1^2 + \kappa^2) = 4E/V_0 = 2.7$, with the two sides joining with no jump in value or in slope
3. for x > 0: a smooth decay $|\psi(0)|^2 e^{(-2\kappa x)}$ with no oscillation, falling by a factor of e in $1/(2\kappa) = 0.080$ nm, where $\kappa = \sqrt{(2m_e(V_0 - E))/\hbar} = 6.27 \times 10^9$ m^{-1}

Common sketches either carry the oscillation on into the step or drop $|\psi|^2$ to zero at x = 0 as though the step were an infinite wall. Both miss the evanescent tail, which is the point of the part.

(e) EXPLAIN · U1 **[2]**

Explain why every electron is reflected when E < V_0, even though $|\psi|^2$ is not zero inside the step.

1. for x > 0, $\psi = te^{(-\kappa x)}$ has the same complex phase at every x, so $\Psi^*\partial\Psi/\partial x = -\kappa|t|^2 e^{(-2\kappa x)}$ is real and $j = (\hbar/m_e)\mathrm{Im}(\Psi^*\partial\Psi/\partial x) = 0$: no probability flows into the step, so T = 0
2. in a stationary state $\partial\rho/\partial t = 0$, so by the continuity equation j is the same at every x; zero on the right means $(\hbar k_1/m_e)(1 - |r|^2) = 0$ on the left, so $R = |r|^2 = 1$, and the density in the step is a standing evanescent penetration rather than a flow of particles through it

Candidates who equate a nonzero density with transmission, or say that some electrons tunnel into the step and get through, are the target. The argument has to run through the current and the continuity equation derived in (a).

University Quantum Mechanics I · Final Examination

The whole of Quantum Mechanics I, weighted towards the second half of the course: the harmonic oscillator by series and by ladder operators, Dirac notation and two-state systems, angular momentum and spin-½, and the Schrödinger equation in three dimensions through to hydrogen. The wavefunction, operators, measurement, wells and tunnelling from the first half are sampled throughout.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**118**	**19**	**10**

SECTION — **UQMI · Final · University Quantum Mechanics I — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 1 · Routine 8 · Demanding 7 · Discriminating 3**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UQMI-01 · UQMI-02 · UQMI-03 · UQMI-04 · UQMI-05 · UQMI-06 · UQMI-07 · UQMI-08 · UQMI-09 · UQMI-10 · UQMI-11 · UQMI-12 · UQMI-13 · UQMI-14 · UQMI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Quantum Mechanics I · Final Examination

Answer all questions. 118 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**

ψ(x) is a normalised wavefunction for a particle moving along the x-axis. A second state of the particle has a position probability density of the same shape as $|\psi(x)|^2$, but compressed by a factor of 2 towards x = 0, so that every feature of the density lies at half its original distance from x = 0. Which wavefunction describes the second state?

A ψ(2x)

B 2ψ(2x)

C √2 ψ(x/2)

D √2 ψ(2x)

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.06 **[1]**

Each option refers to a particle of mass m in one dimension. Which pair of operators commutes, so that the two observables are compatible and can share a complete set of eigenfunctions?

A $\hat{p}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle

B $\hat{x}$ and $\hat{p}$

C $\hat{p}$ and $\hat{H} = \hat{p}^2/2m + \tfrac{1}{2}m\omega^2\hat{x}^2$, for a harmonic oscillator

D $\hat{x}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-05.06 · UQMI-05.07 **[1]**

The simplest model of a wave packet for a free neutron ($m = 1.675\times10^{-27}$ kg) is the sum of two plane waves, $\Psi(x,t) = e^{i(k_1x - \omega_1t)} + e^{i(k_2x - \omega_2t)}$, with $k_1 = 2.90\times10^{10}$ m^{-1} and $k_2 = 3.10\times10^{10}$ m^{-1}, and each frequency given by the free-particle dispersion relation $\omega = \hbar k^2/(2m)$. The probability density $|\Psi(x,t)|^2$ is then a row of fringes about 3.1 nm apart. Take $\hbar = 1.055\times10^{-34}$ J s. At what speed does the fringe pattern of $|\Psi(x,t)|^2$ move?

A 9.4×10^{2} m s^{-1}

B 1.9×10^{3} m s^{-1}

C 3.8×10^{3} m s^{-1}

D 1.2×10^{4} m s^{-1}

Answer ☐

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-07.07 **[1]**

An F-centre is an electron trapped at the site of a missing negative ion in an alkali-halide crystal. Along one axis, model it as an electron ($m = 9.109\times10^{-31}$ kg) in a symmetric finite square well of total width 0.56 nm and depth V_0. In this model the centre's optical absorption lifts the electron from the ground state to the first excited bound state, so the well must bind at least two states. Take $\hbar = 1.055\times10^{-34}$ J s and 1 eV $= 1.602\times10^{-19}$ J. What is the smallest depth V_0 for which the well binds a second state?

A Any depth greater than zero

B 0.30 eV

C 1.2 eV

D 4.8 eV

Answer

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-08.04 · UQMI-08.05 **[1]**

An electron with kinetic energy E meets a rectangular barrier of height $V_0 > E$ and width w. The thick-barrier estimate $T \approx \exp(-2\kappa w)$, with $\kappa = \sqrt{(2m(V_0 - E))}/\hbar$ and the prefactor ignored, gives $T = 1.0\times10^{-3}$. A proton with the same kinetic energy meets the same barrier. Taking $m_p/m_e = 1836$, what does the same estimate give for the proton's transmission probability?

A 2.3×10^{-5}

B 5.4×10^{-7}

C 3×10^{-129}

D 10^{-5508}

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-09.08 · UQMI-09.09 **[1]**

For a one-dimensional harmonic oscillator with energy eigenstates $|n\rangle$, the position operator can be written $\hat{x} = \sqrt{(\hbar/2m\omega)}(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$. Which of the following matrix elements is non-zero?

A $\langle 3|\hat{x}|2\rangle$

B $\langle 2|\hat{x}|2\rangle$

C $\langle 0|\hat{x}|2\rangle$

D $\langle 4|\hat{x}|2\rangle$

Answer

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMI-10.07 · UQMI-10.08 **[1]**

An observable Â is represented in one orthonormal basis by the Hermitian matrix A, and in a second orthonormal basis by A′ = U†AU, where U is the unitary matrix of overlaps between the two bases. Which of the following is in general different for A′ than for A?

A The trace of the matrix

B The set of eigenvalues

C The expectation value ⟨ψ|Â|ψ⟩ for a given state, computed from that state's components in the same basis as the matrix

D The first diagonal element, A_{11}

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-11.02 · UQMI-11.05 **[1]**

A two-state system with orthonormal basis states |1⟩ and |2⟩ is prepared in |ψ⟩ = (|1⟩ + i|2⟩)/√2. Which of the following is the same physical state, giving identical probabilities for every possible measurement?

A (|1⟩ − i|2⟩)/√2

B (i|1⟩ − |2⟩)/√2

C (|1⟩ + |2⟩)/√2

D (i|1⟩ + |2⟩)/√2

Answer ☐

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9 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-14.02 **[1]**

Near the centre of an optical trap, an atom of mass m moves in the potential $V(x, y, z) = \tfrac{1}{2}m\omega^2(x^2 + y^2) + \tfrac{1}{2}m(2\omega)^2z^2$, so that its classical oscillation frequency along z is twice that along x and y. How many independent energy eigenstates share the energy $E = 5\hbar\omega$?

A 4

B 6

C 10

D 12

Answer ☐

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-15.05 **[1]**

The lithium ion Li^{2+} (Z = 3) has a single electron. The energy levels of hydrogen are $E_n = -13.6\ \text{eV}/n^2$. How much energy is needed to ionise Li^{2+} from its first excited state?

A 3.40 eV

B 10.2 eV

C 30.6 eV

D 122 eV

Answer ☐

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11 CORE COURSE

STRUCTURED · ROUTINE · UQMI-04.04 · UQMI-06.03 · UQMI-06.07 · UQMI-06.08 · UQMI-06.09 **[10]**

A single ^{87}Rb atom of mass $m = 1.443\times10^{-25}$ kg moves along an optical box trap whose repulsive light walls are steep and high enough to be treated as an infinite square well on $0 < x < L$, with $L = 20.0$ μm. The stationary states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2E_1$. At $t = 0$ the atom is prepared near the left wall in the state $\Psi(x,0) = (1/\sqrt{2})\psi_1(x) + (1/2)\psi_2(x) + (1/2)\psi_3(x)$, for which $\langle x\rangle = 0.275L$. Take $\hbar = 1.055\times10^{-34}$ J s, $h = 6.626\times10^{-34}$ J s and $k_B = 1.381\times10^{-23}$ J K^{-1}.

(a) CALCULATE **[2]**

Calculate the atom's zero-point energy E_1, and express it as a temperature E_1/k_B.

(b) STATE **[2]**

State the possible results of a measurement of the atom's energy and the probability of each, and state whether these probabilities change as time passes.

(c) CALCULATE **[2]**

Every phase factor $e^{-iE_nt/\hbar}$ in $\Psi(x,t)$ returns to 1 at the same instant, after which the atom's state repeats exactly. Calculate the earliest such time.

(d) EXPLAIN **[2]**

At $t = h/(2E_1)$ the probability density is $|\Psi(L - x, 0)|^2$, the mirror image of the initial density about the centre of the trap. Explain why, and state $\langle x\rangle$ at that instant.

(e) PREDICT **[2]**

Suppose instead that the atom's energy is measured at $t = 0$ and measured again at $t = h/(2E_1)$. Predict the probability that the two results agree, and whether the density at $t = h/(2E_1)$ is then the mirror image of the initial density.

12 CORE COURSE

STRUCTURED · ROUTINE · UQMI-10.05 · UQMI-13.04 · UQMI-13.05 · UQMI-13.06 · UQMI-13.08 **[10]**

A beam of spin-½ atoms is prepared in the spin state $\chi = (1/\sqrt{3})(1, 1 + i)^{T}$, written as a column in the S_z basis $\{|+z\rangle, |-z\rangle\}$. The spin operators are $S_i = (\hbar/2)\sigma_i$, where σ_x, σ_y and σ_z are the Pauli matrices.

(a) STATE **[2]**

State the three Pauli matrices σ_x, σ_y and σ_z in the S_z basis.

(b) CALCULATE **[2]**

Calculate the probability that a measurement of S_z gives $+\hbar/2$, and the expectation value $\langle S_z \rangle$.

(c) DETERMINE **[3]**

Determine the probability that a measurement of S_x gives $+\hbar/2$.

(d) DETERMINE **[3]**

The atoms pass through an S_x analyser, and only those leaving its $+\hbar/2$ port go on to an S_z analyser. Determine the fraction of the original atoms that leave the $-\hbar/2$ port of the S_z analyser, and give the reason it is not related to the 2/3 found in (b).

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13 CORE COURSE

STRUCTURED · DEMANDING · UQMI-02.06 · UQMI-02.08 · UQMI-08.01 · UQMI-08.02 **[12]**

Very slow electrons are strongly reflected by a crystal surface even though the crystal attracts them. Inside the crystal an electron's potential energy is lower than in the vacuum outside by U, the inner potential, so the surface at $x = 0$ is modelled as a downward potential step: $V(x) = 0$ for $x < 0$ (vacuum) and $V(x) = -U$ for $x > 0$ (crystal). Electrons of energy $E = 0.50$ eV arrive from the vacuum, moving in the $+x$ direction at normal incidence. Their stationary scattering state is $\psi(x) = e^{ikx} - (2/3)e^{-ikx}$ for $x < 0$ and $\psi(x) = (1/3)e^{iqx}$ for $x > 0$, where $\hbar k = \sqrt{(2m_e E)}$ and $\hbar q = \sqrt{(2m_e(E + U))}$, but the value of U is not known. The probability current is $j = (\hbar/m_e)\,\mathrm{Im}(\psi^{*}\, d\psi/dx)$. Take $m_e = 9.109\times10^{-31}$ kg, $\hbar = 1.055\times10^{-34}$ J s and 1 eV $= 1.602\times10^{-19}$ J.

(a) DERIVE **[3]**

Stating the matching conditions that ψ must satisfy at x = 0, and why each holds here, derive the ratio q/k for this state and hence the inner potential U in electronvolts.

(b) SHOW THAT **[2]**

Show that the probability current is the same on both sides of the surface, as conservation of probability requires in a stationary state.

(c) CALCULATE **[3]**

A beam of these electrons carrying a current of 2.0 nA strikes the surface. Calculate the reflected and transmitted electron currents, and the speed of the transmitted electrons inside the crystal.

(d) CALCULATE **[2]**

Calculate the electron energy at which the reflected fraction at this surface falls to 1.0%.

(e) EXPLAIN **[2]**

A classical electron crossing the surface would simply speed up, and none would be reflected. Explain why the quantum electron can be reflected by a drop in potential, and why far fewer electrons would be reflected if V fell from 0 to −U smoothly over several nanometres instead of abruptly.

14 CORE COURSE

STRUCTURED · DEMANDING · UQMI-11.01 · UQMI-11.04 · UQMI-11.07 · UQMI-11.08 **[12]**

An electron shared between two neighbouring semiconductor quantum dots, L and R, is modelled as a two-state system in the basis $\{|L\rangle, |R\rangle\}$ of states localised on one dot. Energies are measured from the mean of the two on-dot energies, and in that basis the Hamiltonian is $H = [[\varepsilon/2, -\tau], [-\tau, -\varepsilon/2]]$. The detuning $\varepsilon = 30$ μeV is set by a gate voltage, and the tunnel coupling is $\tau = 20$ μeV. The next single-electron level in each dot lies about 1.5 meV above these states. If the electron starts on dot L at $t = 0$, the probability of finding it on dot R is $P_R(t) = [4\tau^2/(E_+ - E_-)^2] \sin^2[(E_+ - E_-)t/(2\hbar)]$, where E_+ and E_- are the eigenvalues of H. Take $h = 6.626\times10^{-34}$ J s and 1 eV $= 1.602\times10^{-19}$ J.

(a) JUSTIFY **[2]**

Justify the two-state model of the electron, using the energies given.

(b) DETERMINE **[3]**

Determine the eigenvalues E_+ and E_- of H, and hence the splitting $E_+ - E_-$.

(c) DETERMINE **[3]**

Determine the probability of finding the electron on dot R when it is in the ground state.

(d) CALCULATE **[2]**

The electron is placed on dot L at $t = 0$. Calculate the maximum probability of later finding it on dot R, and the period of the oscillation of P_R(t).

(e) PREDICT **[2]**

The gate voltage is adjusted until $\varepsilon = 0$. Predict the maximum probability of transfer to dot R, and the time after $t = 0$ at which it is first reached.

15 CORE COURSE

STRUCTURED · DEMANDING · UQMI-10.07 · UQMI-12.05 · UQMI-12.06 · UQMI-12.08 **[12]**

For orbital angular momentum with $l = 1$, use the ordered basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, in which $L_z = \hbar\,\mathrm{diag}(1, 0, -1)$. The raising and lowering operators $L_\pm = L_x \pm iL_y$ act as $L_\pm|l,m\rangle = \hbar\sqrt{(l(l+1) - m(m\pm1))}\,|l,m\pm1\rangle$. The uncertainty relation for L_x and L_y is $\Delta L_x\,\Delta L_y \geq (\hbar/2)|\langle L_z\rangle|$.

(a) SHOW THAT **[2]**

Show that the only non-zero matrix elements of L_+ in this basis are $\langle 1,1|L_+|1,0\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\,\hbar$.

(b) DETERMINE **[2]**

Determine the 3×3 matrix that represents L_x in this basis.

(c) DETERMINE **[3]**

Determine the eigenvalues of L_x.

(d) CALCULATE **[3]**

A particle is prepared in $|1,1\rangle$. Calculate the probabilities of obtaining $\hbar$, 0 and $-\hbar$ in a measurement of L_x.

(e) SHOW THAT **[2]**

Show that in the state $|1,1\rangle$ the product $\Delta L_x\,\Delta L_y$ is exactly equal to the lower bound $(\hbar/2)|\langle L_z\rangle|$.

16 CORE COURSE

STRUCTURED · DISCRIMINATING · UQMI-14.08 · UQMI-15.04 · UQMI-15.05 · UQMI-15.07 · UQMI-15.08 **[12]**

For a hydrogen atom, the radial function $u(r) = rR(r)$ satisfies $-(\hbar^2/2m)\,d^2u/dr^2 + [-e^2/(4\pi\varepsilon_0 r) + \hbar^2 l(l+1)/(2mr^2)]\,u = Eu$. A proposed solution is $u(r) = N\,r^3\,e^{-r/(3a_0)}$, where N is a normalisation constant and $a_0 = 4\pi\varepsilon_0\hbar^2/(me^2) = 5.29\times10^{-11}$ m is the Bohr radius, so that $e^2/(4\pi\varepsilon_0) = \hbar^2/(ma_0)$ and $\hbar^2/(2ma_0^2) = 13.6$ eV. Treat the proton as infinitely heavy. You may use $\int_0^\infty r^n e^{-\beta r}\,dr = n!/\beta^{n+1}$ for a non-negative integer n and $\beta > 0$.

(a) DETERMINE **[4]**

Determine the value of l for which $u(r)$ satisfies the radial equation, and the corresponding energy E in electronvolts.

(b) STATE **[2]**

State the principal quantum number n of this state, its number of radial nodes, and its spectroscopic label.

(c) DETERMINE **[2]**

Determine the most probable distance of the electron from the proton, in nanometres.

(d) CALCULATE **[2]**

Calculate the expectation value $\langle r\rangle$ in nanometres.

(e) COMPARE **[2]**

Compare the degeneracy of this energy level in hydrogen with the degeneracy an $l = 2$ level would have in a generic central potential, such as a finite spherical well, giving the cause of each.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UQMI-08.04 · UQMI-08.08 **[10]**

Gate leakage in a transistor is caused by electrons tunnelling directly through the thin silicon dioxide layer between the gate and the silicon. To study it, an engineer measures the gate current I of six test capacitors made with different oxide thicknesses t_{ox} and different gate areas A, all at the same small gate voltage. The model to be tested is $J = J_0 \exp(-2\kappa t_{ox})$, where $J = I/A$ is the current density, J_0 is the same for every capacitor, and $\kappa = \sqrt{(2m^*\varphi_B)}/\hbar$, with φ_B the height of the oxide barrier above the energy of the tunnelling electrons and m^* the effective mass of an electron in the oxide. Internal photoemission measurements, made independently, give $\varphi_B = 3.1$ eV. Take $m_e = 9.109\times10^{-31}$ kg, $\hbar = 1.055\times10^{-34}$ J s, 1 eV $= 1.602\times10^{-19}$ J and 1 μm² $= 1\times10^{-12}$ m².

Gate current of six test capacitors at the same gate voltage

Capacitor	Oxide thickness t_{ox} / nm	Gate area A / μm²	Gate current I / A
1	1.40	1.0×10^{2}	2.00×10^{-6}
2	1.60	2.5×10^{2}	5.26×10^{-7}
3	1.80	1.0×10^{3}	2.03×10^{-7}
4	2.00	2.5×10^{3}	5.44×10^{-8}
5	2.20	1.0×10^{4}	2.14×10^{-8}
6	2.40	2.5×10^{4}	5.63×10^{-9}

(a) DESCRIBE **[2]**

Describe how the readings should be processed and plotted so that the model predicts a straight line, including why the currents themselves should not be plotted and what the gradient measures.

(b) DETERMINE **[3]**

Determine κ from the readings, and the reduction in oxide thickness that multiplies the leakage current density by ten.

(c) ESTIMATE **[3]**

Estimate the effective mass m^* of the tunnelling electrons in the oxide, as a multiple of m_e, using $\varphi_B = 3.1$ eV, and state the assumptions on which your estimate rests.

(d) EVALUATE **[2]**

A colleague analyses the same readings taking the electron in the oxide to have the free-electron mass m_e, and reports the barrier height that follows. Evaluate this analysis.

18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-09.04 · UQMI-09.07 · UQMI-09.08 · UQMI-09.09 **[15]**

A particle of mass m moves in the potential $V(x) = \tfrac{1}{2}m\omega^2x^2$, so that $\hat{H} = \hat{p}^2/2m + \tfrac{1}{2}m\omega^2\hat{x}^2$. Define $\hat{a} = \sqrt{(m\omega/2\hbar)}(\hat{x} + i\hat{p}/(m\omega))$ and its adjoint $\hat{a}^\dagger = \sqrt{(m\omega/2\hbar)}(\hat{x} - i\hat{p}/(m\omega))$, and write $\hat{N} = \hat{a}^\dagger\hat{a}$. The canonical commutator is $[\hat{x}, \hat{p}] = i\hbar$. Once $[\hat{a}, \hat{a}^\dagger] = 1$ has been established, you may use its consequences $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(a) DERIVE **[3]**

Derive $[\hat{a}, \hat{a}^\dagger] = 1$ from the canonical commutator.

(b) SHOW THAT **[2]**

Show that $\hat{H} = \hbar\omega(\hat{N} + \tfrac{1}{2})$.

(c) DERIVE **[4]**

Derive the energy spectrum $E_n = (n + \tfrac{1}{2})\hbar\omega$ with $n = 0, 1, 2, \ldots$, using only the commutators and the fact that no vector has a negative squared norm. Do not evaluate any integral.

(d) DETERMINE **[3]**

Determine $\langle\hat{x}^2\rangle$ in the eigenstate $|n\rangle$ using ladder operators alone.

(e) PREDICT **[3]**

The particle is in the ground state $|0\rangle$ when the potential is suddenly tightened to $V'(x) = \tfrac{1}{2}m(2\omega)^2x^2$, so quickly that the wavefunction has no time to change. Predict which energy eigenstates of the new oscillator the particle can then be found in, and its mean energy $\langle\hat{H}'\rangle$ immediately after the change, in terms of $\hbar\omega$.

19 CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UQMI-14.01 · UQMI-14.04 · UQMI-14.08 · UQMI-14.09 **[15]**

A simple model of the deuteron treats the neutron–proton pair as a single particle of reduced mass $\mu = 8.37\times10^{-28}$ kg moving in a spherical square well, with $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, where $a = 2.0\times10^{-15}$ m. Only states with $l = 0$ are considered, so ψ depends on r alone and $\nabla^2\psi = (1/r^2)\,d/dr\,(r^2\,d\psi/dr)$. Primes denote d/dr. Take $\hbar = 1.055\times10^{-34}$ J s and 1 MeV $= 1.602\times10^{-13}$ J. The measured binding energy of the deuteron is 2.22 MeV, and it has no bound excited state.

(a) DERIVE **[3]**

Starting from the time-independent Schrödinger equation in three dimensions, derive the equation $-(\hbar^2/2\mu)u'' + V(r)u = Eu$ satisfied by $u(r) = r\psi(r)$.

(b) JUSTIFY **[2]**

Justify the boundary condition $u(0) = 0$. Note that a wavefunction behaving as $1/r$ near the origin is square-integrable in three dimensions.

(c) SHOW THAT **[3]**

Show that a bound state with energy $E = -E_B$, where $0 < E_B < V_0$, must satisfy $k\cot(ka) = -\kappa$, where $\hbar k = \sqrt{(2\mu(V_0 - E_B))}$ and $\hbar\kappa = \sqrt{(2\mu E_B)}$.

(d) DERIVE **[3]**

Derive, in terms of μ, a and $\hbar$, the minimum depth $V_{0,\min}$ for which the well has any bound state at all.

(e) CALCULATE **[2]**

Calculate $V_{0,\min}$ in MeV for the values given.

(f) SKETCH **[2]**

A well depth of 36.6 MeV reproduces the measured binding energy, giving ka = 1.82 and κa = 0.46. Sketch u(r) for the deuteron's ground state from r = 0 to r = 5a, marking r = a on the axis.

MARK SCHEME

University Quantum Mechanics I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**

ψ(x) is a normalised wavefunction for a particle moving along the x-axis. A second state of the particle has a position probability density of the same shape as $|\psi(x)|^2$, but compressed by a factor of 2 towards x = 0, so that every feature of the density lies at half its original distance from x = 0. Which wavefunction describes the second state?

KEY **D** **√2 ψ(2x)**

Replacing x by 2x compresses the shape by a factor of 2, because every feature of ψ(2x) lies at half its former distance from the origin. The substitution u = 2x gives ∫|ψ(2x)|² dx = ½∫|ψ(u)|² du = ½, so the compressed function must be multiplied by √2 to restore a total probability of 1, and √2 ψ(2x) then meets every requirement on a wavefunction. A is the most tempting wrong answer. It has the right shape, is finite and continuous, and falls off at large |x|, but its total probability is only ½. The particle must be found somewhere, so a function whose |ψ|² integrates to ½ does not describe a state until it is normalised. B doubles the amplitude on the reasoning that a density half as wide must be twice as tall. It is |ψ|² that must be twice as tall, and doubling ψ makes it four times as tall, so the total probability becomes 2. C has the right factor but the wrong substitution: ψ(x/2) is stretched to twice the width, and 2∫|ψ(x/2)|² dx = 4. Normalisation fixes the scale of ψ, so any change of width has to be paid for by a change of height.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.06 **[1]**

Each option refers to a particle of mass m in one dimension. Which pair of operators commutes, so that the two observables are compatible and can share a complete set of eigenfunctions?

KEY **A** **$\hat{p}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle**

When V = 0 the Hamiltonian is a function of $\hat{p}$ alone, and an operator always commutes with a function of itself. The plane waves e^(ikx) are therefore eigenfunctions of both operators, with p = ħk and E = ħ²k²/2m. B is the canonical incompatible pair, $[\hat{x}, \hat{p}] = i\hbar$. It is offered to the candidate who reads 'both Hermitian' as 'compatible'. C is tempting because the kinetic term does commute with $\hat{p}$. The potential term does not: $[\hat{p}, \tfrac{1}{2}m\omega^2\hat{x}^2] = -i\hbar m\omega^2\hat{x}$, which is non-zero, so no oscillator stationary state has a definite momentum. D fails for the mirror-image reason. $[\hat{x}, \hat{p}^2/2m] = i\hbar\hat{p}/m$, so even for a free particle a state of definite position cannot have definite energy. Compatibility is decided by the commutator, never by the fact that each operator on its own represents an observable.

U1

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-05.06 · UQMI-05.07 **[1]**

The simplest model of a wave packet for a free neutron (m = 1.675×10⁻²⁷ kg) is the sum of two plane waves, Ψ(x,t) = e^(i(k₁x − ω₁t)) + e^(i(k₂x − ω₂t)), with k₁ = 2.90×10¹⁰ m⁻¹ and k₂ = 3.10×10¹⁰ m⁻¹, and each frequency given by the free-particle dispersion relation ω = ħk²/(2m). The probability density |Ψ(x,t)|² is then a row of fringes about 3.1 nm apart. Take ħ = 1.055×10⁻³⁴ J s. At what speed does the fringe pattern of |Ψ(x,t)|² move?

KEY **B** **1.9×10³ m s⁻¹**

Multiplying out gives |Ψ|² = 2 + 2cos[(k₂ − k₁)x − (ω₂ − ω₁)t], so the fringes move at (ω₂ − ω₁)/(k₂ − k₁) = ħ(k₂² − k₁²)/(2m(k₂ − k₁)) = ħ(k₁ + k₂)/(2m) = 1.9×10³ m s⁻¹. For two components this result is exact, and it equals the classical speed ħk/m of a neutron with the mean wavenumber 3.00×10¹⁰ m⁻¹. A, 9.4×10² m s⁻¹, is the phase velocity ω/k of a component at the centre of the band, half the correct value. It is chosen by a candidate who takes the speed of the crests inside the fringes for the speed of the fringes themselves. C doubles the correct answer: the candidate writes ħ(k₁ + k₂)/m, treating k₁ + k₂ as the central wavenumber when it is twice that. D uses h in place of ħ, which makes the speed 2π times too large. The probability is carried by the envelope, which moves at the group velocity, and that is why the group velocity and not the phase velocity is the speed of the particle.

U3

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-07.07 **[1]**

An F-centre is an electron trapped at the site of a missing negative ion in an alkali-halide crystal. Along one axis, model it as an electron (m = 9.109×10⁻³¹ kg) in a symmetric finite square well of total width 0.56 nm and depth V₀. In this model the centre's optical absorption lifts the electron from the ground state to the first excited bound state, so the well must bind at least two states. Take ħ = 1.055×10⁻³⁴ J s and 1 eV = 1.602×10⁻¹⁹ J. What is the smallest depth V₀ for which the well binds a second state?

KEY **C** **1.2 eV**

With half-width a = 0.28 nm and z₀ = (a/ħ)√(2mV₀), bound states appear one at a time, even and odd alternately, the n-th once z₀ exceeds (n − 1)π/2. The second state, the first odd one, needs z₀ > π/2, so V₀ > π²ħ²/(8ma²) = 1.92×10⁻¹⁹ J = 1.2 eV. A is true of the ground state but not of the second state. An even bound state exists in a one-dimensional well of any depth, and a candidate who remembers that result without its restriction to the ground state chooses A. B puts the full width, 0.56 nm, into z₀ in place of the half-width, which divides the required depth by 4. D sets z₀ = π, which is the threshold for a third state. It comes from counting as if the n-th state needed z₀ > nπ/2, forgetting that the ground state has no threshold, and it would wrongly rule out the absorption for depths between 1.2 eV and 4.8 eV, where two bound states do exist.

U3

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5 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-08.04 · UQMI-08.05 **[1]**

An electron with kinetic energy E meets a rectangular barrier of height V₀ > E and width w. The thick-barrier estimate T ≈ exp(−2κw), with κ = √(2m(V₀ − E))/ħ and the prefactor ignored, gives T = 1.0×10⁻³. A proton with the same kinetic energy meets the same barrier. Taking m_p/m_e = 1836, what does the same estimate give for the proton's transmission probability?

KEY **C** **3×10⁻¹²⁹**

κ grows as √m, so the proton's exponent is √1836 = 42.8 times the electron's. That gives T_p = (1.0×10⁻³)^42.8 = 10^(−128.5) ≈ 3×10⁻¹²⁹. A divides T by √1836 and B divides it by 1836. Both treat the mass as if it entered a prefactor rather than the exponent, and that is the error this item is set to expose. Mass, width and √(V₀ − E) all sit inside the exponential, so modest changes in any of them change T by orders of magnitude. D raises T to the power 1836 rather than √1836, taking κ proportional to m instead of √m. The neglected prefactor changes T by a factor of order one, so it cannot rescue the proton: at the same energy and barrier, a proton is effectively never transmitted, while one electron in a thousand gets through.

U3

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-09.08 · UQMI-09.09 **[1]**

For a one-dimensional harmonic oscillator with energy eigenstates $|n\rangle$, the position operator can be written $\hat{x} = \sqrt{(\hbar/2m\omega)}(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$. Which of the following matrix elements is non-zero?

KEY **A** **$\langle 3|\hat{x}|2\rangle$**

$\hat{x}$ contains one $\hat{a}$ and one $\hat{a}^\dagger$, so acting on $|2\rangle$ it produces only $|1\rangle$ and $|3\rangle$: $\hat{x}|2\rangle = \sqrt{(\hbar/2m\omega)}(\sqrt{2}|1\rangle + \sqrt{3}|3\rangle)$. Hence $\langle 3|\hat{x}|2\rangle = \sqrt{3}\,\sqrt{(\hbar/2m\omega)}$. B is the expectation value of x in a stationary state. It vanishes because $\hat{x}|2\rangle$ has no $|2\rangle$ component, or equivalently because $|\psi_2|^2$ is even while x is odd. C and D connect states whose n differ by 2. Those are elements of $\hat{x}^2$, which contains $\hat{a}^2$ and $\hat{a}^{\dagger 2}$, and they catch a candidate who remembers that $\hat{x}$ changes n without remembering by how much. Parity agrees: ψ_0, ψ_2 and ψ_4 are all even, so $\int \psi_0\, x\, \psi_2\, dx$ and $\int \psi_4\, x\, \psi_2\, dx$ are integrals of odd functions and vanish.

U1

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMI-10.07 · UQMI-10.08 **[1]**

An observable $\hat{A}$ is represented in one orthonormal basis by the Hermitian matrix A, and in a second orthonormal basis by $A' = U\dagger AU$, where U is the unitary matrix of overlaps between the two bases. Which of the following is in general different for A' than for A?

KEY **D** **The first diagonal element, A_{11}**

A change of basis relabels the components of vectors and matrices without changing the physics. Trace and eigenvalues are unchanged because $U\dagger AU$ has the same characteristic polynomial as A, and $\mathrm{Tr}(U\dagger AU) = \mathrm{Tr}(AUU\dagger) = \mathrm{Tr}\,A$. A candidate who picks A or B has confused a matrix's entries with the operator it represents. C is also invariant: the column of components transforms as $c' = U\dagger c$, so $c'\dagger A' c' = c\dagger UU\dagger AUU\dagger c = c\dagger Ac$. D is the quantity that changes. $A_{11} = \langle 1|\hat{A}|1\rangle$ is the expectation value in the first basis state, and the first basis vector is a different physical state in each basis. The same point is why a Hermitian matrix is diagonal only in its own eigenbasis.

U1

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-11.02 · UQMI-11.05 **[1]**

A two-state system with orthonormal basis states $|1\rangle$ and $|2\rangle$ is prepared in $|\psi\rangle = (|1\rangle + i|2\rangle)/\sqrt{2}$. Which of the following is the same physical state, giving identical probabilities for every possible measurement?

KEY **B** **$(i|1\rangle - |2\rangle)/\sqrt{2}$**

$(i|1\rangle - |2\rangle)/\sqrt{2} = i(|1\rangle + i|2\rangle)/\sqrt{2}$, which is $|\psi\rangle$ multiplied by the global phase i. A global phase cancels from every $|\langle\varphi|\psi\rangle|^2$, so no measurement can detect it. All four options give probability 1/2 for each basis state, which is why a candidate who checks only populations cannot choose between them. A and C change the relative phase between the components, from i to $-i$ and to 1, and relative phase is physical. Measured in the basis $(|1\rangle \pm i|2\rangle)/\sqrt{2}$, $|\psi\rangle$ gives one outcome with certainty, A gives the other outcome with certainty, and C gives each half the time. D is the subtle one. It looks like $|\psi\rangle$ with the i moved to the other term, but it equals $i(|1\rangle - i|2\rangle)/\sqrt{2}$, the same physical state as A.

U1

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9 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-14.02 **[1]**

Near the centre of an optical trap, an atom of mass m moves in the potential $V(x, y, z) = \tfrac{1}{2}m\omega^2(x^2 + y^2) + \tfrac{1}{2}m(2\omega)^2z^2$, so that its classical oscillation frequency along z is twice that along x and y. How many independent energy eigenstates share the energy $E = 5\hbar\omega$?

KEY **B** **6**

The potential is a sum of three one-dimensional oscillator potentials, so $\psi = X(x)Y(y)Z(z)$ and $E = (n_x + \tfrac{1}{2})\hbar\omega + (n_y + \tfrac{1}{2})\hbar\omega + (n_z + \tfrac{1}{2})(2\hbar\omega) = (n_x + n_y + 2n_z + 2)\hbar\omega$, with each quantum number taking the values 0, 1, 2 and so on. $E = 5\hbar\omega$ therefore requires $n_x + n_y + 2n_z = 3$. With $n_z = 0$ the three quanta can be shared between x and y in four ways, $(n_x, n_y) = (3, 0)$, (2, 1), (1, 2) and (0, 3); with $n_z = 1$ the one remaining quantum gives two more, (1, 0) and (0, 1); $n_z = 2$ would already need $4\hbar\omega$ above the zero-point energy. The level holds 4 + 2 = 6 states. A counts only the states with $n_z = 0$, the answer of a candidate who assumes the stiffer direction stays in its ground state, when one quantum along z costs exactly as much as two in the plane. C applies the isotropic-oscillator count $(N + 1)(N + 2)/2$ to N = 3, which would be right only if every quantum cost $\hbar\omega$. D leaves out the zero-point energy, which is $2\hbar\omega$ here, and so counts the solutions of $n_x + n_y + 2n_z = 5$ instead of 3. The coincidence of states with $n_z = 0$ and $n_z = 1$ is not required by the trap's symmetry about the z-axis: it comes from the 2 : 1 ratio of the frequencies, and an irrational ratio would remove it.

U2

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-15.05 **[1]**

The lithium ion Li^{2+} (Z = 3) has a single electron. The energy levels of hydrogen are $E_n = -13.6\ \text{eV}/n^2$. How much energy is needed to ionise Li^{2+} from its first excited state?

KEY **C** **30.6 eV**

For a one-electron ion the levels scale as Z^2: $E_n = -13.6\ \text{eV} \times Z^2/n^2$. The first excited state, n = 2, lies at $-13.6 \times 9/4 = -30.6$ eV, so 30.6 eV removes the electron. A is the hydrogen value for n = 2, the answer of a candidate who ignores Z. B scales by Z instead of Z^2. The Coulomb potential carries one factor of Z and the orbit shrinks by another, so the energy goes as Z^2. D, 122 eV, is the ionisation energy from the ground state, chosen by a candidate who reads 'first excited state' as n = 1.

U3

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11 CORE COURSE

STRUCTURED · ROUTINE · UQMI-04.04 · UQMI-06.03 · UQMI-06.07 · UQMI-06.08 · UQMI-06.09 **[10]**

A single ^{87}Rb atom of mass $m = 1.443\times10^{-25}$ kg moves along an optical box trap whose repulsive light walls are steep and high enough to be treated as an infinite square well on $0 < x < L$, with L = 20.0 µm. The stationary states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2E_1$. At t = 0 the atom is prepared near the left wall in the state $\Psi(x,0) = (1/\sqrt{2})\psi_1(x) + (1/2)\psi_2(x) + (1/2)\psi_3(x)$, for which $\langle x\rangle = 0.275L$. Take $\hbar = 1.055\times10^{-34}$ J s, $h = 6.626\times10^{-34}$ J s and $k_B = 1.381\times10^{-23}$ J K^{-1}.

(a) CALCULATE · U3 **[2]**

Calculate the atom's zero-point energy E_1, and express it as a temperature E_1/k_B.

1. $E_1 = \pi^2\hbar^2/(2mL^2) = \pi^2(1.055\times10^{-34})^2/(2 \times 1.443\times10^{-25} \times (20.0\times10^{-6})^2) = 9.52\times10^{-34}$ J
2. $E_1/k_B = 9.52\times10^{-34}/1.381\times10^{-23} = 6.9\times10^{-11}$ K, about 69 pK

   L must be in metres before it is squared. The result is tiny because the atom is heavy and the box is wide: the level spacing is comparable with the lowest temperatures cold-atom experiments reach, which is why the discrete levels of so large a box matter only for atoms cooled that far.

(b) STATE · U1 **[2]**

State the possible results of a measurement of the atom's energy and the probability of each, and state whether these probabilities change as time passes.

1. E_1, $4E_1$ and $9E_1$, with probabilities $|1/\sqrt{2}|^2 = 1/2$, $|1/2|^2 = 1/4$ and $|1/2|^2 = 1/4$, which sum to 1; no other energy can be obtained
2. the probabilities do not change: each coefficient acquires only the phase factor e^(−iE_nt/ħ), which leaves $|c_n|^2$ unchanged

   Candidates who give the energies as $2E_1$ and $3E_1$ have used a level spacing proportional to n, the pattern of the oscillator rather than the box. Those who say the probabilities oscillate have confused the distribution of energy, which is fixed, with the distribution of position, which is not.

(c) CALCULATE · U3 **[2]**

Every phase factor e^(−iE_nt/ħ) in Ψ(x,t) returns to 1 at the same instant, after which the atom's state repeats exactly. Calculate the earliest such time.

1. $\Psi(x,t) = \Sigma\, c_n\psi_n(x)e^{(-in^2E_1t/\hbar)}$; since every n^2 is an integer, all three factors first equal 1 together when $E_1t/\hbar = 2\pi$ (the relative phases $3E_1t/\hbar$ and $8E_1t/\hbar$ have no common factor that would allow an earlier repeat)
2. $t = 2\pi\hbar/E_1 = h/E_1 = 6.626\times10^{-34}/9.52\times10^{-34} = 0.70$ s

   The time is set by E_1 alone, because every energy is an integer multiple of E_1. A candidate who uses $h/(E_2 - E_1) = 0.23$ s has found when the ψ_1 and ψ_2 terms are back in step, but the ψ_3 term is not yet back in step with them.

(d) EXPLAIN · U1 **[2]**

At $t = h/(2E_1)$ the probability density is $|\Psi(L - x, 0)|^2$, the mirror image of the initial density about the centre of the trap. Explain why, and state $\langle x\rangle$ at that instant.

1. at $t = h/(2E_1)$ each phase factor is $e^{(-i\pi n^2)} = (-1)^n$, while reflection about the centre gives $\psi_n(L - x) = (-1)^{(n+1)}\psi_n(x)$; so $(-1)^n\psi_n(x) = -\psi_n(L - x)$ for every n, and $\Psi(x,t) = -\Psi(L - x, 0)$
2. an overall sign does not change $|\Psi|^2$, so the density is the initial one reflected about $x = L/2$: the atom is now near the right wall, with $\langle x\rangle = L - 0.275L = 0.725L$, although no energy probability has changed

   Candidates who expect $\langle x\rangle$ simply to swing about the centre miss the mechanism. The phases reverse the sign of ψ_2, the only term that is odd about the centre, relative to the two even terms, and reversing the odd part of a function is exactly a reflection.

(e) PREDICT · U1 **[2]**

Suppose instead that the atom's energy is measured at $t = 0$ and measured again at $t = h/(2E_1)$. Predict the probability that the two results agree, and whether the density at $t = h/(2E_1)$ is then the mirror image of the initial density.

1. the first measurement leaves the atom in the single eigenstate ψ_n whose energy it found; that state is stationary, so the second measurement repeats the result with probability 1
2. the density is then $|\psi_n|^2$, constant in time and symmetric about $x = L/2$ with $\langle x\rangle = L/2$, so no mirror image of the initial density appears near the right wall: the measurement removed the other terms whose relative phases produced it

   Candidates who combine the probabilities as independent events, $1/2 \times 1/2 + 1/4 \times 1/4 + 1/4 \times 1/4 = 3/8$, have not updated the state after the first result. Repeatability after collapse is what the part tests.

12

CORE COURSE

STRUCTURED · ROUTINE · UQMI-10.05 · UQMI-13.04 · UQMI-13.05 · UQMI-13.06 · UQMI-13.08 **[10]**

A beam of spin-½ atoms is prepared in the spin state $\chi = (1/\sqrt{3})(1, 1 + i)^{T}$, written as a column in the S_z basis $\{|+z\rangle, |-z\rangle\}$. The spin operators are $S_i = (\hbar/2)\sigma_i$, where σ_x, σ_y and σ_z are the Pauli matrices.

(a) STATE · U1 **[2]**

State the three Pauli matrices σ_x, σ_y and σ_z in the S_z basis.

1. $\sigma_x = [[0, 1], [1, 0]]$ and $\sigma_z = [[1, 0], [0, -1]]$
2. $\sigma_y = [[0, -i], [i, 0]]$, with $-i$ in the upper right

The sign in σ_y is the one most often lost. It is fixed by $\sigma_x\sigma_y = i\sigma_z$, which takes a line to check.

(b) CALCULATE · U3 **[2]**

Calculate the probability that a measurement of S_z gives $+\hbar/2$, and the expectation value $\langle S_z\rangle$.

1. $P(+\hbar/2) = |1/\sqrt{3}|^2 = 1/3$ (and $P(-\hbar/2) = |1 + i|^2/3 = 2/3$)
2. $\langle S_z\rangle = (\hbar/2)(1/3 - 2/3) = -\hbar/6$

$|1 + i|^2 = 2$, not $(1 + i)^2 = 2i$. A candidate who squares the complex number instead of taking its squared modulus finds probabilities that do not sum to 1 and should stop there.

(c) DETERMINE · U2 **[3]**

Determine the probability that a measurement of S_x gives $+\hbar/2$.

1. the $+\hbar/2$ eigenspinor of S_x is $|+x\rangle = (1/\sqrt{2})(1, 1)^{T}$
2. $\langle +x|\chi\rangle = (1/\sqrt{6})(1 + 1 + i) = (2 + i)/\sqrt{6}$, with the bra's components conjugated (they are real here)
3. $P = |2 + i|^2/6 = 5/6$

The probability is the squared modulus of the overlap with the S_x eigenspinor. A candidate who looks for 'the x component of χ' is looking for something that does not exist, because spinor components are amplitudes along z only.

(d) DETERMINE · U1 **[3]**

The atoms pass through an S_x analyser, and only those leaving its $+\hbar/2$ port go on to an S_z analyser. Determine the fraction of the original atoms that leave the $-\hbar/2$ port of the S_z analyser, and give the reason it is not related to the 2/3 found in (b).

1. the atoms kept after the S_x analyser are in $|+x\rangle$, and they are a fraction 5/6 of the original beam
2. $|\langle -z|+x\rangle|^2 = 1/2$, so the fraction leaving the $-\hbar/2$ port is $(5/6)(1/2) = 5/12$
3. the S_x measurement replaces χ by $|+x\rangle$ and erases the earlier S_z amplitudes; since $[S_x, S_z] \neq 0$ no state has definite values of both, so the loss comes from the change of state, not from the magnet mechanically disturbing the atoms

Candidates multiply 5/6 by 2/3, carrying the original S_z statistics through a measurement that destroyed them. The cascade is set to catch exactly that.

13

CORE COURSE

STRUCTURED · DEMANDING · UQMI-02.06 · UQMI-02.08 · UQMI-08.01 · UQMI-08.02 **[12]**

Very slow electrons are strongly reflected by a crystal surface even though the crystal attracts them. Inside the crystal an electron's potential energy is lower than in the vacuum outside by U, the inner potential, so the surface at x = 0 is modelled as a downward potential step: V(x) = 0 for x < 0 (vacuum) and V(x) = −U for x > 0 (crystal). Electrons of energy E = 0.50 eV arrive from the vacuum, moving in the +x direction at normal incidence. Their stationary scattering state is ψ(x) = e^(ikx) − (2/3)e^(−ikx) for x < 0 and ψ(x) = (1/3)e^(iqx) for x > 0, where $\hbar k = \sqrt{(2m_e E)}$ and $\hbar q = \sqrt{(2m_e(E + U))}$, but the value of U is not known. The probability current is $j = (\hbar/m_e)\,\mathrm{Im}(\psi^* \, d\psi/dx)$. Take $m_e = 9.109\times10^{-31}$ kg, $\hbar = 1.055\times10^{-34}$ J s and 1 eV = 1.602×10^{-19} J.

(a) DERIVE · U2 **[3]**

Stating the matching conditions that ψ must satisfy at x = 0, and why each holds here, derive the ratio q/k for this state and hence the inner potential U in electronvolts.

1. ψ is continuous at x = 0, because a jump in ψ would put the derivative of a delta function into ψ″, which no finite V can balance in the Schrödinger equation: the two sides give 1 − 2/3 = 1/3, so the amplitudes given satisfy this condition
2. dψ/dx is continuous at x = 0, because the step in V is finite and so ψ″ has at most a finite jump: ik(1 + 2/3) = iq(1/3), giving q = 5k
3. $(q/k)^2 = (E + U)/E = 25$, so U = 24E = 12.0 eV

The reflected amplitude is negative, so the slope condition contains 1 + 2/3 = 5/3. A candidate who writes 1 − 2/3 there finds q = k, which would mean no step at all and contradicts the reflected wave in the same state.

(b) SHOW THAT · U2 **[2]**

Show that the probability current is the same on both sides of the surface, as conservation of probability requires in a stationary state.

1. for x < 0 the cross terms between e^(ikx) and e^(−ikx) contribute nothing to Im(ψ* dψ/dx), so $j = (\hbar k/m_e)(1 - 4/9) = (5/9)\hbar k/m_e$
2. for x > 0, $j = (\hbar q/m_e)(1/3)^2 = (5\hbar k/m_e)(1/9) = (5/9)\hbar k/m_e$, equal to the current for x < 0, so no probability builds up at the surface

Candidates who compare $|\psi|^2$ on the two sides, 1/9 inside against 1 for the incident wave, conclude that probability is lost at the surface. Density is not current: the transmitted electrons move five times faster, so a smaller density carries the same flux.

(c) CALCULATE · U3 **[3]**

A beam of these electrons carrying a current of 2.0 nA strikes the surface. Calculate the reflected and transmitted electron currents, and the speed of the transmitted electrons inside the crystal.

1. R = |−2/3|² = 4/9, so the reflected current is (4/9) × 2.0 nA = 0.89 nA
2. T = (q/k)|t|² = 5 × (1/9) = 5/9, so the transmitted current is (5/9) × 2.0 nA = 1.1 nA, and R + T = 1
3. inside the crystal $v = \hbar q/m_e = 5\sqrt{(2E/m_e)} = 5 \times 4.19\times10^5$ m s⁻¹ = 2.1×10^6 m s⁻¹

T is not $|t|^2 = 1/9$. A candidate who uses $|t|^2$ finds only 0.22 nA transmitted and 0.89 nA reflected, and should notice that 0.9 nA of the beam has disappeared.

(d) CALCULATE · U3 **[2]**

Calculate the electron energy at which the reflected fraction at this surface falls to 1.0%.

1. R = 0.010 requires |r| = (q − k)/(q + k) = 0.10, so q/k = 1.10/0.90 = 11/9
2. $(q/k)^2 = 1 + U/E = 121/81$, giving E = 81U/40 = 24 eV (24.3 eV)

Candidates who set $|r| = 0.010$ instead of $|r|^2 = 0.010$ find about 290 eV. The reflection from a drop is appreciable only while E is not large compared with U.

(e) EXPLAIN · U1 **[2]**

A classical electron crossing the surface would simply speed up, and none would be reflected. Explain why the quantum electron can be reflected by a drop in potential, and why far fewer electrons would be reflected if V fell from 0 to −U smoothly over several nanometres instead of abruptly.

1. at an abrupt step the wavenumber jumps from k to q = 5k; a transmitted wave alone would need 1 = t from continuity of ψ and k = qt from continuity of dψ/dx, which cannot both hold when q ≠ k, so a reflected wave is required: a wave effect, like the partial reflection of light where the refractive index changes suddenly, with no counterpart for a classical particle
2. if V changes over a distance much larger than the reduced wavelength 1/k (0.28 nm in vacuum at 0.50 eV, and 1/q = 0.055 nm inside), the local wavelength adjusts gradually and ψ follows it without a mismatch at any point, so the reflected amplitude becomes exponentially small and is negligible for a change spread over several nanometres; reflection is appreciable only if V changes within a distance of order 1/k or less, a fraction of a nanometre

Candidates who say the electron bounces off the edge of the crystal are using a classical picture, and that picture predicts no reflection at all. The reflection comes from the abruptness of the change in wavelength, not from a force pushing the electron back.

14

CORE COURSE

STRUCTURED · DEMANDING · UQMI-11.01 · UQMI-11.04 · UQMI-11.07 · UQMI-11.08 **[12]**

An electron shared between two neighbouring semiconductor quantum dots, L and R, is modelled as a two-state system in the basis $\{|L\rangle, |R\rangle\}$ of states localised on one dot. Energies are measured from the mean of the two on-dot energies, and in that basis the Hamiltonian is $H = [[\varepsilon/2, -\tau], [-\tau, -\varepsilon/2]]$. The detuning $\varepsilon = 30$ μeV is set by a gate voltage, and the tunnel coupling is $\tau = 20$ μeV. The next single-electron level in each dot lies about 1.5 meV above these states. If the electron starts on dot L at $t = 0$, the probability of finding it on dot R is $P_R(t) = [4\tau^2/(E_+ - E_-)^2] \sin^2[(E_+ - E_-)t/(2\hbar)]$, where E_+ and E_- are the eigenvalues of H. Take $h = 6.626\times10^{-34}$ J s and 1 eV $= 1.602\times10^{-19}$ J.

(a) JUSTIFY · U4 **[2]**

Justify the two-state model of the electron, using the energies given.

1. the omitted levels lie about 1.5 meV away, some 30 times the 50 μeV scale of ε, τ and the level splitting, so they mix only weakly into the two localised states
2. nothing supplies the 1.5 meV needed to reach those levels: the gate voltage must change slowly compared with h/(1.5 meV) ≈ 3 ps, and the thermal energy k_BT must be much less than 1.5 meV (either condition, stated as a comparison with the 1.5 meV spacing, earns the mark)

A justification must compare energies. 'Two dots, so two states' earns nothing, because each dot has many states; what matters is that only two of them are close together.

(b) DETERMINE · U2 **[3]**

Determine the eigenvalues E_+ and E_- of H, and hence the splitting $E_+ - E_-$.

1. $\det(H - EI) = 0$ gives $(\varepsilon/2 - E)(-\varepsilon/2 - E) - \tau^2 = 0$, so $E^2 = \varepsilon^2/4 + \tau^2$
2. $E\pm = \pm\sqrt{(\varepsilon^2/4 + \tau^2)} = \pm\sqrt{(225 + 400)}$ μeV $= \pm25$ μeV
3. $E_+ - E_- = \sqrt{(\varepsilon^2 + 4\tau^2)} = 50$ μeV, larger than the 30 μeV detuning alone because the coupling pushes the levels apart

Candidates who take the eigenvalues to be the diagonal elements ±15 μeV have ignored the coupling, and those who add ε and 2τ get 70 μeV. The sum in quadrature is the avoided crossing.

(c) DETERMINE · U2 **[3]**

Determine the probability of finding the electron on dot R when it is in the ground state.

1. for the ground state, $E_- = -25$ μeV, writing $v = a|L\rangle + b|R\rangle$, $(H - E_-I)v = 0$ gives $(15 + 25)a - 20b = 0$ with energies in μeV
2. so $b = 2a$, and the normalised ground state is $(|L\rangle + 2|R\rangle)/\sqrt{5}$
3. $P_R = 4/5 = 0.80$ (equivalently, the mixing angle with $\tan 2\chi = 2\tau/\varepsilon = 4/3$ gives $\sin^2\chi = 0.20$ on dot L)

The ground state leans towards dot R, which has the lower on-dot energy, but not completely. A candidate who answers 1 has ignored the coupling; one who answers 1/2 has ignored the detuning.

(d) CALCULATE · U3 **[2]**

The electron is placed on dot L at $t = 0$. Calculate the maximum probability of later finding it on dot R, and the period of the oscillation of $P_R(t)$.

1. maximum $P_R = 4\tau^2/(E_+ - E_-)^2 = 1600/2500 = 0.64$
2. $\sin^2$ repeats when $(E_+ - E_-)t/(2\hbar)$ advances by π, so the period is $2\pi\hbar/(E_+ - E_-) = h/(E_+ - E_-) = 6.626\times10^{-34}/(50\times10^{-6} \times 1.602\times10^{-19}) = 8.3\times10^{-11}$ s

$\sin^2\theta$ has period π, not 2π. A candidate who misses this doubles the period to 1.7×10^{-10} s.

(e) PREDICT · U1 **[2]**

The gate voltage is adjusted until $\varepsilon = 0$. Predict the maximum probability of transfer to dot R, and the time after $t = 0$ at which it is first reached.

1. with $\varepsilon = 0$ the eigenstates are $(|L\rangle \pm |R\rangle)/\sqrt{2}$, with equal weight on each dot, and $E_+ - E_- = 2\tau = 40$ μeV, so the maximum $P_R = 4\tau^2/(2\tau)^2 = 1$: complete transfer
2. P_R first reaches 1 when $(2\tau)t/(2\hbar) = \pi/2$, at $t = \pi\hbar/(2\tau) = h/(4\tau) = 5.2\times10^{-11}$ s

Removing the detuning makes the transfer complete but the oscillation slower, because the splitting falls from 50 μeV to 40 μeV. Candidates who expect a more complete transfer to be a faster one have confused the amplitude with the frequency.

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15 CORE COURSE

STRUCTURED · DEMANDING · UQMI-10.07 · UQMI-12.05 · UQMI-12.06 · UQMI-12.08 **[12]**

For orbital angular momentum with $l = 1$, use the ordered basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, in which $L_z = \hbar\,\mathrm{diag}(1, 0, -1)$. The raising and lowering operators $L_\pm = L_x \pm iL_y$ act as $L_\pm|l,m\rangle = \hbar\sqrt{(l(l+1) - m(m\pm1))}\,|l,m\pm1\rangle$. The uncertainty relation for L_x and L_y is $\Delta L_x\,\Delta L_y \geq (\hbar/2)|\langle L_z\rangle|$.

(a) SHOW THAT · U2 **[2]**

Show that the only non-zero matrix elements of L_+ in this basis are $\langle 1,1|L_+|1,0\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\,\hbar$.

1. $L_+|1,0\rangle = \hbar\sqrt{(2-0)}|1,1\rangle = \sqrt{2}\hbar|1,1\rangle$ and $L_+|1,-1\rangle = \hbar\sqrt{(2-(-1)(0))}|1,0\rangle = \sqrt{2}\hbar|1,0\rangle$
2. $L_+|1,1\rangle = \hbar\sqrt{(2-2)} = 0$, and each $L_+|1,m\rangle$ is proportional to $|1,m+1\rangle$, so by orthonormality every other element $\langle 1,m'|L_+|1,m\rangle$ vanishes

The ladder stops at $m = l$ because the square root vanishes there. Candidates should see the zero come out of the formula rather than assert that $m = 2$ is not allowed.

(b) DETERMINE · U2 **[2]**

Determine the 3×3 matrix that represents L_x in this basis.

1. $L_- = L_+\dagger$, with non-zero elements $\langle 1,0|L_-|1,1\rangle = \langle 1,-1|L_-|1,0\rangle = \sqrt{2}\hbar$
2. $L_x = (L_+ + L_-)/2 = (\hbar/\sqrt{2})[[0, 1, 0], [1, 0, 1], [0, 1, 0]]$

Candidates who forget the ½ in $L_x = (L_+ + L_-)/2$ carry a matrix with eigenvalues $\pm2\hbar$ into (c), which cannot be right for a component of $l = 1$ angular momentum.

(c) DETERMINE · U2 **[3]**

Determine the eigenvalues of L_x.

1. sets up $\det(L_x - \lambda I) = 0$ using the matrix from (b)
2. expanding gives $-\lambda^3 + \lambda\hbar^2 = 0$, that is $-\lambda(\lambda^2 - \hbar^2) = 0$
3. $\lambda = \hbar$, 0 and $-\hbar$, the same values as L_z, as they must be, since nothing distinguishes the x axis from the z axis

The comparison with L_z is a built-in check. A candidate who finds any other eigenvalues should look for a lost factor in (b) before going on.

(d) CALCULATE · U3 **[3]**

A particle is prepared in $|1,1\rangle$. Calculate the probabilities of obtaining $\hbar$, 0 and $-\hbar$ in a measurement of L_x.

1. the normalised eigenvectors of L_x are $(1, \sqrt{2}, 1)/2$ for $\hbar$, $(1, 0, -1)/\sqrt{2}$ for 0, and $(1, -\sqrt{2}, 1)/2$ for $-\hbar$
2. their overlaps with $|1,1\rangle = (1, 0, 0)$ are the first components, $1/2$, $1/\sqrt{2}$ and $1/2$
3. the probabilities are $1/4$, $1/2$ and $1/4$, which sum to 1

A state with definite L_z has no definite L_x. Candidates who answer '$\hbar$ with certainty' have assumed that the largest z component forces the largest x component too.

(e) SHOW THAT · U2 **[2]**

Show that in the state $|1,1\rangle$ the product $\Delta L_x\,\Delta L_y$ is exactly equal to the lower bound $(\hbar/2)|\langle L_z\rangle|$.

1. from (d), $\langle L_x\rangle = 0$ and $\langle L_x^2\rangle = (1/4)\hbar^2 + (1/2)(0) + (1/4)\hbar^2 = \hbar^2/2$, so $\Delta L_x = \hbar/\sqrt{2}$; the state is unchanged except for a phase by rotation about z, so $\Delta L_y = \hbar/\sqrt{2}$ as well (or $\langle L_x^2\rangle + \langle L_y^2\rangle = \langle L^2\rangle - \langle L_z^2\rangle = 2\hbar^2 - \hbar^2 = \hbar^2$)
2. $\Delta L_x\,\Delta L_y = \hbar^2/2$, and $(\hbar/2)|\langle L_z\rangle| = (\hbar/2)(\hbar) = \hbar^2/2$, so the bound is met with equality

Candidates who say that $|1,1\rangle$ has no uncertainty because it is an eigenstate are thinking of L_z alone. The state is exactly as sharp in L_x and L_y as the commutator permits, and no sharper.

16 CORE COURSE

STRUCTURED · DISCRIMINATING · UQMI-14.08 · UQMI-15.04 · UQMI-15.05 · UQMI-15.07 · UQMI-15.08 **[12]**

For a hydrogen atom, the radial function $u(r) = rR(r)$ satisfies $-(\hbar^2/2m)\,d^2u/dr^2 + [-e^2/(4\pi\varepsilon_0 r) + \hbar^2 l(l+1)/(2mr^2)]\,u = Eu$. A proposed solution is $u(r) = N\,r^3\,e^{-r/(3a_0)}$, where N is a normalisation constant and $a_0 = 4\pi\varepsilon_0\hbar^2/(me^2) = 5.29\times10^{-11}$ m is the Bohr radius, so that $e^2/(4\pi\varepsilon_0) = \hbar^2/(ma_0)$ and $\hbar^2/(2ma_0^2) = 13.6$ eV. Treat the proton as infinitely heavy. You may use $\int_0^\infty r^n e^{-\beta r}\,dr = n!/\beta^{n+1}$ for a non-negative integer n and $\beta > 0$.

(a) DETERMINE · U2 **[4]**

Determine the value of l for which u(r) satisfies the radial equation, and the corresponding energy E in electronvolts.

1. $d^2u/dr^2 = N[6r - 2r^2/a_0 + r^3/(9a_0^2)]\,e^{-r/(3a_0)}$
2. writes the Coulomb term as $-(\hbar^2/2m)(2/(a_0 r))u$, whose r^2 contribution $-(\hbar^2/2m)(2r^2/a_0)Ne^{-r/(3a_0)}$ cancels the $+(\hbar^2/2m)(2r^2/a_0)Ne^{-r/(3a_0)}$ from $-(\hbar^2/2m)u''$
3. the terms linear in r are $-(\hbar^2/2m)(6r)$ from the kinetic term and $+(\hbar^2/2m)l(l+1)r$ from the centrifugal term (each times $Ne^{-r/(3a_0)}$), which cancel only if $l(l+1) = 6$, so $l = 2$
4. the surviving r^3 term gives $-(\hbar^2/2m)(1/(9a_0^2))u = Eu$, so $E = -\hbar^2/(18ma_0^2) = -13.6$ eV$/9 = -1.51$ eV

   l is not given; it is found only by insisting that the equation hold at every r, power by power. A candidate who tries l = 1 finds an 'energy' that depends on r, which means the guess is wrong, not that the energy varies.

(b) STATE · U1 **[2]**

State the principal quantum number n of this state, its number of radial nodes, and its spectroscopic label.

1. $E = -13.6$ eV$/n^2 = -1.51$ eV gives $n = 3$
2. u has no zero for $r > 0$, so there are no radial nodes ($n - l - 1 = 0$), and the state is 3d

   Candidates who count the zero of u at r = 0 as a node report one radial node. That zero is imposed at the origin for every state and is never counted.

(c) DETERMINE · U3 **[2]**

Determine the most probable distance of the electron from the proton, in nanometres.

1. the radial probability density is $P(r) = |u|^2 \propto r^6 e^{-2r/(3a_0)}$, and $dP/dr = 0$ gives $6/r = 2/(3a_0)$
2. $r = 9a_0 = 0.476$ nm

   The function to maximise is $|u|^2 = r^2|R|^2$, which includes the r^2 from the volume element. Maximising $|R|^2$ instead gives $6a_0$, which is the peak of the density per unit volume along a line, not of the probability per unit radius.

(d) CALCULATE · U3 **[2]**

Calculate the expectation value $\langle r\rangle$ in nanometres.

1. $\langle r\rangle = \int_0^\infty r^7 e^{-\beta r}\,dr \,/\, \int_0^\infty r^6 e^{-\beta r}\,dr$ with $\beta = 2/(3a_0)$, which is $(7!/\beta^8)/(6!/\beta^7) = 7/\beta$
2. $\langle r\rangle = 21a_0/2 = 10.5a_0 = 0.555$ nm, larger than the most probable distance because P(r) has a long tail at large r

   N cancels in the ratio and never needs to be found. A candidate who normalises first loses no marks but loses several minutes.

(e) COMPARE · U1 **[2]**

Compare the degeneracy of this energy level in hydrogen with the degeneracy an $l = 2$ level would have in a generic central potential, such as a finite spherical well, giving the cause of each.

1. in hydrogen E depends on n alone, so the level contains every $n = 3$ state, 3s, 3p and 3d: $1 + 3 + 5 = n^2 = 9$ spatial states (18 with spin); the extra degeneracy between different l belongs to the 1/r form of the Coulomb potential, not to spherical symmetry
2. in a generic central potential such as a finite spherical well, E depends on l as well as on the radial quantum number, so only the $2l + 1 = 5$ states with $m = -2$ to 2 share the energy; that degeneracy is guaranteed by rotational symmetry, because m does not appear in the radial equation

Candidates often attribute the ninefold degeneracy to spherical symmetry. Spherical symmetry accounts only for the fivefold degeneracy in m, which every central potential shares; the degeneracy between 3s, 3p and 3d is special to 1/r. The comparison is made with a generic well because the isotropic harmonic oscillator has an accidental degeneracy of its own.

17

CORE COURSE

DATA ANALYSIS · DEMANDING · UQMI-08.04 · UQMI-08.08 **[10]**

Gate leakage in a transistor is caused by electrons tunnelling directly through the thin silicon dioxide layer between the gate and the silicon. To study it, an engineer measures the gate current I of six test capacitors made with different oxide thicknesses t_{ox} and different gate areas A, all at the same small gate voltage. The model to be tested is $J = J_0 \exp(-2\kappa t_{ox})$, where $J = I/A$ is the current density, J_0 is the same for every capacitor, and $\kappa = \sqrt{(2m^*\varphi_B)}/\hbar$, with φ_B the height of the oxide barrier above the energy of the tunnelling electrons and m^* the effective mass of an electron in the oxide. Internal photoemission measurements, made independently, give $\varphi_B = 3.1$ eV. Take $m_e = 9.109\times10^{-31}$ kg, $\hbar = 1.055\times10^{-34}$ J s, 1 eV $= 1.602\times10^{-19}$ J and 1 μm² $= 1\times10^{-12}$ m².

(a) DESCRIBE · U4 **[2]**

Describe how the readings should be processed and plotted so that the model predicts a straight line, including why the currents themselves should not be plotted and what the gradient measures.

1. divide each current by its gate area, converted to m², to obtain $J = I/A$: the capacitors have different areas and the model predicts current per unit area, so ln I would contain a different ln A for each capacitor and would not follow the model
2. plot $\ln(J / \text{A m}^{-2})$ against t_{ox}; the model gives $\ln J = \ln J_0 - 2\kappa t_{ox}$, a straight line of gradient -2κ, with J_0 affecting only the intercept

Plotting ln I against t_{ox} gives a gradient of about -5.8 nm^{-1}, half the true value, because the larger capacitors were used for the thicker oxides. A candidate who does not remove the area has measured the design of the test chip, not the barrier.

(b) DETERMINE · U4 **[3]**

Determine κ from the readings, and the reduction in oxide thickness that multiplies the leakage current density by ten.

1. $J = 2.00\times10^4$, 2.10×10^3, 203, 21.8, 2.14 and 0.225 A m^{-2}, so $\ln(J / \text{A m}^{-2}) = 9.90$, 7.65, 5.31, 3.08, 0.76 and -1.49 for capacitors 1 to 6
2. the best-fit gradient of ln J against t_{ox} is -1.14×10^{10} m^{-1} (accept -1.10×10^{10} to -1.18×10^{10} m^{-1}, or -11.4 nm^{-1}), so $\kappa = 5.7\times10^9$ m^{-1} (accept 5.5×10^9 to 5.9×10^9 m^{-1})
3. a tenfold rise in J needs a thinning of $\Delta t_{ox} = \ln 10/(2\kappa) = 2.303/(1.14\times10^{10}\ \text{m}^{-1}) = 2.0\times10^{-10}$ m, so each 0.20 nm removed from the oxide raises the leakage tenfold

Candidates who leave the areas in μm² shift every ln J by the same amount, which does not change the gradient; those who leave the areas out altogether halve it. Checking that ln J falls by about 2.3 for each 0.20 nm of oxide catches the second slip.

(c) ESTIMATE · U4 **[3]**

Estimate the effective mass m^* of the tunnelling electrons in the oxide, as a multiple of m_e, using $\varphi_B = 3.1$ eV, and state the assumptions on which your estimate rests.

1. rearranges $\kappa = \sqrt{(2m^*\varphi_B)}/\hbar$ to $m^* = \hbar^2\kappa^2/(2\varphi_B)$
2. $m^* = (1.055\times10^{-34})^2 \times (5.7\times10^{9})^2/(2 \times 3.1 \times 1.602\times10^{-19}) = 3.6\times10^{-31}$ kg $\approx 0.40m_e$ (accept $0.37m_e$ to $0.43m_e$)
3. states at least two assumptions, for example a rectangular barrier of fixed height φ_B that the small gate voltage does not tilt appreciably, a single effective mass independent of energy and thickness, the same J_0 for every capacitor, or oxide thicknesses that are accurate and uniform across each gate

   A number without its assumptions earns two of the three marks. The assumption about J_0 matters most, because any prefactor that varies with t_{ox} changes the gradient and therefore m^*.

(d) EVALUATE · U4 **[2]**

A colleague analyses the same readings taking the electron in the oxide to have the free-electron mass m_e, and reports the barrier height that follows. Evaluate this analysis.

1. with $m^* = m_e$ the gradient gives $\varphi = \hbar^2\kappa^2/(2m_e) = 2.0\times10^{-19}$ J ≈ 1.2 eV (accept 1.15 to 1.33 eV), less than half the 3.1 eV measured independently by photoemission
2. the gradient fixes only the product $m^*\varphi_B$, so the readings alone cannot decide between the two analyses; judged against the independent barrier height the free-electron analysis must be rejected, and the data require an effective mass near $0.4m_e$ rather than a barrier near 1.2 eV

   Candidates who accept 1.2 eV because the straight line fits well have tested the exponential form, not the parameters. A good fit cannot separate two quantities that enter the model only as a product.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-09.04 · UQMI-09.07 · UQMI-09.08 · UQMI-09.09 **[15]**

A particle of mass m moves in the potential $V(x) = \tfrac{1}{2}m\omega^2x^2$, so that $\hat{H} = \hat{p}^2/2m + \tfrac{1}{2}m\omega^2\hat{x}^2$. Define $\hat{a} = \sqrt{(m\omega/2\hbar)}(\hat{x} + i\hat{p}/(m\omega))$ and its adjoint $\hat{a}^\dagger = \sqrt{(m\omega/2\hbar)}(\hat{x} - i\hat{p}/(m\omega))$, and write $\hat{N} = \hat{a}^\dagger\hat{a}$. The canonical commutator is $[\hat{x}, \hat{p}] = i\hbar$. Once $[\hat{a}, \hat{a}^\dagger] = 1$ has been established, you may use its consequences $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(a) DERIVE · U2 **[3]**

Derive $[\hat{a}, \hat{a}^\dagger] = 1$ from the canonical commutator.

1. $[\hat{a}, \hat{a}^\dagger] = (m\omega/2\hbar)[\hat{x} + i\hat{p}/(m\omega), \hat{x} - i\hat{p}/(m\omega)]$, and $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$ leaves only the cross terms
2. these are $-(i/(m\omega))[\hat{x}, \hat{p}] + (i/(m\omega))[\hat{p}, \hat{x}] = -(2i/(m\omega))[\hat{x}, \hat{p}]$
3. substituting $[\hat{x}, \hat{p}] = i\hbar$ gives $(m\omega/2\hbar)(2\hbar/(m\omega)) = 1$

   The result is a pure number. A candidate who finds $i\hbar$ or $\hbar/(m\omega)$ has dropped a factor in the definition of $\hat{a}$; $\hat{a}$ is dimensionless, and checking that catches the slip at once.

(b) SHOW THAT · U2 **[2]**

Show that $\hat{H} = \hbar\omega(\hat{N} + \tfrac{1}{2})$.

1. $\hat{a}^\dagger\hat{a} = (m\omega/2\hbar)[\hat{x}^2 + \hat{p}^2/(m^2\omega^2) + (i/(m\omega))(\hat{x}\hat{p} - \hat{p}\hat{x})] = (m\omega/2\hbar)[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - \tfrac{1}{2}$, using $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$
2. multiplying by $\hbar\omega$ gives $\hbar\omega\hat{a}^\dagger\hat{a} = \hat{p}^2/2m + \tfrac{1}{2}m\omega^2\hat{x}^2 - \tfrac{1}{2}\hbar\omega = \hat{H} - \tfrac{1}{2}\hbar\omega$, so $\hat{H} = \hbar\omega(\hat{N} + \tfrac{1}{2})$

   The $\tfrac{1}{2}$ comes entirely from the operators failing to commute. A candidate who factorises $x^2 + p^2$ as if they were numbers loses the zero-point energy.

(c) DERIVE · U2 **[4]**

Derive the energy spectrum $E_n = (n + \tfrac{1}{2})\hbar\omega$ with $n = 0, 1, 2, \ldots$, using only the commutators and the fact that no vector has a negative squared norm. Do not evaluate any integral.

1. if $\hat{N}|\nu\rangle = \nu|\nu\rangle$, then $\hat{N}(\hat{a}|\nu\rangle) = (\hat{a}\hat{N} - \hat{a})|\nu\rangle = (\nu - 1)\hat{a}|\nu\rangle$, so $\hat{a}|\nu\rangle$ is either zero or an eigenstate with eigenvalue $\nu - 1$; likewise $\hat{a}^\dagger|\nu\rangle$ has eigenvalue $\nu + 1$
2. for normalised $|\nu\rangle$, $\nu = \langle\nu|\hat{a}^\dagger\hat{a}|\nu\rangle = \|\hat{a}|\nu\rangle\|^2 \geq 0$, so no eigenvalue of $\hat{N}$ is negative
3. repeated lowering would reach a negative eigenvalue unless it ends on a state $|0\rangle$ with $\hat{a}|0\rangle = 0$, and that state has $\nu = \|\hat{a}|0\rangle\|^2 = 0$; a non-integer ν would step past zero without ending, so ν must be a non-negative integer
4. $\|\hat{a}^\dagger|n\rangle\|^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle = n + 1 > 0$, so raising from $\nu = 0$ never terminates and reaches every $\nu = n = 0, 1, 2, \ldots$; then $\hat{H} = \hbar\omega(\hat{N} + \tfrac{1}{2})$ gives $E_n = (n + \tfrac{1}{2})\hbar\omega$, with ground-state energy $\tfrac{1}{2}\hbar\omega$

The positive norm is what makes the argument work. Candidates who announce that the ladder has a bottom rung have assumed the result, and the terminating state must be shown to have $\nu = 0$, not merely to exist.

(d) DETERMINE · U2 **[3]**

Determine $\langle\hat{x}^2\rangle$ in the eigenstate $|n\rangle$ using ladder operators alone.

1. $\hat{x} = \sqrt{(\hbar/2m\omega)}(\hat{a} + \hat{a}^\dagger)$, so $\hat{x}^2 = (\hbar/2m\omega)(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$
2. $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ change n by -2 and $+2$, so $\langle n|\hat{a}^2|n\rangle = \langle n|\hat{a}^{\dagger 2}|n\rangle = 0$
3. $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$ gives $\langle n|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle = 2n + 1$, so $\langle\hat{x}^2\rangle = (2n + 1)\hbar/(2m\omega) = E_n/(m\omega^2)$

Candidates who expand $(\hat{a} + \hat{a}^\dagger)^2$ as $\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}$ have commuted $\hat{a}$ past $\hat{a}^\dagger$ and lost the $+1$ that keeps $\langle x^2\rangle$ non-zero in the ground state. Writing the answer as $E_n/(m\omega^2)$ shows half the energy is potential, a check from the virial theorem.

(e) PREDICT · U1 **[3]**

The particle is in the ground state $|0\rangle$ when the potential is suddenly tightened to $V'(x) = \tfrac{1}{2}m(2\omega)^2x^2$, so quickly that the wavefunction has no time to change. Predict which energy eigenstates of the new oscillator the particle can then be found in, and its mean energy $\langle\hat{H}'\rangle$ immediately after the change, in terms of $\hbar\omega$.

1. V and V' are both even, so the new eigenstates have parity $(-1)^n$; the unchanged state $|0\rangle$ is even, so its overlap with every odd-n state of the new oscillator vanishes, and only $n = 0, 2, 4, \ldots$ can be found, with energies $(n + \tfrac{1}{2})(2\hbar\omega) = \hbar\omega, 5\hbar\omega, 9\hbar\omega, \ldots$
2. the state is still the old $|0\rangle$, so from (d) $\langle\hat{x}^2\rangle = \hbar/(2m\omega)$; then $\langle V\rangle = \tfrac{1}{2}m\omega^2\langle\hat{x}^2\rangle = \tfrac{1}{4}\hbar\omega$ is half of $E_0 = \tfrac{1}{2}\hbar\omega$, so the kinetic energy is $\langle\hat{p}^2\rangle/2m = \tfrac{1}{4}\hbar\omega$, and neither expectation changes at the instant of the change
3. $\langle\hat{H}'\rangle = \langle\hat{p}^2\rangle/2m + \tfrac{1}{2}m(2\omega)^2\langle\hat{x}^2\rangle = \tfrac{1}{4}\hbar\omega + \hbar\omega = (5/4)\hbar\omega$, above the new ground energy $\hbar\omega$, so the particle is not certain to be found in the new ground state

Candidates who say the energy is still $\tfrac{1}{2}\hbar\omega$, or that the particle simply occupies the new ground state, have confused an unchanged wavefunction with an unchanged energy: the Hamiltonian has changed while the state has not. A candidate who allows the new $|1\rangle$ has not used the parity that both potentials share.

19

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UQMI-14.01 · UQMI-14.04 · UQMI-14.08 · UQMI-14.09 **[15]**

A simple model of the deuteron treats the neutron–proton pair as a single particle of reduced mass $\mu = 8.37\times10^{-28}$ kg moving in a spherical square well, with $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, where $a = 2.0\times10^{-15}$ m. Only states with $l = 0$ are considered, so ψ depends on r alone and $\nabla^2\psi = (1/r^2)\,d/dr\,(r^2\,d\psi/dr)$. Primes denote d/dr. Take $\hbar = 1.055\times10^{-34}$ J s and 1 MeV $= 1.602\times10^{-13}$ J. The measured binding energy of the deuteron is 2.22 MeV, and it has no bound excited state.

(a) DERIVE · U2 **[3]**

Starting from the time-independent Schrödinger equation in three dimensions, derive the equation $-(\hbar^2/2\mu)u'' + V(r)u = Eu$ satisfied by $u(r) = r\psi(r)$.

1. writes $-(\hbar^2/2\mu)\nabla^2\psi + V(r)\psi = E\psi$, with $\nabla^2\psi = (1/r^2)(r^2\psi')'$ for $l = 0$
2. with $\psi = u/r$, $\psi' = u'/r - u/r^2$, so $r^2\psi' = ru' - u$ and $(r^2\psi')' = u' + ru'' - u' = ru''$, giving $\nabla^2\psi = u''/r$
3. substituting and multiplying through by r gives $-(\hbar^2/2\mu)u'' + V(r)u = Eu$

The point of $u = r\psi$ is that the first-derivative terms cancel exactly, turning a three-dimensional problem into a one-dimensional one. Candidates who drop the u' terms without showing that they cancel have asserted the result.

(b) JUSTIFY · U1 **[2]**

Justify the boundary condition $u(0) = 0$. Note that a wavefunction behaving as $1/r$ near the origin is square-integrable in three dimensions.

1. if $u(0) \neq 0$ then $\psi \approx u(0)/r$ near $r = 0$, and the volume element $r^2\,dr$ keeps $\int|\psi|^2 r^2\,dr$ finite there, so normalisability alone does not exclude it
2. but $\nabla^2(1/r) = -4\pi\delta^3(r)$, so such a ψ satisfies the Schrödinger equation at the origin only if V contains a delta function there; the finite well contains none, so $u(0) = 0$ is required

The tempting answer is that ψ would blow up. It does, but harmlessly for normalisation; the real objection is that $1/r$ is not a solution at the origin, which is why the prompt supplies the hint.

(c) SHOW THAT · U2 **[3]**

Show that a bound state with energy $E = -E_B$, where $0 < E_B < V_0$, must satisfy $k\cot(ka) = -\kappa$, where $\hbar k = \sqrt{(2\mu(V_0 - E_B))}$ and $\hbar\kappa = \sqrt{(2\mu E_B)}$.

1. inside, $u'' = -k^2u$ with $u(0) = 0$ gives $u = A\sin(kr)$; outside, $u'' = \kappa^2u$ and normalisability removes $e^{(+\kappa r)}$, leaving $u = B\,e^{(-\kappa r)}$
2. u and u' are continuous at $r = a$ because V is finite there: $A\sin(ka) = B\,e^{(-\kappa a)}$ and $Ak\cos(ka) = -\kappa B\,e^{(-\kappa a)}$
3. dividing the second condition by the first gives $k\cot(ka) = -\kappa$

This is the odd-parity condition of a one-dimensional finite well of width 2a, with $u(0) = 0$ playing the part of odd symmetry. Candidates who write $u = A\cos(kr)$ inside have lost the result of (b).

(d) DERIVE · U2 **[3]**

Derive, in terms of μ, a and $\hbar$, the minimum depth $V_{0,\min}$ for which the well has any bound state at all.

1. with $k > 0$ and $\kappa > 0$, $k\cot(ka) = -\kappa$ needs $\cot(ka) < 0$, so the lowest state requires $ka > \pi/2$; for $ka \leq \pi/2$ there is no solution
2. the bound state first appears at threshold, $E_B \to 0$, where $\kappa \to 0$ and the condition becomes $\cot(ka) = 0$, with $ka = \pi/2$
3. at $E_B = 0$, $\hbar k = \sqrt{(2\mu V_0)}$, so $V_{0,\min} = \pi^2\hbar^2/(8\mu a^2)$

In one dimension any attractive well binds a state. In three dimensions $u(0) = 0$ removes the even solutions and with them that guarantee, and candidates who carry the one-dimensional result over never look for a minimum depth.

(e) CALCULATE · U3 **[2]**

Calculate $V_{0,\min}$ in MeV for the values given.

1. $V_{0,\min} = \pi^2(1.055\times10^{-34})^2/(8 \times 8.37\times10^{-28} \times (2.0\times10^{-15})^2) = 4.10\times10^{-12}$ J
2. $V_{0,\min} = 25.6$ MeV

Squaring 2.0×10^{-15} m is where powers of ten go astray. Tens of MeV is the scale of nuclear binding, and an answer in eV or GeV signals a slip.

(f) SKETCH · U1 **[2]**

A well depth of 36.6 MeV reproduces the measured binding energy, giving ka = 1.82 and κa = 0.46. Sketch u(r) for the deuteron's ground state from r = 0 to r = 5a, marking r = a on the axis.

1. for r < a, u rises from u(0) = 0 as a sine, reaches its maximum just inside the wall (at kr = π/2, r ≈ 0.86a) and has begun to fall by r = a
2. at r = a, u and its slope join smoothly onto a slowly decaying exponential with decay length 1/κ ≈ 2.2a, still clearly non-zero at r = 5a, so that most of the area under u^2 lies outside the well

Candidates tend to draw a bound state as a hump confined to the well. A barely bound state spends most of its time outside the range of the force that binds it (here about 64% of the probability lies beyond r = a), and the sketch must show that tail.

University Quantum Mechanics II · Midterm Examination

The first eight units of Quantum Mechanics II on one paper: the spherical harmonics and the hydrogen radial functions derived rather than quoted, spin and charge in magnetic fields, composite systems and density operators, rotations and the addition of angular momentum, time-independent perturbation theory, identical particles, and the fine structure of hydrogen. Every degenerate perturbation is diagonalised in the basis its symmetry picks out, and the results the course quotes rather than derives, such as the Thomas factor and the Darwin term, are given in the stems.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**74**	**14**	**8**

SECTION	**UQMII · Midterm · University Quantum Mechanics II — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Recall 2 · Routine 5 · Demanding 6 · Discriminating 1** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UQMII-01 · UQMII-02 · UQMII-03 · UQMII-04 · UQMII-05 · UQMII-06 · UQMII-07 · UQMII-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Quantum Mechanics II · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-01.01 · UQMII-01.02 **[1]**

Separating the angular equation for m = 0 and writing x = cos θ gives Legendre's equation, $(1 - x^2)y'' - 2xy' + \lambda y = 0$. Substituting $y = \Sigma\, a_k x^k$ gives the recursion $a_{k+2} = [k(k + 1) - \lambda]a_k/[(k + 1)(k + 2)]$, so the even and the odd coefficients form two independent series. Take λ = 6. Which statement is correct?

A Both series terminate, the even one as $1 - 3x^2$ and the odd one as $x - (2/3)x^3$, so λ = 6 has two independent polynomial solutions.

B Neither series terminates, but both converge for $|x| < 1$, so λ = 6 is as acceptable as any other value; λ is quantised only by the boundary condition on the φ equation.

C The even series terminates as $1 - 3x^2$; the odd series does not, and because its coefficients fall off only as 1/k it diverges logarithmically at $x = \pm 1$, so its coefficient must be set to zero.

D The even series terminates as $1 - 3x^2$; the odd series does not, but its coefficients alternate in sign, so it converges at $x = \pm 1$ and any combination of the two solutions is acceptable.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-02.08 **[1]**

For hydrogen, with a_0 the Bohr radius, compare the 3s state (n = 3, l = 0) with a 3d state (n = 3, l = 2). Which expectation value is the same for the two states?

A $\langle 1/r \rangle$, equal to $1/(9a_0)$ in both, because the virial theorem fixes $\langle V \rangle = 2E_3$

B $\langle r \rangle$, equal to $9a_0$ in both, the radius of the third Bohr orbit

C $\langle 1/r^2 \rangle$, because the centrifugal term averages to the same value in every state of a given n

D $\langle r^2 \rangle$, because the mean-square radius is fixed by the energy through $\langle T \rangle = -E_3$

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-03.06 · UQMII-03.07 **[1]**

A particle of charge q and mass m moves in the potentials φ and A with Hamiltonian $\hat{H} = (\hat{p} - qA)^2/(2m) + q\varphi$, where $\hat{p} = -i\hbar\nabla$. Under the gauge transformation $A \to A + \nabla\Lambda$, $\varphi \to \varphi - \partial\Lambda/\partial t$, with Λ(r, t) any real function, the Schrödinger equation keeps its form provided $\psi \to e^{(iq\Lambda/\hbar)}\psi$. Which expectation value is the same in every gauge, whatever Λ is?

A ⟨p̂⟩, because $\hat{p} = -i\hbar\nabla$ is the same operator in every gauge

B ⟨Ĥ⟩, because the Hamiltonian generates the same physical time evolution in every gauge

C The phase of ψ at a fixed point r_0, because multiplying ψ by $e^{(iq\Lambda/\hbar)}$ leaves $|\psi|^2$ unchanged

D ⟨p̂ − qA⟩, which equals m times the rate of change of ⟨r⟩

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-04.03 · UQMII-04.04 · UQMII-04.05 **[1]**

A source emits spin-½ particles, each prepared either in $|+z\rangle$ or in $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$, with probability ½ each, and the record of which state was prepared is discarded. What is the purity $\mathrm{Tr}(\rho^2)$ of the ensemble?

A 0.50

B 0.75

C 0.85

D 1.00

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-05.06 · UQMII-05.07 **[1]**

An electron in a 2p orbital of hydrogen is in the coupled state with l = 1, s = ½, j = ½ and m_j = ½. The stretched state is $|j = 3/2, m_j = 3/2\rangle = |m_l = 1\rangle|\uparrow\rangle$, and $J_- = L_- + S_-$, with $L_-|l, m\rangle = \hbar\sqrt{(l(l+1) - m(m-1))}\,|l, m-1\rangle$ and $S_-|\uparrow\rangle = \hbar|\downarrow\rangle$. What is the probability that a measurement of S_z on the j = ½ state gives $+\hbar/2$?

A 1/2

B 2/3

C 1/3

D 1

Answer

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-06.05 · UQMII-06.06 · UQMII-06.07 **[1]**

A particle of mass m moves in the isotropic two-dimensional oscillator $H_0 = (p_x^2 + p_y^2)/(2m) + \tfrac{1}{2}m\omega^2(x^2 + y^2)$. Its second excited level, at $3\hbar\omega$, is spanned by $|2, 0\rangle$, $|1, 1\rangle$ and $|0, 2\rangle$, the labels counting quanta along x and along y. Here $x = \sqrt{\hbar/(2m\omega)}(a_x + a_x^\dagger)$, with y written likewise, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. A weak perturbation $H' = \varepsilon xy$ is added. What are the first-order shifts of this level, and which good state is left unshifted?

A All three shifts are zero: every diagonal element $\langle n_x, n_y|H'|n_x, n_y\rangle$ vanishes, so the level stays threefold degenerate at first order.

B $+\varepsilon\hbar/(m\omega)$, 0 and $-\varepsilon\hbar/(m\omega)$, with $|1, 1\rangle$ left unshifted

C $+\varepsilon\hbar/(\sqrt{2}\,m\omega)$, 0 and $-\varepsilon\hbar/(\sqrt{2}\,m\omega)$, with $(|2, 0\rangle - |0, 2\rangle)/\sqrt{2}$ left unshifted

D $+\varepsilon\hbar/(m\omega)$, 0 and $-\varepsilon\hbar/(m\omega)$, with $(|2, 0\rangle - |0, 2\rangle)/\sqrt{2}$ left unshifted

Answer

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-07.09 **[1]**

The ground configuration of oxygen is $1s^2 2s^2 2p^4$. The $2p^4$ shell has the same terms as $2p^2$, namely ^1S, ^1D and ^3P. According to Hund's rules, which level is the ground level of oxygen?

A $^3P_2$

B $^3P_0$

C $^1D_2$

D $^1S_0$

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-08.01 · UQMII-08.08 **[1]**

Three small splittings are measured in the n = 2 level of hydrogen: the fine-structure splitting $2P_{3/2}$–$2P_{1/2}$, the Lamb shift $2S_{1/2}$–$2P_{1/2}$, and the hyperfine splitting of $2S_{1/2}$. Using $mc^2 = 511$ keV, $\alpha = 1/137$, $m_e/m_p = 1/1836$, and the powers of α and of m_e/m_p that set the size of each effect, which ordering, largest first, is correct?

A fine structure, then hyperfine splitting, then Lamb shift

B fine structure, then Lamb shift, then hyperfine splitting

C hyperfine splitting, then fine structure, then Lamb shift

D Lamb shift, then fine structure, then hyperfine splitting

Answer

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9 CORE COURSE

STRUCTURED · ROUTINE · UQMII-01.05 · UQMII-01.06 · UQMII-01.07 · UQMII-01.08 **[9]**

On the unit sphere the angular-momentum operators act as $L_z = -i\hbar\,\partial/\partial\varphi$ and $L_\pm = \pm\hbar e^{\pm i\varphi}(\partial/\partial\theta \pm i\cot\theta\,\partial/\partial\varphi)$, and $L_\pm Y_l^m = \hbar\sqrt{(l(l+1) - m(m\pm1))}\,Y_l^{m\pm1}$. The Condon–Shortley convention is used throughout, in which $Y_1^1 = -\sqrt{(3/(8\pi))}\sin\theta\,e^{i\varphi}$.

(a) DERIVE **[2]**

Write $Y_2^2 = g(\theta)e^{2i\varphi}$. Starting from the condition $L_+ Y_2^2 = 0$, derive $g(\theta)$ up to a constant factor.

(b) SHOW THAT **[2]**

Show that normalising Y_2^2 over the sphere, with C real and positive, gives $C = (1/4)\sqrt{(15/(2\pi))}$.

(c) DETERMINE **[2]**

Determine Y_2^1 by applying L_- to Y_2^2, keeping the sign that the operator produces.

(d) STATE **[1]**

State the parity of Y_2^1 and the number of values of θ strictly between 0 and π at which it vanishes.

(e) EXPLAIN **[2]**

In general the recursion for $\cos\theta\,Y_l^m$ gives a combination of Y_{l+1}^m and Y_{l-1}^m. Explain why $\cos\theta\,Y_1^1$ is a multiple of Y_2^1 alone, find the multiple, and say what this implies for a z-polarised electric-dipole transition from an $l = 1$, $m = 1$ state.

10 CORE COURSE

STRUCTURED · DEMANDING · UQMII-02.01 · UQMII-02.02 · UQMII-02.03 · UQMII-02.04 · UQMII-02.05 · UQMII-02.08 **[11]**

In a crystal of cuprous oxide, an electron in the conduction band and a hole in the valence band bind into a hydrogen-like exciton. Model it as two particles of charges $-e$ and $+e$ with effective masses $0.99m_e$ (electron) and $0.58m_e$ (hole), interacting through the Coulomb potential screened by the relative permittivity $\varepsilon_r = 7.5$, $V(r) = -e^2/(4\pi\varepsilon_0\varepsilon_r r)$. Once the centre-of-mass motion is separated, the relative motion is that of one particle of reduced mass μ, whose radial function $u(r) = rR(r)$ obeys $-(\hbar^2/2\mu)u'' + [V(r) + \hbar^2 l(l+1)/(2\mu r^2)]u = Eu$. For a bound state write $\kappa = \sqrt{(-2\mu E)}/\hbar$, $\rho = \kappa r$, $a^* = 4\pi\varepsilon_0\varepsilon_r\hbar^2/(\mu e^2)$ and $\rho_0 = 2/(\kappa a^*)$. Substituting $u = \rho^{l+1}e^{-\rho}v(\rho)$ gives $\rho v'' + 2(l+1-\rho)v' + [\rho_0 - 2(l+1)]v = 0$. For hydrogen with an infinitely heavy nucleus, $\hbar^2/(2m_e a_0^2) = 13.606$ eV and $a_0 = 0.05292$ nm.

(a) DERIVE **[2]**

Substituting $v(\rho) = \sum_j c_j\rho^j$, derive the recursion relation between c_{j+1} and c_j.

(b) EXPLAIN **[2]**

The series solves the differential equation for any value of E. Explain why a physical solution must nevertheless have a terminating series, and why this condition fixes the energy.

(c) SHOW THAT **[2]**

Show that the bound-state energies are $E_n = -(\mu/m_e)(13.606\text{ eV})/(\varepsilon_r^2 n^2)$, with $n = N + l + 1$, where N is the degree of the terminated polynomial, and that $l \le n - 1$.

(d) CALCULATE **[3]**

Calculate μ as a multiple of m_e, the exciton Rydberg energy $R^* = -E_1$ in meV, and the effective Bohr radius a^* in nm.

(e) DETERMINE **[2]**

For these states Kramers' relation reads $((s+1)/n^2)\langle r^s\rangle - (2s+1)a^*\langle r^{s-1}\rangle + (s/4)[(2l+1)^2 - s^2]a^{*2}\langle r^{s-2}\rangle = 0$. Absorption lines of the exciton P series ($l = 1$) have been resolved up to $n = 25$. Determine $\langle r\rangle$ for the $n = 25$, $l = 1$ exciton.

11 CORE COURSE

STRUCTURED · DEMANDING · UQMII-03.01 · UQMII-03.02 · UQMII-03.03 · UQMII-03.04 · UQMII-05.02 **[10]**

The neutron is a spin-½ particle whose magnetic moment is $\mu = \gamma S$, with $\gamma = -1.832 \times 10^8$ rad s^{-1} T^{-1}, so the moment points opposite to the spin. In a magnetic field B its spin Hamiltonian is $H = -\mu \cdot B = -\gamma B \cdot S$, and the spin operators obey $[S_x, S_y] = i\hbar S_z$ and its cyclic permutations. In every part the neutrons travel at 2200 m/s, and their motion through the apparatus is treated classically, so that a neutron spends a time L/v in a region of length L.

(a) DERIVE **[2]**

For a uniform static field $B = B\hat{z}$, derive from the Heisenberg equation of motion the equations of motion of S_x, S_y and S_z, written as one vector equation, and hence the angular frequency ω_L at which $\langle S \rangle$ precesses about B and the sense of the precession.

(b) CALCULATE **[2]**

In one arm of a neutron interferometer the neutrons cross a region 2.0 cm long in which a uniform field B is applied perpendicular to their path. Calculate the time each neutron spends in the field, and the field $B_{2\pi}$ for which its spin precesses through exactly one turn.

(c) PREDICT **[2]**

The two arms are recombined, and the count rate at one output port is recorded as B is raised from zero; at B = 0 this port is at an interference maximum. In the field the spin state is acted on by the rotation operator $\exp(-i\varphi \hat{n} \cdot \sigma/2)$, where φ is the precession angle and $\hat{n}$ the field direction. Predict the count rate at $B = B_{2\pi}$, and the smallest nonzero field at which the port returns to its B = 0 count rate.

(d) DESCRIBE **[2]**

A resonance spin flipper is a region with a static field B_0 along z and a field of magnitude $B_1 \ll B_0$ rotating in the xy-plane at angular frequency ω, in the same sense as the precession. Describe how transforming to the frame that rotates with B_1 reduces the problem to precession about a static effective field, giving that field's components and the flip probability that follows.

(e) CALCULATE **[2]**

The flipper is 5.0 cm long and $B_0 = 10.0$ mT. Calculate the frequency $\omega/2\pi$ at which B_1 must rotate, and the smallest value of B_1 that turns a neutron spin polarised along +z into one polarised along −z.

12 CORE COURSE

STRUCTURED · DEMANDING · UQMII-06.02 · UQMII-06.05 · UQMII-06.06 · UQMII-07.01 · UQMII-07.02 · UQMII-07.03 · UQMII-07.04 · UQMII-07.05 **[11]**

Two atoms of mass m are held in a narrow one-dimensional box trap, modelled as an infinite square well on $0 < x < L$. The single-particle states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2E_1$, where $E_1 = \pi^2\hbar^2/(2mL^2)$. The atoms interact weakly through a short-range repulsion $H' = g\delta(x_1 - x_2)$, with $g > 0$, which is treated as a perturbation of $H_0 = H_1 + H_2$. P_{12} exchanges every label of the two atoms, coordinates and internal states alike. You may use $\int_0^L \psi_1(x)^4\,dx = 3/(2L)$.

(a) EXPLAIN **[2]**

P_{12} commutes with $H_0 + H'$. Explain why this alone does not require the two-atom state to be symmetric or antisymmetric under exchange, and state what does.

(b) SHOW THAT **[2]**

Show that $\int_0^L \psi_1(x)^2\psi_2(x)^2\,dx = 1/L$.

(c) DETERMINE **[3]**

For two distinguishable atoms the unperturbed level $5E_1$ is twofold degenerate, spanned by $|12\rangle = \psi_1(x_1)\psi_2(x_2)$ and $|21\rangle = \psi_2(x_1)\psi_1(x_2)$. Determine the matrix of H' in this subspace, its first-order energy shifts and the corresponding good states.

(d) CALCULATE **[2]**

Take $g = 0.10E_1L$. Calculate, to first order in g and in units of E_1, the ground-state energy of two identical spinless bosons, and of two identical fermions in the same internal state.

(e) COMPARE **[2]**

Two spin-½ fermions, for example atoms in two internal states labelled ↑ and ↓, occupy ψ_1 and ψ_2 in the same trap, H' does not act on spin, and $g = 0.10E_1L$ as in (d). Compare the first-order energies of the states with total spin $S = 0$ and $S = 1$, and account for the difference.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UQMII-04.07 · UQMII-04.08 · UQMII-05.04 **[10]**

A source emits pairs of spin-½ particles intended to be in the singlet state. Alice measures the spin component of one particle along a direction at angle a in the xz-plane and Bob that of the other along angle b, each recording +1 or −1. For each setting the correlation E is the mean of the product of the two results; for the ideal singlet E(a, b) = −cos(a − b). Every local hidden-variable model obeys |S| ≤ 2, where S = E(a, b) − E(a, b′) + E(a′, b) + E(a′, b′). The run used a = 0°, a′ = 90°, b = 45° and b′ = 135°, and the table gives the number of detected pairs with each combination of results, Alice's sign first. Treat each detected pair as an independent trial.

Coincidence counts at the four analyser settings

Settings	N_{++} (counts)	N_{+-} (counts)	N_{-+} (counts)	N_{--} (counts)
a = 0°, b = 45°	43	215	205	50
a = 0°, b′ = 135°	192	48	46	196
a′ = 90°, b = 45°	52	214	196	58
a′ = 90°, b′ = 135°	48	196	189	49

(a) CALCULATE **[2]**

Calculate E for each of the four settings.

(b) DETERMINE **[2]**

Determine S, and the fraction of the ideal-singlet value that it reaches.

(c) ESTIMATE **[2]**

Estimate the standard uncertainty in S from counting statistics, stating the assumptions you make.

(d) EVALUATE **[2]**

Evaluate whether these data rule out every local hidden-variable model, and say what else a conclusive test must secure.

(e) JUSTIFY **[2]**

Quantum mechanics predicts that Alice's results alone cannot reveal which setting Bob chose. Justify this from the singlet's reduced density operator, and test it against the data for Alice's setting a.

14 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMII-05.07 · UQMII-06.02 · UQMII-08.02 · UQMII-08.03 · UQMII-08.04 · UQMII-08.05 · UQMII-08.06 · UQMII-08.08 **[15]**

A hydrogen-like ion has one electron of mass m bound to a point nucleus of charge Ze, treated as infinitely heavy, so that $V(r) = -Ze^2/(4\pi\varepsilon_0 r)$ and the unperturbed energies $E_n = -(Z\alpha)^2 mc^2/(2n^2)$ are degenerate in l, m_l and m_s. With $a = a_0/Z$, so that $\hbar^2/(ma) = Ze^2/(4\pi\varepsilon_0)$ and $E_n = -Ze^2/(8\pi\varepsilon_0 a n^2)$, the unperturbed states give $\langle 1/r\rangle = 1/(n^2 a)$, $\langle 1/r^2\rangle = 1/[n^3(l + \tfrac{1}{2})a^2]$, $\langle 1/r^3\rangle = 1/[n^3 l(l + \tfrac{1}{2})(l + 1)a^3]$ for $l \geq 1$, and $|\psi_{n00}(0)|^2 = 1/(\pi n^3 a^3)$. The fine-structure Hamiltonian is $H_{rel} + H_{SO} + H_D$, with $H_{rel} = -p^4/(8m^3c^2)$; $H_{SO} = [1/(2m^2c^2)](1/r)(dV/dr)\, \mathbf{L}\cdot\mathbf{S}$, which already contains the Thomas factor of ½, quoted from relativistic kinematics; and the Darwin term $H_D = [\pi\hbar^2/(2m^2c^2)][Ze^2/(4\pi\varepsilon_0)]\delta^3(r)$, quoted from the Dirac equation. Take $mc^2 = 0.5110$ MeV, $\alpha = 1/137.04$ and $h = 4.136 \times 10^{-15}$ eV s.

(a) JUSTIFY **[2]**

Each level n is degenerate, yet each fine-structure term can be evaluated with the non-degenerate formula $E^{(1)} = \langle H'\rangle$ in the coupled basis $|n, l, j, m_j\rangle$. Justify this, naming the operators each term commutes with.

(b) SHOW THAT **[3]**

Starting from $p^2/(2m) = H_0 - V$, show that the relativistic shift is $E_{rel} = -[E_n^2/(2mc^2)][4n/(l + \tfrac{1}{2}) - 3]$.

(c) DERIVE **[3]**

Derive the first-order spin–orbit shift E_{SO} of a state with $l \geq 1$, expressed in terms of E_n, mc^2, n, l and j.

(d) SHOW THAT **[3]**

Show that, once the Darwin term is included for $l = 0$, the three terms sum to $E_{fs} = [E_n^2/(2mc^2)][3 - 4n/(j + \tfrac{1}{2})]$ for every state, so that the energy depends on n and j only.

(e) SKETCH **[2]**

For He^+ ($Z = 2$), sketch the fine structure of the $n = 2$ level: draw the unperturbed level and the fine-structure levels labelled $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$, marking each shift in μeV and the $2P_{3/2}$–$2P_{1/2}$ splitting in GHz.

(f) EVALUATE **[2]**

The measured $2S_{1/2}$–$2P_{1/2}$ Lamb shifts are 1.058 GHz in hydrogen and 14.04 GHz in He^+. Welton's argument gives, for an nS level, $\Delta E \propto \alpha(Z\alpha)^4 mc^2 \ln[1/(Z\alpha)]/n^3$, with a prefactor, and an argument of the logarithm, that it fixes only roughly. Evaluate how well this scaling accounts for the ratio of the two shifts.

MARK SCHEME

University Quantum Mechanics II · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-01.01 · UQMII-01.02 **[1]**

Separating the angular equation for m = 0 and writing x = cos θ gives Legendre's equation, $(1 - x^2)y'' - 2xy' + \lambda y = 0$. Substituting $y = \Sigma\, a_k x^k$ gives the recursion $a_{k+2} = [k(k + 1) - \lambda]a_k/[(k + 1)(k + 2)]$, so the even and the odd coefficients form two independent series. Take λ = 6. Which statement is correct?

KEY **C** **The even series terminates as $1 - 3x^2$; the odd series does not, and because its coefficients fall off only as 1/k it diverges logarithmically at x = ±1, so its coefficient must be set to zero.**

A stops the odd recursion one step too early. $a_3 = (2 - 6)a_1/6 = -(2/3)a_1$ is nonzero, and so is the next coefficient, $a_5 = (12 - 6)a_3/20$; the factor $k(k + 1) - 6$ vanishes only at k = 2, an even index, so only the even series is cut off. B is right that both series converge inside the interval, but the poles θ = 0 and θ = π, where x = ±1, are part of the sphere and an angular function must be finite there. It is regularity at the poles that quantises λ; the φ equation quantises m. D invents a rescue: after $a_3 = -(2/3)a_1$ the odd coefficients are $-0.200a_1$, $-0.114a_1$ and $-0.079a_1$, all of one sign, falling like 1/k as the coefficients of $\tfrac{1}{2}\ln[(1 + x)/(1 - x)]$ do, so the series grows without bound at the poles. C is correct: at large k the ratio $a_{k+2}/a_k \approx 1 - 2/k$, the terminated even series is $P_2 \propto 3x^2 - 1$, and $\lambda = 6 = l(l + 1)$ with l = 2.

U2

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2 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-02.08 **[1]**

For hydrogen, with a_0 the Bohr radius, compare the 3s state (n = 3, l = 0) with a 3d state (n = 3, l = 2). Which expectation value is the same for the two states?

KEY **A** **$\langle 1/r \rangle$, equal to $1/(9a_0)$ in both, because the virial theorem fixes $\langle V \rangle = 2E_3$**

B carries the Bohr model over: n^2a_0 is neither state's mean radius. Kramers' relation gives $\langle r \rangle = (a_0/2)[3n^2 - l(l + 1)]$, which is $13.5a_0$ for 3s and $10.5a_0$ for 3d; the 3d electron, with no radial node, stays closer to a circular-orbit radius. C is the reverse of the truth: $\langle 1/r^2 \rangle = 1/[n^3(l + \tfrac{1}{2})a_0^2]$ is $0.0741/a_0^2$ for 3s and $0.0148/a_0^2$ for 3d, five times smaller, and that dependence on l is exactly why the relativistic correction separates levels of different l. D starts from a true statement, $\langle T \rangle = -E_3$ for every state of the level, and draws a false conclusion: T fixes an average of p^2, not of r^2, and $\langle r^2 \rangle = (n^2a_0^2/2)[5n^2 + 1 - 3l(l + 1)]$ is $207a_0^2$ for 3s and $126a_0^2$ for 3d. A is correct: for the Coulomb potential the virial theorem gives $\langle V \rangle = 2E_n$, which depends on n alone, so $\langle 1/r \rangle = 1/(n^2a_0)$ for every l.

U1

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-03.06 · UQMII-03.07 **[1]**

A particle of charge q and mass m moves in the potentials φ and A with Hamiltonian $\hat{H} = (\hat{p} - qA)^2/(2m) + q\varphi$, where $\hat{p} = -i\hbar\nabla$. Under the gauge transformation $A \to A + \nabla\Lambda$, $\varphi \to \varphi - \partial\Lambda/\partial t$, with Λ(r, t) any real function, the Schrödinger equation keeps its form provided $\psi \to e^{iq\Lambda/\hbar}\psi$. Which expectation value is the same in every gauge, whatever Λ is?

KEY **D** **$\langle \hat{p} - qA \rangle$, which equals m times the rate of change of $\langle r \rangle$**

A confuses the operator with its expectation value. The operator $-i\hbar\nabla$ is indeed the same, but the state is not: $e^{-iq\Lambda/\hbar}\hat{p}e^{iq\Lambda/\hbar} = \hat{p} + q\nabla\Lambda$, so $\langle \hat{p} \rangle$ moves by $q\langle\nabla\Lambda\rangle$, and canonical momentum is not an observable once a vector potential is present. B fails whenever Λ depends on time: the new Hamiltonian contains $q\varphi - q\partial\Lambda/\partial t$, so $\langle \hat{H} \rangle$ shifts by $-q\langle\partial\Lambda/\partial t\rangle$ although every measurable prediction is unchanged; it survives only for a time-independent Λ. C reads a statement about $|\psi|^2$ as a statement about ψ: the modulus is unchanged, but the phase at r_0 moves by $q\Lambda(r_0, t)/\hbar$, which is why the phase of ψ at one point carries no physical meaning. D is correct: $e^{-iq\Lambda/\hbar}(\hat{p} - qA - q\nabla\Lambda)e^{iq\Lambda/\hbar} = \hat{p} - qA$, so the mechanical momentum, which the Heisenberg equation identifies as $m\,d\hat{r}/dt$, is gauge-invariant.

U1

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-04.03 · UQMII-04.04 · UQMII-04.05 **[1]**

A source emits spin-½ particles, each prepared either in $|+z\rangle$ or in $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$, with probability ½ each, and the record of which state was prepared is discarded. What is the purity $\mathrm{Tr}(\rho^2)$ of the ensemble?

KEY **B** **0.75**

A treats any equal mixture of two different states as maximally mixed. That holds only when the two states are orthogonal, so that their Bloch vectors cancel; here $|\langle +z|+x\rangle|^2 = ½$, and the Bloch vector is $r = ½(0, 0, 1) + ½(1, 0, 0) = (½, 0, ½)$. C is the larger eigenvalue of ρ, $(1 + |r|)/2 = (1 + 0.707)/2 = 0.854$: the largest probability $\langle\psi|\rho|\psi\rangle$ with which any one pure state, here spin up along the bisector of the z- and x-axes, is found in the ensemble, not its purity. D is what the superposition $(|+z\rangle + |+x\rangle)/\sqrt{(2 + \sqrt{2})}$ would give: a pure state pointing along that bisector. Mixing, unlike superposing, puts the Bloch vector inside the ball, here at length 0.707. B is correct: $\mathrm{Tr}(\rho^2) = (1 + |r|^2)/2 = (1 + ½)/2 = 0.75$, which equals the sum of the squared eigenvalues, $0.854^2 + 0.146^2$.

U3

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-05.06 · UQMII-05.07 **[1]**

An electron in a 2p orbital of hydrogen is in the coupled state with $l = 1$, $s = ½$, $j = ½$ and $m_j = ½$. The stretched state is $|j = 3/2, m_j = 3/2\rangle = |m_l = 1\rangle|\uparrow\rangle$, and $J_- = L_- + S_-$, with $L_-|l, m\rangle = \hbar\sqrt{(l(l + 1) - m(m - 1))}\,|l, m - 1\rangle$ and $S_-|\uparrow\rangle = \hbar|\downarrow\rangle$. What is the probability that a measurement of S_z on the $j = ½$ state gives $+\hbar/2$?

KEY **C** **1/3**

B is the weight of $|m_l = 0\rangle|\uparrow\rangle$ in the other state of the pair. Lowering the stretched state gives $|3/2, ½\rangle = \sqrt{(2/3)}|0\rangle|\uparrow\rangle + \sqrt{(1/3)}|1\rangle|\downarrow\rangle$, because L_- acting on $m_l = 1$ brings a factor $\sqrt{2}$ while S_- brings 1; a candidate who stops there has found the $j = 3/2$ state, not the $j = ½$ one. A assumes that the two uncoupled states contribute equally, which nothing in the algebra supports. D assumes that $m_j = ½$ can only mean $m_l = 0$ with $m_s = +½$, ignoring the $|m_l = 1\rangle|\downarrow\rangle$ component, which has the same $m_l + m_s$. C is correct: $|½, ½\rangle$ must be orthogonal to $|3/2, ½\rangle$ inside the two-dimensional $m_j = ½$ subspace, so $|½, ½\rangle = \sqrt{(2/3)}|1\rangle|\downarrow\rangle - \sqrt{(1/3)}|0\rangle|\uparrow\rangle$ in the Condon–Shortley phase, and $P(S_z = +\hbar/2) = 1/3$.

U2

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6 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-06.05 · UQMII-06.06 · UQMII-06.07 **[1]**

A particle of mass m moves in the isotropic two-dimensional oscillator $H_0 = (p_x^2 + p_y^2)/(2m) + ½m\omega^2(x^2 + y^2)$. Its second excited level, at $3\hbar\omega$, is spanned by $|2, 0\rangle$, $|1, 1\rangle$ and $|0, 2\rangle$, the labels counting quanta along x and along y. Here $x = \sqrt{(\hbar/(2m\omega))}(a_x + a_x\dagger)$, with y written likewise, $a\dagger|n\rangle = \sqrt{(n + 1)}|n + 1\rangle$ and $a|n\rangle = \sqrt{n}|n - 1\rangle$. A weak perturbation $H' = \varepsilon xy$ is added. What are the first-order shifts of this level, and which good state is left unshifted?

KEY **D** **$+\varepsilon\hbar/(m\omega)$, 0 and $-\varepsilon\hbar/(m\omega)$, with $(|2, 0\rangle - |0, 2\rangle)/\sqrt{2}$ left unshifted**

A applies the non-degenerate formula to a degenerate level. The diagonal elements do vanish, because xy changes the number of quanta along each axis by one, but $\langle 2, 0|H'|1, 1\rangle = \varepsilon\langle 2|x|1\rangle\langle 0|y|1\rangle$ and $\langle 1, 1|H'|0, 2\rangle = \varepsilon\langle 1|x|0\rangle\langle 1|y|2\rangle$ do not, and inside a degenerate subspace it is the whole matrix that must be diagonalised. C has the unshifted state right but the outer shifts too small by $\sqrt{2}$: it comes from dropping the $\sqrt{2}$ in $\langle 2|x|1\rangle = \sqrt{2}\,\sqrt{(\hbar/(2m\omega))}$, or from diagonalising only the block of $|2, 0\rangle$ and $|1, 1\rangle$ and so missing the coupling of $|1, 1\rangle$ to $|0, 2\rangle$. B has the right shifts but gives the middle eigenvalue to the middle label: $|1, 1\rangle$ is coupled to both other states and is not a good state at all. The exchange $x \leftrightarrow y$ commutes with H_0 and with xy, and $(|2, 0\rangle - |0, 2\rangle)/\sqrt{2}$ is the only state of the level that is odd under it, so it cannot mix with the other two; its shift is zero because H' has no diagonal elements and no element between $|2, 0\rangle$ and $|0, 2\rangle$. D is correct: with $\delta = \varepsilon\hbar/(2m\omega)$ the matrix in the order $|2, 0\rangle$, $|1, 1\rangle$, $|0, 2\rangle$ is $\sqrt{2}\delta[[0, 1, 0], [1, 0, 1], [0, 1, 0]]$, with eigenvalues $+2\delta$, 0 and -2δ, the outer ones belonging to $(|2, 0\rangle \pm \sqrt{2}|1, 1\rangle + |0, 2\rangle)/2$. There is an exact check: in $u = (x + y)/\sqrt{2}$ and $v = (x - y)/\sqrt{2}$, $xy = (u^2 - v^2)/2$, so to first order the u frequency rises and the v frequency falls by $\varepsilon/(2m\omega)$, and the states with two u quanta, one of each, and two v quanta shift by $+2\delta$, 0 and -2δ.

U2

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-07.09 **[1]**

The ground configuration of oxygen is $1s^2 2s^2 2p^4$. The $2p^4$ shell has the same terms as $2p^2$, namely ^1S, ^1D and ^3P. According to Hund's rules, which level is the ground level of oxygen?

KEY **A** **$^3P_2$**

B is the ground level of carbon, $2p^2$, and carrying the less-than-half-filled rule over to oxygen is the error to catch. Four electrons fill more than half of the six p spin-orbitals, the shell then behaves as two holes whose spin-orbit coupling has the opposite sign, and the largest J lies lowest. C puts the rule for L ahead of the rule for S: ^1D has the larger L, but Hund's first rule, maximum S, comes first, because the antisymmetric spatial state of a triplet keeps the electrons apart and lowers their repulsion; in oxygen ^1D lies about 2 eV above the ground term. D treats $2p^4$ as a closed shell because its spins can be paired, but only a full p^6 shell is obliged to be $^1S_0$. A is correct: the term is ^3P, and $J = L + S = 2$ because the shell is more than half full.

U1

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-08.01 · UQMII-08.08 **[1]**

Three small splittings are measured in the n = 2 level of hydrogen: the fine-structure splitting 2P_(3/2)–2P_(1/2), the Lamb shift 2S_(1/2)–2P_(1/2), and the hyperfine splitting of 2S_(1/2). Using mc^2 = 511 keV, $\alpha = 1/137$, $m_e/m_p = 1/1836$, and the powers of α and of m_e/m_p that set the size of each effect, which ordering, largest first, is correct?

KEY **B** **fine structure, then Lamb shift, then hyperfine splitting**

A counts powers of α and nothing else. Fine and hyperfine structure both scale as $\alpha^4 mc^2$, but the hyperfine interaction couples the electron to the nuclear magnetic moment and so carries a further factor of about $g_p\, m_e/m_p \approx 3 \times 10^{-3}$, whereas the Lamb shift sits only one power of α, softened by a logarithm, below the fine structure: $\alpha^5 mc^2 \ln(1/\alpha)/n^3 \approx 6$ μeV against $\alpha^4 mc^2/n^3 \approx 0.2$ meV. C mistakes the most famous hyperfine line, the 21 cm transition of 1s hydrogen, for a large one; at 5.9 μeV it is small, and the 2S splitting is eight times smaller again. D takes the Lamb shift's historical importance for size: it was hard to find precisely because it is ten times smaller than the fine-structure splitting it sits beside. B is correct, and the measured values bear out the estimate: 45 μeV (10.9 GHz), 4.4 μeV (1.06 GHz) and 0.73 μeV (178 MHz).

U4

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9 CORE COURSE

STRUCTURED · ROUTINE · UQMII-01.05 · UQMII-01.06 · UQMII-01.07 · UQMII-01.08 **[9]**

On the unit sphere the angular-momentum operators act as $L_z = -i\hbar\,\partial/\partial\varphi$ and $L_\pm = \pm\hbar e^{(\pm i\varphi)}(\partial/\partial\theta \pm i\cot\theta\,\partial/\partial\varphi)$, and $L_\pm Y_l^m = \hbar\sqrt{(l(l+1) - m(m\pm1))}\,Y_l^{(m\pm1)}$. The Condon–Shortley convention is used throughout, in which $Y_1^1 = -\sqrt{(3/(8\pi))}\sin\theta\,e^{(i\varphi)}$.

(a) DERIVE · U2 **[2]**

Write $Y_2^2 = g(\theta)e^{(2i\varphi)}$. Starting from the condition $L_+Y_2^2 = 0$, derive $g(\theta)$ up to a constant factor.

1 L_+ acting on $g(\theta)e^{(2i\varphi)}$ gives $\hbar e^{(3i\varphi)}[g' - 2\cot\theta\,g]$, since $\partial/\partial\varphi$ brings down $2i$ and $i \times 2i = -2$, so $L_+Y_2^2 = 0$ requires $dg/d\theta = 2\cot\theta\,g$

2 integrating, $\ln g = 2\ln\sin\theta + \text{constant}$, so $g = C\sin^2\theta$, which is finite at both poles as a spherical harmonic must be

The sign of the $i\cot\theta\,\partial/\partial\varphi$ term acting on $e^{(2i\varphi)}$ is where answers go wrong: $g' = -2\cot\theta\,g$ leads to $g \propto 1/\sin^2\theta$, which diverges at the poles. That divergence is the signal to recheck, because regularity is what the ladder route relies on.

(b) SHOW THAT · U2 **[2]**

Show that normalising Y_2^2 over the sphere, with C real and positive, gives $C = (1/4)\sqrt{(15/(2\pi))}$.

1 $\int|Y_2^2|^2\,d\Omega = C^2\int_0^{(2\pi)}d\varphi\int_0^{\pi}\sin^5\theta\,d\theta$, and with $u = \cos\theta$, $\int_0^{\pi}\sin^5\theta\,d\theta = \int_{-1}^{1}(1-u^2)^2\,du = 2 - 4/3 + 2/5 = 16/15$

2 setting $2\pi C^2 \times 16/15 = 1$ gives $C^2 = 15/(32\pi)$, so $C = \sqrt{(15/(32\pi))} = (1/4)\sqrt{(15/(2\pi))} = 0.386$

Dropping the $\sin\theta$ of the solid-angle element $d\Omega = \sin\theta\,d\theta\,d\varphi$, and integrating $\sin^4\theta$ instead, gives $3\pi/8$ in place of $16/15$; the answer is printed so that this slip shows.

(c) DETERMINE · U2 **[2]**

Determine Y_2^1 by applying L_- to Y_2^2, keeping the sign that the operator produces.

1 $L_-(\sin^2\theta\,e^{(2i\varphi)}) = -\hbar e^{(-i\varphi)}[2\sin\theta\cos\theta - i\cot\theta\,(2i)\sin^2\theta]e^{(2i\varphi)} = -4\hbar\sin\theta\cos\theta\,e^{(i\varphi)}$

2 dividing by $\hbar\sqrt{(6-2)} = 2\hbar$ and multiplying by C: $Y_2^1 = -(1/2)\sqrt{(15/(2\pi))}\sin\theta\cos\theta\,e^{(i\varphi)} = -\sqrt{(15/(8\pi))}\sin\theta\cos\theta\,e^{(i\varphi)}$, the minus sign being the Condon–Shortley phase that the lowering generates

Candidates who drop the overall minus sign of L_-, or who normalise Y_2^1 afresh and choose it positive, lose the phase that makes the Y_l^m a consistent set; every later Clebsch–Gordan coefficient or dipole matrix element inherits the error.

(d) STATE · U1 **[1]**

State the parity of Y_2^1 and the number of values of θ strictly between 0 and π at which it vanishes.

1 even parity, $(-1)^l = +1$, since under $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$ both $\sin\theta\cos\theta$ and $e^{(i\varphi)}$ change sign; and one value, $\theta = \pi/2$, where $\cos\theta = 0$, a nodal plane at the equator, as $l - |m| = 1$ requires

(e) EXPLAIN · U1 **[2]**

In general the recursion for $\cos\theta\,Y_l^m$ gives a combination of Y_{l+1}^m and Y_{l-1}^m. Explain why $\cos\theta\,Y_1^1$ is a multiple of Y_2^1 alone, find the multiple, and say what this implies for a z-polarised electric-dipole transition from an $l = 1$, $m = 1$ state.

1 $\cos\theta\,Y_1^1 = -\sqrt{(3/(8\pi))}\sin\theta\cos\theta\,e^{(i\varphi)} = (1/\sqrt{5})Y_2^1$, using $Y_2^1 = -\sqrt{(15/(8\pi))}\sin\theta\cos\theta\,e^{(i\varphi)}$; the $l - 1$ partner would be Y_0 with $m = 1$, which does not exist because $|m|$ cannot exceed l

2 so $\langle l', m'|\cos\theta|1, 1\rangle$ is nonzero only for $l' = 2$, $m' = 1$: a z-polarised dipole transition from a p state with $m = 1$ can reach d states but no s state, although $\Delta l = -1$ is allowed in general

Candidates who quote the selection rule $\Delta l = \pm1$ conclude that p to s is always open. The recursion makes the rule a list of what may occur, and m decides which of its two terms survives.

10 CORE COURSE

STRUCTURED · DEMANDING · UQMII-02.01 · UQMII-02.02 · UQMII-02.03 · UQMII-02.04 · UQMII-02.05 · UQMII-02.08 **[11]**

In a crystal of cuprous oxide, an electron in the conduction band and a hole in the valence band bind into a hydrogen-like exciton. Model it as two particles of charges $-e$ and $+e$ with effective masses $0.99m_e$ (electron) and $0.58m_e$ (hole), interacting through the Coulomb potential screened by the relative permittivity $\varepsilon_r = 7.5$, $V(r) = -e^2/(4\pi\varepsilon_0\varepsilon_r r)$. Once the centre-of-mass motion is separated, the relative motion is that of one particle of reduced mass μ, whose radial function $u(r) = rR(r)$ obeys $-(\hbar^2/2\mu)u'' + [V(r) + \hbar^2 l(l+1)/(2\mu r^2)]u = Eu$. For a bound state write $\kappa = \sqrt{(-2\mu E)}/\hbar$, $\rho = \kappa r$, $a^* = 4\pi\varepsilon_0\varepsilon_r\hbar^2/(\mu e^2)$ and $\rho_0 = 2/(\kappa a^*)$. Substituting $u = \rho^{l+1}e^{-\rho}v(\rho)$ gives $\rho v'' + 2(l+1-\rho)v' + [\rho_0 - 2(l+1)]v = 0$. For hydrogen with an infinitely heavy nucleus, $\hbar^2/(2m_e a_0^2) = 13.606$ eV and $a_0 = 0.05292$ nm.

(a) DERIVE · U2 **[2]**

Substituting $v(\rho) = \sum_j c_j\rho^j$, derive the recursion relation between c_{j+1} and c_j.

1. with every sum re-indexed to carry ρ^j: $\rho v''$ gives $\sum j(j+1)c_{j+1}\rho^j$, $2(l+1)v'$ gives $\sum 2(l+1)(j+1)c_{j+1}\rho^j$, and $-2\rho v'$ gives $\sum(-2j)c_j\rho^j$
2. equating coefficients of ρ^j: $(j+1)(j+2l+2)c_{j+1} = [2(j+l+1) - \rho_0]c_j$, so $c_{j+1} = [2(j+l+1) - \rho_0]c_j/[(j+1)(j+2l+2)]$

The index shift is where marks are lost: $\rho v''$ and v' each lower the power of ρ by one, while $-2\rho v'$ does not. Candidates who equate coefficients without re-indexing produce a relation that mixes three consecutive coefficients.

(b) EXPLAIN · U1 **[2]**

The series solves the differential equation for any value of E. Explain why a physical solution must nevertheless have a terminating series, and why this condition fixes the energy.

1. for large j, $c_{j+1}/c_j \to 2/j$, the ratio for $e^{2\rho} = \Sigma(2\rho)^j/j!$, so an unterminated v grows like $e^{2\rho}$ times a power of ρ, and $u = \rho^{l+1}e^{-\rho}v$ grows like e^{ρ} times a power of ρ: it diverges at large r and cannot be normalised
2. the series must therefore stop at some c_N, which needs $2(N+l+1) - \rho_0 = 0$ for an integer $N \geq 0$; $\rho_0 = 2/(\kappa a^*)$ depends on E through κ, so only discrete energies satisfy the condition

The weak answer says that an infinite series must diverge. It need not: the series converges for every ρ. What fails is normalisability, because $e^{2\rho}$ overwhelms the factor $e^{-\rho}$ already removed, and a candidate who cannot name the condition being imposed has not explained why the energy is quantised. The ratio test fixes the exponential but not the power of ρ in front of it; the often-written $u \sim \rho^{l+1}e^{\rho}$ has the wrong power, though the growth that matters is e^{ρ} either way.

(c) SHOW THAT · U2 **[2]**

Show that the bound-state energies are $E_n = -(\mu/m_e)(13.606\text{ eV})/(\varepsilon_r^2 n^2)$, with $n = N + l + 1$, where N is the degree of the terminated polynomial, and that $l \leq n-1$.

1. termination requires $\rho_0 = 2(N+l+1) \equiv 2n$, and since $N \geq 0$, $n \geq l+1$, that is $l \leq n-1$
2. $\kappa = 2/(\rho_0 a^*) = 1/(na^*)$, so $E = -\hbar^2\kappa^2/(2\mu) = -\hbar^2/(2\mu a^{*2}n^2)$; with $a^* = a_0\varepsilon_r(m_e/\mu)$ this is $-[\hbar^2/(2m_e a_0^2)](\mu/m_e)/(\varepsilon_r^2 n^2) = -(\mu/m_e)(13.606\text{ eV})/(\varepsilon_r^2 n^2)$

Writing a^* with μ/m_e instead of m_e/μ, or keeping a single factor of ε_r, gives an energy that scales wrongly. One factor of ε_r comes from the weaker attraction and the second from the larger orbit that results.

(d) CALCULATE · U3 **[3]**

Calculate μ as a multiple of m_e, the exciton Rydberg energy $R^* = -E_1$ in meV, and the effective Bohr radius a^* in nm.

1. $\mu = (0.99 \times 0.58/1.57)m_e = 0.366m_e$
2. $R^* = 0.366 \times 13.606\text{ eV}/7.5^2 = 0.0885\text{ eV} = 88.5\text{ meV}$
3. $a^* = 0.05292\text{ nm} \times 7.5/0.366 = 1.09\text{ nm}$

Using the electron's effective mass alone in place of μ overestimates R^* by a factor of 2.7. The two masses are comparable, so neither particle can be treated as fixed, as the proton nearly can in hydrogen.

(e) DETERMINE · U3 **[2]**

For these states Kramers' relation reads $((s+1)/n^2)\langle r^s\rangle - (2s+1)a^*\langle r^{s-1}\rangle + (s/4)[(2l+1)^2 - s^2]a^{*2}\langle r^{s-2}\rangle = 0$. Absorption lines of the exciton P series ($l = 1$) have been resolved up to $n = 25$. Determine $\langle r\rangle$ for the $n = 25$, $l = 1$ exciton.

1. $s = 0$ gives $\langle 1/r\rangle = 1/(n^2a^*)$; $s = 1$ then gives $(2/n^2)\langle r\rangle - 3a^* + l(l+1)a^*/n^2 = 0$, so $\langle r\rangle = (a^*/2)[3n^2 - l(l+1)]$
2. $\langle r\rangle = (1.09\ \text{nm}/2)(3 \times 625 - 2) = 1.02 \times 10^3$ nm, about 1.0 μm

   The error to catch is using the $s = 1$ relation without first finding $\langle 1/r\rangle$ from $s = 0$. Candidates who take n^2a^* as the size get 0.68 μm; the P-state mean radius is half as large again, because $\langle r\rangle \approx (3/2)n^2a^*$ when $l \ll n$.

11

CORE COURSE

STRUCTURED · DEMANDING · UQMII-03.01 · UQMII-03.02 · UQMII-03.03 · UQMII-03.04 · UQMII-05.02 **[10]**

The neutron is a spin-½ particle whose magnetic moment is $\mu = \gamma S$, with $\gamma = -1.832 \times 10^8$ rad s^{-1} T^{-1}, so the moment points opposite to the spin. In a magnetic field B its spin Hamiltonian is $H = -\mu\cdot B = -\gamma B\cdot S$, and the spin operators obey $[S_x, S_y] = i\hbar S_z$ and its cyclic permutations. In every part the neutrons travel at 2200 m/s, and their motion through the apparatus is treated classically, so that a neutron spends a time L/v in a region of length L.

(a) DERIVE · U2 **[2]**

For a uniform static field $B = B\hat{z}$, derive from the Heisenberg equation of motion the equations of motion of S_x, S_y and S_z, written as one vector equation, and hence the angular frequency ω_L at which $\langle S\rangle$ precesses about B and the sense of the precession.

1. $dS_x/dt = (i/\hbar)[H, S_x] = (i/\hbar)(-\gamma B)[S_z, S_x] = (i/\hbar)(-\gamma B)(i\hbar S_y) = \gamma B S_y$; likewise $dS_y/dt = -\gamma B S_x$ and $dS_z/dt = 0$, which together are $dS/dt = \gamma S \times B$
2. taking expectation values, $\langle S_z\rangle$ is constant while $\langle S_x\rangle$ and $\langle S_y\rangle$ rotate about B at $\omega_L = |\gamma|B$; because $\gamma < 0$, $dS/dt = |\gamma|B \times S$, a right-handed rotation about B

   A sign slip in $[S_z, S_x] = i\hbar S_y$ reverses the sense of precession while leaving $|\gamma|B$ intact, which is why the part asks for the vector equation rather than the frequency alone.

(b) CALCULATE · U3 **[2]**

In one arm of a neutron interferometer the neutrons cross a region 2.0 cm long in which a uniform field B is applied perpendicular to their path. Calculate the time each neutron spends in the field, and the field $B_{2\pi}$ for which its spin precesses through exactly one turn.

1. $t = 0.020\ \text{m}/(2200\ \text{m s}^{-1}) = 9.09 \times 10^{-6}$ s
2. $B_{2\pi} = 2\pi/(|\gamma|t) = 2\pi/(1.832 \times 10^8\ \text{rad s}^{-1}\ \text{T}^{-1} \times 9.09 \times 10^{-6}\ \text{s}) = 3.77 \times 10^{-3}$ T (3.77 mT)

   Treating γ as a frequency in hertz per tesla, or dropping the 2π, gives a field wrong by a factor of 2π; with γ in rad s^{-1} T^{-1} the precession angle is $|\gamma|Bt$ in radians.

(c) PREDICT · U1 **[2]**

The two arms are recombined, and the count rate at one output port is recorded as B is raised from zero; at $B = 0$ this port is at an interference maximum. In the field the spin state is acted on by the rotation operator $\exp(-i\varphi\hat{n}\cdot\sigma/2)$, where φ is the precession angle and $\hat{n}$ the field direction. Predict the count rate at $B = B_{2\pi}$, and the smallest nonzero field at which the port returns to its $B = 0$ count rate.

1. the recombined intensity is proportional to $|\chi + e\text{^}(-i\varphi\hat{n}\cdot\sigma/2)\chi|^2 = 2 + 2\cos(\varphi/2)$ for any spin state χ, since $\langle\chi|\hat{n}\cdot\sigma|\chi\rangle$ is real and contributes only to the imaginary part of the overlap; at $\varphi = 2\pi$ the operator is $-I$, so the port is at an interference minimum, ideally dark, although every spin points as it did at $B = 0$
2. the count rate first returns to its $B = 0$ value at $\varphi = 4\pi$, that is $B = 2B_{2\pi} = 7.55$ mT

   The expected answer is that a spin turned through a full circle is unchanged, so the count rate recovers at 3.77 mT. The prediction must rest on the rotation operator for spin ½, which is $-I$ at 2π; the sign is invisible in a single beam and becomes observable only against the unrotated arm.

(d) DESCRIBE · U2 **[2]**

A resonance spin flipper is a region with a static field B_0 along z and a field of magnitude $B_1 \ll B_0$ rotating in the xy-plane at angular frequency ω, in the same sense as the precession. Describe how transforming to the frame that rotates with B_1 reduces the problem to precession about a static effective field, giving that field's components and the flip probability that follows.

1. in the frame rotating at ω about z, B_1 is static along an axis x′, and the transformation adds $-\omega S_z$ to the Hamiltonian, which acts as a fictitious field $\omega/|\gamma|$ opposing B_0: $B_{eff} = (B_0 - \omega/|\gamma|)\hat{z} + B_1\hat{x}'$
2. the spin precesses about B_{eff} at $\Omega = \sqrt{(\Delta^2 + \omega_1^2)}$, with $\omega_0 = |\gamma|B_0$, $\omega_1 = |\gamma|B_1$ and $\Delta = \omega - \omega_0$, so a spin starting along z flips with probability $P = (\omega_1^2/\Omega^2)\sin^2(\Omega t/2)$; on resonance $B_{eff} = B_1\hat{x}'$ and $P = \sin^2(\omega_1 t/2)$

   Candidates who stay in the laboratory frame face a time-dependent Hamiltonian they cannot integrate. Going to the frame that turns with B_1 makes the field static whichever way B_1 rotates; the sense decides only the sign of the fictitious field, which cancels B_0 at $\omega = \omega_0$ when B_1 turns with the precession. A field turning the other way sees $B_0 + \omega/|\gamma|$ and never comes to resonance, which is why the counter-rotating half of an oscillating field is what the rotating-wave approximation discards.

(e) CALCULATE · U3 **[2]**

The flipper is 5.0 cm long and $B_0 = 10.0$ mT. Calculate the frequency $\omega/2\pi$ at which B_1 must rotate, and the smallest value of B_1 that turns a neutron spin polarised along +z into one polarised along −z.

1. on resonance, $\omega/2\pi = |\gamma|B_0/2\pi = 1.832 \times 10^8$ rad s^{-1} T^{-1} × 0.0100 T/2π = 2.92×10^5 Hz (292 kHz)
2. $t = 0.050$ m/(2200 m s^{-1}) = 2.27×10^{-5} s, and a flip needs $\omega_1 t = \pi$, so $B_1 = \pi/(|\gamma|t) = 7.55 \times 10^{-4}$ T (0.755 mT)

   Setting $\omega_1 t = 2\pi$, a full Rabi cycle, returns the spin to where it started; a flip needs half a cycle. For a truly rotating B_1 the rotating-frame result is exact at any strength; the smallness of 0.755 mT against $B_0 = 10.0$ mT matters when the flipper is driven by an oscillating field instead, whose counter-rotating half is then far enough off resonance to neglect.

12

CORE COURSE

STRUCTURED · DEMANDING · UQMII-06.02 · UQMII-06.05 · UQMII-06.06 · UQMII-07.01 · UQMII-07.02 · UQMII-07.03 · UQMII-07.04 · UQMII-07.05 **[11]**

Two atoms of mass m are held in a narrow one-dimensional box trap, modelled as an infinite square well on $0 < x < L$. The single-particle states are $\psi_n(x) = \sqrt{(2/L)}\sin(n\pi x/L)$, with energies $E_n = n^2E_1$, where $E_1 = \pi^2\hbar^2/(2mL^2)$. The atoms interact weakly through a short-range repulsion $H' = g\delta(x_1 - x_2)$, with $g > 0$, which is treated as a perturbation of $H_0 = H_1 + H_2$. P_{12} exchanges every label of the two atoms, coordinates and internal states alike. You may use $\int_0^L \psi_1(x)^4\,dx = 3/(2L)$.

(a) EXPLAIN · U1 **[2]**

P_{12} commutes with $H_0 + H'$. Explain why this alone does not require the two-atom state to be symmetric or antisymmetric under exchange, and state what does.

1. $[P_{12}, H] = 0$ means only that exchange symmetry, if a state has it, is conserved in time; states with no exchange symmetry, such as $\psi_1(x_1)\psi_2(x_2)$, are perfectly good states of the two-particle space, as they are for distinguishable atoms
2. the symmetrisation postulate: identical particles must be in states symmetric under P_{12} (bosons, integer spin) or antisymmetric (fermions, half-integer spin), the link to spin being the spin-statistics connection, quoted from relativistic field theory rather than derived

   The error to catch is the argument that a symmetric Hamiltonian forces symmetric eigenstates. A commuting operator allows the eigenstates to be chosen symmetric or antisymmetric; only the postulate makes the choice compulsory, and only for identical particles.

(b) SHOW THAT · U2 **[2]**

Show that $\int_0^L \psi_1(x)^2\psi_2(x)^2\,dx = 1/L$.

1. with $u = \pi x/L$, $\sin^2 u \sin^2 2u = \tfrac{1}{4}(1-\cos 2u)(1-\cos 4u) = \tfrac{1}{4}[1 - \tfrac{1}{2}\cos 2u - \cos 4u + \tfrac{1}{2}\cos 6u]$, using $\cos 2u \cos 4u = \tfrac{1}{2}(\cos 2u + \cos 6u)$
2. each cosine integrates to zero over $0 < u < \pi$, leaving $(2/L)^2 \times \tfrac{1}{4} \times L = 1/L$

Replacing each $\sin^2$ by its average of ½ happens to give 1/L here, but the same shortcut gives 1/L for $\int\psi_1^4\,dx$, whose true value is 3/(2L); the marks are for the product expansion, which is what makes the result reliable.

(c) DETERMINE · U2 **[3]**

For two distinguishable atoms the unperturbed level $5E_1$ is twofold degenerate, spanned by $|12\rangle = \psi_1(x_1)\psi_2(x_2)$ and $|21\rangle = \psi_2(x_1)\psi_1(x_2)$. Determine the matrix of H' in this subspace, its first-order energy shifts and the corresponding good states.

1. diagonal (direct) elements: $\langle 12|H'|12\rangle = \langle 21|H'|21\rangle = g\int\psi_1(x)^2\psi_2(x)^2\,dx = g/L$, the delta function setting $x_1 = x_2 = x$
2. off-diagonal (exchange) element: $\langle 12|H'|21\rangle = g\int\psi_1\psi_2\psi_2\psi_1\,dx = g/L$, the same integral, because a contact interaction samples both orderings at the same point
3. the matrix $(g/L)[[1, 1], [1, 1]]$ has eigenvalues $2g/L$ and 0, with good states $(|12\rangle + |21\rangle)/\sqrt{2}$ and $(|12\rangle - |21\rangle)/\sqrt{2}$, the eigenstates of P_{12} with eigenvalues $+1$ and -1

Applying the non-degenerate formula gives both states the single shift g/L and misses the splitting. The equality of the exchange and direct integrals is special to a contact interaction; for a finite-range repulsion the exchange integral is smaller, but P_{12} still fixes the good states.

(d) CALCULATE · U3 **[2]**

Take $g = 0.10E_1L$. Calculate, to first order in g and in units of E_1, the ground-state energy of two identical spinless bosons, and of two identical fermions in the same internal state.

1. bosons: both atoms in ψ_1, so $E = 2E_1 + g\int\psi_1^4\,dx = 2E_1 + 3g/(2L) = 2.15E_1$
2. fermions: antisymmetry forbids both in ψ_1, so the lowest state is $(|12\rangle - |21\rangle)/\sqrt{2}$ at $5E_1$ with shift $J - K = 0$, giving $E = 5.00E_1$, which is exact to all orders because every antisymmetric spatial state vanishes at $x_1 = x_2$, where the delta function acts

Candidates give the fermions the direct shift g/L alone, forgetting the exchange term that cancels it. First order is safe for the bosons: the largest coupling of the ground state to another level, $\langle 11|H'|22\rangle = g/L = 0.10E_1$, is small against the gap of $6E_1$.

(e) COMPARE · U1 **[2]**

Two spin-½ fermions, for example atoms in two internal states labelled ↑ and ↓, occupy ψ_1 and ψ_2 in the same trap, H' does not act on spin, and $g = 0.10E_1L$ as in (d). Compare the first-order energies of the states with total spin $S = 0$ and $S = 1$, and account for the difference.

1. the whole state must be antisymmetric, so $S = 0$ (antisymmetric spin singlet) goes with the symmetric spatial state, at $5E_1 + 2g/L = 5.20E_1$, and $S = 1$ (symmetric spin triplet) goes with the antisymmetric spatial state and stays at $5.00E_1$: the triplet lies lower by $2g/L = 0.20E_1$
2. the splitting is an exchange correlation, not a spin-dependent force: the triplet's spatial wavefunction vanishes where the atoms meet, so the repulsion never acts, just as orthohelium lies below parahelium and as Hund's first rule favours maximum S

The error to catch is placing the singlet lower because paired spins seem to attract, or giving the two states one energy because H' ignores spin. The energy depends on spin only through the spatial symmetry that antisymmetry imposes.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UQMII-04.07 · UQMII-04.08 · UQMII-05.04 **[10]**

A source emits pairs of spin-½ particles intended to be in the singlet state. Alice measures the spin component of one particle along a direction at angle a in the xz-plane and Bob that of the other along angle b, each recording +1 or −1. For each setting the correlation E is the mean of the product of the two results; for the ideal singlet $E(a, b) = -\cos(a - b)$. Every local hidden-variable model obeys $|S| \le 2$, where $S = E(a, b) - E(a, b') + E(a', b) + E(a', b')$. The run used $a = 0°$, $a' = 90°$, $b = 45°$ and $b' = 135°$, and the table gives the number of detected pairs with each combination of results, Alice's sign first. Treat each detected pair as an independent trial.

(a) CALCULATE · U4 **[2]**

Calculate E for each of the four settings.

1. $E = (N_{++} + N_{--} - N_{+-} - N_{-+})/N$, with N the total for that setting: $E(a, b) = -327/513 = -0.637$ and $E(a, b') = +294/482 = +0.610$
2. $E(a', b) = -300/520 = -0.577$ and $E(a', b') = -288/482 = -0.598$

Dividing every row by one round total, such as 500, introduces errors of up to 4% that survive into S; each correlation must be divided by its own number of pairs, which here runs from 482 to 520.

(b) DETERMINE · U4 **[2]**

Determine S, and the fraction of the ideal-singlet value that it reaches.

1. $S = -0.637 - 0.610 - 0.577 - 0.598 = -2.42$, the minus sign in front of the positive $E(a, b')$ making all four terms negative
2. for the ideal singlet each $|E| = \cos 45° = 0.707$ and $S = -2\sqrt{2} = -2.83$, so the data reach $2.42/2.83 = 0.86$ of it, as for a singlet mixed with uncorrelated noise, $E = -V\cos(a - b)$ with visibility $V \approx 0.86$

Adding $E(a, b')$ instead of subtracting it gives $S = -1.20$ and an apparent agreement with local realism. The combination is fixed by the inequality, and the angles were chosen so that every term pushes S the same way.

(c) ESTIMATE · U4 **[2]**

Estimate the standard uncertainty in S from counting statistics, stating the assumptions you make.

1. assumes that each pair is an independent trial whose product ±1 has mean E and variance $1 - E^2$, and that the four settings are sampled independently, so $\sigma_E = \sqrt{((1 - E^2)/N)} \approx \sqrt{((1 - 0.37)/500)} \approx 0.036$ for each setting
2. $\sigma_S = \sqrt{(\Sigma \sigma_E^2)} \approx 2 \times 0.036 = 0.071$

Adding the four σ_E linearly gives 0.14 and halves the apparent significance; independent errors add in quadrature. Taking the spread of the four $|E|$ values as the uncertainty is no better, since four numbers fix a spread only roughly and any real difference between settings would be mistaken for noise.

(d) EVALUATE · U4 **[2]**

Evaluate whether these data rule out every local hidden-variable model, and say what else a conclusive test must secure.

1. $|S| - 2 = 0.42$, about 5.9σ with $\sigma_S = 0.071$: statistically decisive, far beyond what counting fluctuations about any $|S| \le 2$ could produce
2. but the conclusion also needs a large enough fraction of pairs detected, or a fair-sampling assumption (the detection loophole), setting choices made at random and at space-like separation from the other side's measurement (the locality loophole), and a check of systematic errors such as analyser-angle calibration, which counting statistics do not include

Strong answers separate statistical significance from the assumptions behind the inequality. A violation by many standard deviations shows that no local model reproduces these counts only if the detected pairs are a fair sample and neither side's setting could influence the other.

(e) JUSTIFY · U4 **[2]**

Quantum mechanics predicts that Alice's results alone cannot reveal which setting Bob chose. Justify this from the singlet's reduced density operator, and test it against the data for Alice's setting a.

1. tracing Bob's particle out of the singlet gives ρ_A = I/2, and Bob's measurement, averaged over his results, which Alice does not know, leaves ρ_A unchanged, so Alice obtains +1 with probability ½ at every a, whatever b is
2. from the data, P_A(+1) = (43 + 215)/513 = 0.503 with Bob at b and (192 + 48)/482 = 0.498 with Bob at b′; each has standard uncertainty √(0.25/N) ≈ 0.022, so both agree with ½ and with each other: no signal

Candidates look for the no-signalling test in the correlations E, which do depend on both settings. It lies in the marginal counts, which the table gives only implicitly, and the comparison means nothing without an uncertainty.

14

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMII-05.07 · UQMII-06.02 · UQMII-08.02 · UQMII-08.03 · UQMII-08.04 · UQMII-08.05 · UQMII-08.06 · UQMII-08.08 **[15]**

A hydrogen-like ion has one electron of mass m bound to a point nucleus of charge Ze, treated as infinitely heavy, so that $V(r) = -Ze^2/(4\pi\varepsilon_0 r)$ and the unperturbed energies $E_n = -(Z\alpha)^2 mc^2/(2n^2)$ are degenerate in l, m_l and m_s. With $a = a_0/Z$, so that $\hbar^2/(ma) = Ze^2/(4\pi\varepsilon_0)$ and $E_n = -Ze^2/(8\pi\varepsilon_0 a n^2)$, the unperturbed states give $\langle 1/r\rangle = 1/(n^2 a)$, $\langle 1/r^2\rangle = 1/[n^3(l+\tfrac{1}{2})a^2]$, $\langle 1/r^3\rangle = 1/[n^3 l(l+\tfrac{1}{2})(l+1)a^3]$ for $l \geq 1$, and $|\psi_{n00}(0)|^2 = 1/(\pi n^3 a^3)$. The fine-structure Hamiltonian is H_rel + H_SO + H_D, with $H_{rel} = -p^4/(8m^3c^2)$; $H_{SO} = [1/(2m^2c^2)](1/r)(dV/dr)\, L\cdot S$, which already contains the Thomas factor of ½, quoted from relativistic kinematics; and the Darwin term $H_D = [\pi\hbar^2/(2m^2c^2)][Ze^2/(4\pi\varepsilon_0)]\delta^3(r)$, quoted from the Dirac equation. Take $mc^2 = 0.5110$ MeV, $\alpha = 1/137.04$ and $h = 4.136 \times 10^{-15}$ eV s.

(a) JUSTIFY · U1 **[2]**

Each level n is degenerate, yet each fine-structure term can be evaluated with the non-degenerate formula $E^{(1)} = \langle H'\rangle$ in the coupled basis $|n, l, j, m_j\rangle$. Justify this, naming the operators each term commutes with.

1. H_rel and H_D are spin-independent scalars: they commute with L^2, L_z, S^2 and S_z, so they have no matrix elements between states of different l, m_l or m_s, and are diagonal in the uncoupled basis and equally in the coupled one
2. $H_{SO} \propto L\cdot S = (J^2 - L^2 - S^2)/2$ commutes with J^2, J_z, L^2 and S^2 but not with L_z or S_z, so it is diagonal only in the coupled basis; there all three terms are diagonal, the degenerate states are already the good states, and $E^{(1)} = \langle H'\rangle$ holds state by state

Answers that call the coupled basis conventional miss the point: it is forced by H_SO. The argument is complete only when it also shows that H_rel cannot mix different l inside the degenerate level, which a less symmetric perturbation could.

(b) SHOW THAT · U2 **[3]**

Starting from $p^2/(2m) = H_0 - V$, show that the relativistic shift is $E_{rel} = -[E_n^2/(2mc^2)][4n/(l+\tfrac{1}{2}) - 3]$.

1. $H_{rel} = -(1/(2mc^2))(p^2/2m)^2$, and because p^2 is Hermitian $\langle p^4\rangle$ is the squared norm of $p^2|\psi\rangle$, so $E_{rel} = -(1/(2mc^2))\langle (E_n - V)^2\rangle = -(1/(2mc^2))[E_n^2 - 2E_n\langle V\rangle + \langle V^2\rangle]$
2. $\langle V\rangle = -[Ze^2/(4\pi\varepsilon_0)]\langle 1/r\rangle = -Ze^2/(4\pi\varepsilon_0 n^2 a) = 2E_n$, and $\langle V^2\rangle = [Ze^2/(4\pi\varepsilon_0)]^2\langle 1/r^2\rangle = [Ze^2/(4\pi\varepsilon_0 a)]^2/[n^3(l+\tfrac{1}{2})] = 4nE_n^2/(l+\tfrac{1}{2})$, using $Ze^2/(4\pi\varepsilon_0 a) = -2n^2E_n$
3. so $E_{rel} = -(1/(2mc^2))[E_n^2 - 4E_n^2 + 4nE_n^2/(l+\tfrac{1}{2})] = -[E_n^2/(2mc^2)][4n/(l+\tfrac{1}{2}) - 3]$

Candidates who expand p^4 in coordinates meet derivatives of the Coulomb singularity; the route through $(E_n - V)^2$ avoids them, and it is legitimate only because p^2 can act to the left on the bra. The sign of $\langle V\rangle = 2E_n$, which is negative, is the slip that turns −3 into +5, and since the result is printed, a candidate who forces agreement without it earns nothing for the last line.

(c) DERIVE · U2 **[3]**

Derive the first-order spin–orbit shift E_{SO} of a state with $l \geq 1$, expressed in terms of E_n, mc^2, n, l and j.

1. $(1/r)dV/dr = [Ze^2/(4\pi\varepsilon_0)]/r^3$, so $H_{SO} = [Ze^2/(4\pi\varepsilon_0)]\, L\cdot S/(2m^2c^2r^3)$
2. in the coupled basis $L\cdot S = (J^2 - L^2 - S^2)/2$ has eigenvalue $(\hbar^2/2)[j(j+1) - l(l+1) - \tfrac{3}{4}]$, and $\langle 1/r^3\rangle = 1/[n^3 l(l+\tfrac{1}{2})(l+1)a^3]$
3. $E_{SO} = [Ze^2/(4\pi\varepsilon_0)](\hbar^2/(ma))[j(j+1) - l(l+1) - \tfrac{3}{4}]/[4mc^2a^2n^3l(l+\tfrac{1}{2})(l+1)]$, and with $\hbar^2/(ma) = Ze^2/(4\pi\varepsilon_0)$ and $[Ze^2/(4\pi\varepsilon_0 a)]^2 = 4n^4E_n^2$ this is $[E_n^2/(mc^2)]\, n[j(j+1) - l(l+1) - \tfrac{3}{4}]/[l(l+\tfrac{1}{2})(l+1)]$

   Evaluating $\langle L\cdot S\rangle$ in the uncoupled basis as $m_l\, m_s\, \hbar^2$ takes the diagonal elements of a matrix that is not diagonal there, and gives a shift that depends on m_l; the result must depend on j instead.

(d) SHOW THAT · U2 **[3]**

Show that, once the Darwin term is included for $l = 0$, the three terms sum to $E_{fs} = [E_n^2/(2mc^2)][3 - 4n/(j+\tfrac{1}{2})]$ for every state, so that the energy depends on n and j only.

1. $j = l + \tfrac{1}{2}$: the bracket in E_{SO} is l, so $E_{SO} = nE_n^2/[mc^2(l+\tfrac{1}{2})(l+1)]$, and $E_{rel} + E_{SO} = [E_n^2/(2mc^2)][3 - 4n/(l+\tfrac{1}{2}) + 2n/((l+\tfrac{1}{2})(l+1))] = [E_n^2/(2mc^2)][3 - 4n/(l+1)]$, with $l + 1 = j + \tfrac{1}{2}$
2. $j = l - \tfrac{1}{2}$: the bracket is $-(l+1)$, so $E_{SO} = -nE_n^2/[mc^2 l(l+\tfrac{1}{2})]$, and $E_{rel} + E_{SO} = [E_n^2/(2mc^2)][3 - 4n/(l+\tfrac{1}{2}) - 2n/(l(l+\tfrac{1}{2}))] = [E_n^2/(2mc^2)][3 - 4n/l]$, with $l = j + \tfrac{1}{2}$
3. $l = 0$: H_{SO} contributes nothing, $E_D = [\pi\hbar^2/(2m^2c^2)][Ze^2/(4\pi\varepsilon_0)]/(\pi n^3a^3) = 2nE_n^2/(mc^2)$, and $E_{rel} = -[E_n^2/(2mc^2)](8n - 3)$, so the sum is $[E_n^2/(2mc^2)](3 - 4n)$, the formula with $j = \tfrac{1}{2}$

   This part catches candidates who apply the spin–orbit formula at $l = 0$, where it is 0/0, instead of recognising that $L\cdot S$ vanishes there and that the Darwin term, a contact term acting only where $|\psi(0)|^2$ is nonzero, takes its place. Because the result is printed, each case must be carried to it explicitly.

(e) SKETCH · U1 **[2]**

For He^+ ($Z = 2$), sketch the fine structure of the $n = 2$ level: draw the unperturbed level and the fine-structure levels labelled $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$, marking each shift in μeV and the $2P_{3/2}$–$2P_{1/2}$ splitting in GHz.

1. $E_2 = -(2\alpha)^2mc^2/8 = -13.6$ eV and $E_2^2/(2mc^2) = 181.1$ μeV, so $2S_{1/2}$ and $2P_{1/2}$ are drawn as one level $5 \times 181.1 = 906$ μeV below the unperturbed level and $2P_{3/2}$ as a level 181 μeV below it: every state lies below E_2
2. the $2P_{3/2}$–$2P_{1/2}$ splitting is marked as 4×181.1 μeV $= 724$ μeV, which is 724×10^{-6} eV$/(4.136 \times 10^{-15}$ eV s$) = 175$ GHz, sixteen times hydrogen's 10.9 GHz

   Common sketches put $2P_{3/2}$ above the unperturbed level, as a spin–orbit-only picture would because its E_{SO} is positive, or separate $2S_{1/2}$ from $2P_{1/2}$. The relativistic term lowers every state, and in this theory the two $j = \tfrac{1}{2}$ states coincide exactly.

(f) EVALUATE · U4 **[2]**

The measured 2S_(1/2)–2P_(1/2) Lamb shifts are 1.058 GHz in hydrogen and 14.04 GHz in He^{+}. Welton's argument gives, for an nS level, $\Delta E \propto \alpha(Z\alpha)^4 mc^2 \ln[1/(Z\alpha)]/n^3$, with a prefactor, and an argument of the logarithm, that it fixes only roughly. Evaluate how well this scaling accounts for the ratio of the two shifts.

1. the scaling predicts a ratio $2^4 \ln[1/(2\alpha)]/\ln(1/\alpha) = 16 \times 4.23/4.92 = 13.7$, below the $Z^4 = 16$ of the fine structure because the logarithm falls with Z; the measured ratio is 14.04/1.058 = 13.3
2. judgement: agreement to about 4%, so the scaling captures the physics, a correction proportional to $Z|\psi(0)|^2 \propto Z^4$ that acts on s states only; but the argument of the logarithm is fixed only to an order of magnitude, so agreement at a few per cent is partly fortunate, since the ratio cancels the prefactor but not that uncertainty

   Scaling the hydrogen value by Z^4 alone predicts 16.9 GHz, 20% too high. A strong answer shows the logarithm doing measurable work, then limits its claim to what an order-of-magnitude argument can support.

University Quantum Mechanics II · Final Examination

The whole of Quantum Mechanics II, weighted towards the second half of the course: external fields and hyperfine structure, the variational principle, WKB, time-dependent perturbation theory and radiation, Berry's phase, and scattering. The spherical harmonics, hydrogen, spin in fields, density operators, angular momentum, perturbation theory and identical particles from the first half are sampled throughout.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**118**	**19**	**10**

SECTION — **UQMII · Final · University Quantum Mechanics II — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 1 · Routine 8 · Demanding 7 · Discriminating 3**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UQMII-01 · UQMII-02 · UQMII-03 · UQMII-04 · UQMII-05 · UQMII-06 · UQMII-07 · UQMII-08 · UQMII-09 · UQMII-10 · UQMII-11 · UQMII-12 · UQMII-13 · UQMII-14 · UQMII-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Quantum Mechanics II · Final Examination

Answer all questions. 118 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-01.02 **[1]**

A linear molecule attached by one end to a flat surface is modelled as a rigid rotor, of moment of inertia I about the attached end, that turns freely except that its axis cannot point into the surface. The polar angle θ of the axis, measured from the outward normal to the surface, is therefore confined to $0 \le \theta \le \pi/2$, and the surface acts as a hard wall: the wavefunction must vanish at $\theta = \pi/2$, where the axis would lie in the surface. The rotor's Hamiltonian is $H = L^2/(2I)$. For $m = 0$, with $x = \cos\theta$ and $\lambda = 2IE/\hbar^2$, the angular equation is Legendre's equation, and of its power-series solutions about $x = 0$ only the series of odd powers of x vanishes at $x = 0$, as the wall requires. Which values of λ are allowed?

A $\lambda = l(l+1)$ with $l = 0, 1, 2, \ldots$, the free rotor's values, because the wall changes a boundary condition but not Legendre's equation.

B $\lambda = l(l+1)$ with $l = 0, 2, 4, \ldots$, because only the even polynomials are symmetric about the equator, where the wall lies.

C $\lambda = l(l+1)$ with $l = 1, 3, 5, \ldots$, because the odd series must still terminate to stay finite at $\theta = 0$, where the axis lies along the surface normal.

D Any $\lambda \ge 0$, because the series need only be finite for $0 \le \theta \le \pi/2$, and the pole $\theta = \pi$, where an unterminated series diverges, lies outside that range.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-02.09 · UQMII-07.07 **[1]**

A crude model of a valence electron outside a closed-shell core replaces the Coulomb potential by $V(r) = -k/r - \beta/r^2$, where $k = e^2/(4\pi\varepsilon_0)$ and β is a small positive constant representing the extra attraction felt by an electron that penetrates the core. Compared with hydrogen, which statement about the bound-state spectrum is correct?

A The degeneracy in m is lifted, because the extra term breaks the symmetry that protected it.

B Levels of the same n but different l separate, with the lowest l pushed down most, while each keeps its $(2l+1)$-fold degeneracy in m; the quantum Runge-Lenz vector is no longer conserved.

C No degeneracy is lifted: the extra term simply adds to the centrifugal term $\hbar^2 l(l+1)/(2mr^2)$, and so shifts every level of a given n by the same amount.

D The l-degeneracy survives at first order in β, because $\langle 1/r^2 \rangle$ is the same for every state of a given n, and is lifted only at second order.

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-03.01 · UQMII-03.03 **[1]**

In a nuclear magnetic resonance magnet whose field B points along +z, the spins of protons at rest precess at 400.0 MHz. A particle at rest whose magnetic moment is $\mu = \gamma S$ has the Hamiltonian $H = -\gamma B S_z$, and each spin component obeys the Heisenberg equation $dS/dt = (i/\hbar)[H, S]$; $\gamma = +2.675 \times 10^{8}$ rad s^{-1} T^{-1} for the proton and $\gamma = -1.761 \times 10^{11}$ rad s^{-1} T^{-1} for the electron. A free electron is now held at rest in the same magnet. Which statement about the precession of its spin is correct?

A It precesses at 400.0 MHz, in the same sense as the proton spins, because the Larmor frequency is set by the field alone.

B It precesses at 263.3 GHz, in the same sense as the proton spins, because every moment precesses in the sense fixed by the direction of B.

C It precesses at 131.7 GHz, in the opposite sense to the proton spins, because the two spinor components acquire phases of only $\pm\gamma B t/2$.

D It precesses at 263.3 GHz, in the opposite sense to the proton spins, while S_z, which commutes with H, stays constant.

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-08.01 · UQMII-08.06 **[1]**

In hydrogen the fine-structure shifts are $E_{fs} = -[E_n^2/(2mc^2)][4n/(j + \tfrac{1}{2}) - 3]$, where $E_n = -13.6\ \text{eV}/n^2$, so they are smaller than the Bohr energies by a factor of order α^2 and depend on n and j only. The $2p_{3/2}$–$2p_{1/2}$ splitting is measured to be 10.97 GHz. What splitting does the formula then predict between the $3d_{5/2}$ and $3d_{3/2}$ levels?

A 1.08 GHz

B 3.25 GHz

C 4.88 GHz

D 0.722 GHz

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-09.01 · UQMII-09.02 **[1]**

A hydrogen atom in the $2p_{1/2}$ level is placed in a magnetic field of 0.10 T, which is weak compared with the 10.97 GHz fine-structure splitting of n = 2. By what frequency are its two m_j sublevels split? Take $\mu_B = 5.788 \times 10^{-5}$ eV T^{-1} and $h = 4.136 \times 10^{-15}$ eV s.

A 2.80 GHz

B 1.40 GHz

C 0.933 GHz

D 1.87 GHz

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-10.01 **[1]**

A system has exact ground and first excited energies $E_0 = -1.00$ eV and $E_1 = -0.25$ eV, with normalised eigenstates ψ_0 and ψ_1. A normalised trial state is $\varphi = \sqrt{(1-\varepsilon^2)}\,\psi_0 + \varepsilon\psi_1$ with $\varepsilon = 0.10$, so that 1% of its weight lies in the wrong state. What is $\langle\varphi|H|\varphi\rangle$?

A -1.020 eV

B -0.9925 eV

C -0.925 eV

D -1.000 eV

Answer

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-11.06 · UQMII-11.07 **[1]**

A cold atom of mass m is held along x in a harmonic trap whose angular frequency ω satisfies $\omega/2\pi = 2.0$ kHz. A thin repulsive light sheet is then switched on at the trap centre and is modelled as an impenetrable wall at $x = 0$, so that the atom moves in $V(x) = \tfrac{1}{2}m\omega^2x^2$ for $x > 0$ only. WKB quantisation is applied using the connection-formula phase at the smooth turning point and the condition $\psi = 0$ at the wall. What energy does it predict for the first excited state of this half-trap, expressed as the frequency E/h and measured from the bottom of the trap?

A 3.0 kHz

B 5.0 kHz

C 6.0 kHz

D 7.0 kHz

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-14.04 **[1]**

A two-level system is swept at a constant rate through an avoided crossing. The probability of a diabatic passage, in which the system stays in its initial diabatic state and so jumps the gap, is measured to be 0.20. According to the Landau-Zener formula, what is this probability when the sweep rate is doubled?

A 0.45

B 0.40

C 0.10

D 0.040

Answer

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9 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-12.09 · UQMII-13.05 · UQMII-13.08 **[1]**

A hydrogen atom is prepared in the $2s_{1/2}$ state, isolated and in zero field. It lives for about 0.12 s, whereas 2p lives for 1.6 ns. Which process is responsible for its decay?

A Emission of one electric-dipole photon to 1s, slowed because the 2s radial function has a node.

B Emission of one electric-dipole photon to $2p_{1/2}$, which lies 1.06 GHz below $2s_{1/2}$ because of the Lamb shift.

C Emission of two electric-dipole photons together, through virtual intermediate p states, sharing the 10.2 eV between them.

D Emission of one magnetic-dipole photon to 1s, allowed because parity is unchanged and $\Delta l = 0$.

Answer

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-15.04 · UQMII-15.07 **[1]**

Slow electrons scatter from atoms of a noble gas, which are modelled as spherically symmetric attractive wells. At the energies of interest only the s wave is appreciably scattered, and at one energy its phase shift passes through $\delta_0 = \pi$. What is observed near that energy?

A A peak in the total cross-section, reaching the unitarity limit $4\pi/k^2$.

B A deep minimum in the total cross-section, because $(4\pi/k^2)\sin^2\delta_0$ vanishes, so the gas becomes nearly transparent to the electrons.

C A new bound state appearing at exactly that energy, because Levinson's theorem requires one whenever δ_0 reaches π.

D A cross-section equal to the geometric value πR^2, because a phase shift of π reflects the wave as a hard sphere of the well's radius would.

Answer

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11 CORE COURSE

STRUCTURED · ROUTINE · UQMII-04.04 · UQMII-04.07 · UQMII-05.04 **[11]**

Two electrons, each held in one of a pair of neighbouring quantum dots, interact through the exchange Hamiltonian $H = (J/\hbar^2)\mathbf{S}_1 \cdot \mathbf{S}_2$, with $J = 2.0$ μeV. Their orbital states are frozen, so only the spins evolve. Write $|\uparrow\downarrow\rangle$ for spin 1 up and spin 2 down along z, and similarly for the other product states. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) SHOW THAT **[2]**

Show that the eigenvalues of H are J/4 for each of the three triplet states and $-3J/4$ for the singlet.

(b) DETERMINE **[3]**

At $t = 0$ the spins are prepared in $|\uparrow\downarrow\rangle$. By expressing this state in the coupled basis, determine the probability $P(t)$ that a measurement at time t finds the spins in $|\downarrow\uparrow\rangle$.

(c) CALCULATE **[1]**

Calculate the shortest time after which the two spins have completely exchanged their states.

(d) DETERMINE **[3]**

Determine the reduced density matrix of spin 1 at time t and its purity $\mathrm{Tr}\,\rho_1^2$, and hence the earliest time at which spin 1 is maximally mixed.

(e) EXPLAIN **[2]**

At that time an experimenter measures spin 1 alone, along any axis they choose. Explain why no such measurement can distinguish it from a spin drawn from an ensemble prepared half in $|\uparrow\rangle$ and half in $|\downarrow\rangle$, and what kind of measurement would reveal the difference.

12 CORE COURSE

STRUCTURED · DEMANDING · UQMII-06.05 · UQMII-06.06 · UQMII-06.07 **[11]**

An electron in a cubic InAs nanocrystal is modelled as a particle of charge $-e$ and effective mass $m^* = 0.023m_e$ in a cubical infinite well of side $L = 20$ nm centred on the origin, $-L/2 < x, y, z < L/2$. Spin is ignored, since its Zeeman energy is the same for every orbital state. The unperturbed states are $\psi_{n_1n_2n_3} = u_{n_1}(x)u_{n_2}(y)u_{n_3}(z)$, with energies $(n_1^2 + n_2^2 + n_3^2)\varepsilon$, where $\varepsilon = \pi^2\hbar^2/(2m^*L^2)$, $u_1(x) = \sqrt{(2/L)}\cos(\pi x/L)$ and $u_2(x) = \sqrt{(2/L)}\sin(2\pi x/L)$. A uniform magnetic field $B = 1.5$ T is applied along z. In the symmetric gauge, and to first order in B, it adds $H' = (\mu^*B/\hbar)L_z$, where $\mu^* = e\hbar/(2m^*)$ is the effective magneton and $L_z = -i\hbar(x\,\partial/\partial y - y\,\partial/\partial x)$; the term in B^2 is neglected. You may use $\int u_1(x)\,x\,u_2(x)\,dx = 16L/(9\pi^2)$ and $\int u_1(du_2/dx)\,dx = 8/(3L) = -\int u_2(du_1/dx)\,dx$, each integral running from $-L/2$ to $L/2$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) CALCULATE **[2]**

Calculate ε in meV, and the energies and degeneracies of the ground level and of the first excited level.

(b) CALCULATE **[2]**

Calculate μ^*B in meV, and its ratio to the spacing 3ε that separates the first excited level from the levels on either side of it.

(c) DETERMINE **[3]**

Using the integrals given, determine the matrix of H' in the first excited level, its eigenvalues in meV, and the good states.

(d) EXPLAIN **[2]**

Explain how the rotation through 90° about the z-axis identifies these good states without any diagonalisation.

(e) SHOW THAT **[2]**

The field is turned, at the same strength, to lie along the body diagonal $\hat{n} = (1, 1, 1)/\sqrt{3}$, so that $H' = (\mu^*B/\hbar)\hat{n}\cdot L$. Show that the first excited level splits in exactly the same way, into shifts of +3.63 meV, 0 and −3.63 meV.

13 CORE COURSE

STRUCTURED · DEMANDING · UQMII-05.07 · UQMII-09.06 · UQMII-13.05 **[12]**

A hydrogen atom in the n = 2 shell is placed in a uniform static electric field F along z; hyperfine structure is ignored. Without the field, $2s_{1/2}$ lies above $2p_{1/2}$ by the Lamb shift, L = h × 1057.8 MHz, and $2p_{3/2}$ lies h × 10.97 GHz above $2p_{1/2}$. The perturbation is $H' = eFz$. In the uncoupled basis $\langle 2s|z|2p, m_l = 0\rangle = -3a_0$, with $a_0 = 5.29 \times 10^{-11}$ m, and in the convention used here $|2p_{1/2}, m_j = +\tfrac{1}{2}\rangle = \sqrt{(2/3)}|m_l = 1\rangle|{\downarrow}\rangle - \sqrt{(1/3)}|m_l = 0\rangle|{\uparrow}\rangle$ and $|2s_{1/2}, m_j = +\tfrac{1}{2}\rangle = |2s\rangle|{\uparrow}\rangle$. The 2p lifetime is 1.60 ns, and an isolated 2s atom lives for 0.122 s. Take $h = 6.626 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) SHOW THAT **[2]**

Show that H' couples $|2s_{1/2}, +\tfrac{1}{2}\rangle$ to $|2p_{1/2}, +\tfrac{1}{2}\rangle$ and to no other $2p_{1/2}$ state, with a matrix element of magnitude $\sqrt{3}\, ea_0F$.

(b) EXPLAIN **[2]**

Explain why the $2s_{1/2}$ level has no linear Stark shift in a weak field, although the n = 2 level of hydrogen is often said to show a linear Stark effect of $\pm 3ea_0F$.

(c) DETERMINE **[3]**

Neglecting $2p_{3/2}$, determine the energies of the two $m_j = +\tfrac{1}{2}$ states as functions of F, measured from the midpoint between $2s_{1/2}$ and $2p_{1/2}$, and the field at which the coupling equals half the Lamb shift.

(d) CALCULATE **[3]**

The 2s-like state decays through the $2p_{1/2}$ component that the field mixes into it. Calculate its lifetime in a field of 2.0 V cm^{-1}.

(e) JUSTIFY **[2]**

Justify the neglect in (d) of the $2p_{3/2}$ level and of the natural width of the 2p state.

14 CORE COURSE

STRUCTURED · DEMANDING · UQMII-10.01 · UQMII-10.02 · UQMII-10.06 **[12]**

A heavy quark and antiquark bound in a meson are modelled non-relativistically as one particle of reduced mass μ, with $\mu c^2 = 0.75$ GeV, in the linear confining potential $V(r) = \sigma r$, with $\sigma = 0.90$ GeV fm^{-1}. Only $l = 0$ is considered. The exact ground-state energy is $E_0 = 2.3381\varepsilon_0$, where $\varepsilon_0 = (\hbar^2\sigma^2/2\mu)^{(1/3)}$ and -2.3381 is the first zero of the Airy function Ai; for any s state of this potential it can also be shown that $|\psi(0)|^2 = \mu\sigma/(2\pi\hbar^2)$ exactly. Take $\hbar c = 0.1973$ GeV fm, and $\int_0^{\infty} r^n e^{(-\beta r)}\, dr = n!/\beta^{n+1}$.

(a) SHOW THAT **[2]**

Show that for the normalised trial function $\psi_\beta(r) = \sqrt{(\beta^3/\pi)}\, e^{(-\beta r)}$, $\langle H\rangle = \hbar^2\beta^2/(2\mu) + 3\sigma/(2\beta)$.

(b) DETERMINE **[3]**

Determine the value of β that minimises $\langle H\rangle$, and show that the resulting bound is $E_{exp} = 3(3/4)^{(2/3)}\varepsilon_0 = 2.476\varepsilon_0$.

(c) CALCULATE **[2]**

Calculate ε_0, the exponential bound and the exact ground-state energy in GeV, and the percentage by which the bound exceeds the exact value.

(d) EXPLAIN **[2]**

A Gaussian trial function $e^{(-\alpha r^2)}$, optimised in the same way, gives $2.3448\varepsilon_0$, within 0.3% of the exact value, although an exponential is the exact ground state of hydrogen. Explain why the Gaussian does so much better in this potential.

(e) EVALUATE **[3]**

The meson's decay rate to a lepton pair is proportional to $|\psi(0)|^2$. The optimal Gaussian trial, $\psi = (2\alpha/\pi)^{(3/4)}\, e^{(-\alpha r^2)}$, has $\alpha^{(3/2)} = (\mu\sigma/3\hbar^2)\sqrt{(2/\pi)}$. A student proposes using it to predict $|\psi(0)|^2$, arguing that an energy within 0.3% of the exact value shows the trial state to be accurate. Evaluate this proposal.

15 CORE COURSE

STRUCTURED · DEMANDING · UQMII-13.03 · UQMII-13.06 · UQMII-13.07 **[11]**

For spontaneous emission by an electron from an upper state $|b\rangle$ to a lower state $|a\rangle$ at angular frequency ω, Einstein's detailed-balance argument with the Planck spectrum gives $A = \omega^3 |\langle a|er|b\rangle|^2/(3\pi\varepsilon_0\hbar c^3)$. The absorption oscillator strength from a non-degenerate state $|a\rangle$ to a level b is $f = (2m_e\omega/3\hbar)\Sigma|\langle b|r|a\rangle|^2$, summed over the degenerate states of b, and for one electron the Thomas-Reiche-Kuhn sum rule gives $\Sigma f = 1$ over all levels b. As a model, take an electron bound in the isotropic three-dimensional harmonic potential $\tfrac{1}{2}m_e\omega^2 r^2$, with states $|n_x, n_y, n_z\rangle$ and $\langle 0|x|1\rangle = \sqrt{(\hbar/2m_e\omega)}$ for each Cartesian direction. Take $e = 1.602 \times 10^{-19}$ C, $m_e = 9.109 \times 10^{-31}$ kg, $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, $c = 2.998 \times 10^8$ m s^{-1}, $k_B = 1.381 \times 10^{-23}$ J K^{-1} and $1\ \text{u} = 1.661 \times 10^{-27}$ kg.

(a) EXPLAIN **[2]**

Explain why an atom treated by perturbation theory in a classical field would, in an excited state with no applied field, never decay, and how Einstein's argument nevertheless fixes A.

(b) SHOW THAT **[2]**

Show that for the oscillator the rate from $|1, 0, 0\rangle$ to $|0, 0, 0\rangle$ is $A = e^2\omega^2/(6\pi\varepsilon_0 m_e c^3)$, and comment on the absence of $\hbar$.

(c) DETERMINE **[2]**

Determine the absorption oscillator strength from $|0, 0, 0\rangle$ to the first excited level, and state what the Thomas-Reiche-Kuhn sum rule then implies for every other transition from the ground state.

(d) CALCULATE **[3]**

The sodium resonance lines near 589.0 nm (3s to 3p) have statistical weights in the same 1 : 3 ratio as the oscillator's, so for them $A = f \times e^2\omega^2/(6\pi\varepsilon_0 m_e c^3)$. The measured 3p lifetime is 16.2 ns. Calculate the oscillator's lifetime at 589.0 nm, and hence the oscillator strength of the sodium resonance transition.

(e) COMPARE **[2]**

Compare the natural linewidth of the sodium line with its Doppler width in a vapour at 500 K, for which the full width at half maximum is $\Delta\nu_D = (\nu/c)\sqrt{(8k_B T \ln 2/M)}$, with $M = 22.99$ u.

16 CORE COURSE

STRUCTURED · DISCRIMINATING · UQMII-14.02 · UQMII-14.05 · UQMII-14.06 · UQMII-14.07 **[11]**

A superconducting qubit is driven close to resonance. In the frame rotating with the drive its Hamiltonian is $H(\varphi) = (\hbar/2)[\Delta\sigma_z + \Omega(\cos\varphi\,\sigma_x + \sin\varphi\,\sigma_y)]$, where the detuning is $\Delta/2\pi = 3.0$ MHz, the Rabi frequency is $\Omega/2\pi = 4.0$ MHz, and φ is the phase of the drive, which can be ramped. Equivalently $H = (\hbar\Omega_{\text{eff}}/2)\,\hat{n}\cdot\sigma$, with $\Omega_{\text{eff}} = \sqrt{(\Delta^2 + \Omega^2)}$ and $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. In the experiment φ is ramped at a constant rate from 0 to 2π in a time $T = 4.0$ μs, so that $\hat{n}$ traces a circle on a cone about the z-axis.

(a) STATE **[2]**

State the eigenvalues of H, and the angle θ between $\hat{n}$ and the z-axis.

(b) SHOW THAT **[2]**

Show that the adiabatic condition for the ramp becomes $T \gg \pi\Omega/(\Delta^2 + \Omega^2)$, and evaluate the right-hand side.

(c) CALCULATE **[3]**

Calculate the geometric phase acquired in one circuit by each of the two eigenstates, and the difference between their dynamic phases.

(d) PREDICT **[2]**

In a spin-echo version, the qubit starts in an equal superposition of the two eigenstates at $\varphi = 0$, is taken once round the circuit, has its two eigenstates exchanged by a π pulse, and is then taken round the circuit in the opposite sense in the same time T. Predict the relative phase between the two components at the end, and how it changes if T is doubled or if $\Delta/2\pi$ is raised to 6.0 MHz with Ω unchanged.

(e) EXPLAIN **[2]**

The same circuit also bounds the rest of the sphere, a cap of solid angle 4π minus the one used in (c). Explain why that cap gives the same phase factor, and what this has to do with the degeneracy of H at $\Delta = \Omega = 0$.

17 CORE COURSE

DATA ANALYSIS · DEMANDING · UQMII-09.07 · UQMII-09.08 · UQMII-09.09 **[10]**

A hydrogen maser oscillates on the transition from F = 1, m_F = 0 to F = 0 in the hyperfine structure of the ground state, whose zero-field frequency is ν_hfs. A small static field B along z is applied by a solenoid around the storage bulb, but the ambient field is not fully cancelled. The field inside the bulb is measured by the Zeeman frequency f_Z, the low-frequency transition between neighbouring F = 1 sublevels, which in these weak fields is f_Z = μ_B B/h. For each solenoid current the maser's output is compared with a reference oscillator, giving a frequency offset δν that also contains a constant but unknown contribution from the reference and from collisions with the bulb wall. With g_e = 2 and the proton's magnetic moment neglected, the ground-state Hamiltonian is $H = A\,\mathbf{S}_e\cdot\mathbf{S}_p/\hbar^2 + (2\mu_B B/\hbar)S_z$, where A = hν_hfs. Take $\mu_B = 9.274\times10^{-24}$ J T^{-1} and $h = 6.626\times10^{-34}$ J s.

Maser frequency offset and in-situ Zeeman frequency for six solenoid currents

Run	Solenoid current I / mA	Zeeman frequency f_Z / kHz	Frequency offset δν / mHz
1	1.0	0.40	3.74
2	2.0	0.70	4.17
3	3.0	1.00	4.92
4	4.0	1.30	5.87
5	5.0	1.60	7.13
6	6.0	1.90	8.57

(a) SHOW THAT **[3]**

Working in the m_F = 0 subspace spanned by $|{\uparrow}{\Downarrow}\rangle$ and $|{\downarrow}{\Uparrow}\rangle$ (electron spin first), show that the frequency of the F = 1, m_F = 0 to F = 0 transition is $\nu_c = \sqrt{\nu_{hfs}^2 + 4f_Z^2} \approx \nu_{hfs} + 2f_Z^2/\nu_{hfs}$.

(b) DESCRIBE **[2]**

Describe how the readings should be plotted to test this model and to extract ν_hfs, and explain why the solenoid current should not be used as the measure of the field.

(c) DETERMINE **[3]**

Determine the gradient of your graph and hence a value of ν_hfs, and comment on its agreement with the accepted 1420.406 MHz.

(d) ESTIMATE **[2]**

The maser is normally operated at f_Z = 0.80 kHz. Estimate how constant f_Z, and hence the field, must be kept if field fluctuations are to change the maser frequency by less than 1 part in 10^{15}, stating your assumptions.

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18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMII-12.06 · UQMII-15.02 · UQMII-15.05 · UQMII-15.08 · UQMII-15.09 **[15]**

A particle of mass m and energy $E = \hbar^2k^2/2m$ is scattered elastically by a potential V(r) that vanishes at large r. To apply Fermi's golden rule, the system is placed in a cubic box of side L with periodic boundary conditions, so that the plane-wave states ⟨r|k⟩ = e^(ik·r)/L^(3/2) are normalised and the incident state carries a probability current density $\hbar k/(mL^3)$. The golden rule gives the rate $\Gamma = (2\pi/\hbar)|\langle f|V|i\rangle|^2\rho(E)$ into a group of final states whose density per unit energy is ρ(E). Write $\tilde{V}(q) = \int e^{-iq\cdot r}V(r)\,d^3r$. Take $m_e = 9.109 \times 10^{-31}$ kg, e = 1.602 × 10^{-19} C, c = 2.998 × 10^8 m s^{-1} and α = 1/137.0.

(a) DERIVE **[3]**

Derive, from the periodic boundary conditions, the density of final states per unit energy ρ(E) for plane waves whose wavevectors point into a small solid angle dΩ.

(b) DERIVE **[3]**

Using the golden rule, derive the differential cross-section dσ/dΩ for elastic scattering in terms of $\tilde{V}(q)$, where q = k_f − k_i, and confirm that it does not depend on L.

(c) SHOW THAT **[2]**

For the screened Coulomb potential V(r) = (g/r)e^(−r/a), show that $\tilde{V}(q) = 4\pi g/(q^2 + 1/a^2)$.

(d) DETERMINE **[2]**

Take the limit a → ∞ with $g = Z_1Z_2e^2/(4\pi\varepsilon_0)$. Determine dσ/dΩ as a function of the scattering angle θ, and compare it with Rutherford's classical result, $d\sigma/d\Omega = g^2/(16E^2\sin^4(\theta/2))$.

(e) EXPLAIN **[2]**

The first Born amplitude, $f(\theta) = -(m/2\pi\hbar^2)\tilde{V}(q)$, is real for this potential, so $\mathrm{Im}\, f(0) = 0$, yet the total cross-section it gives is positive. Explain why this does not contradict the optical theorem, $\sigma = (4\pi/k)\,\mathrm{Im}\, f(0)$.

(f) ESTIMATE **[3]**

For a Coulomb-like potential of strength $g = Ze^2/(4\pi\varepsilon_0)$, the first Born approximation is reliable when $g/(\hbar v) = Z\alpha c/v \ll 1$, where v is the speed of the incident particle. Estimate whether the Born cross-section can be trusted for electrons of kinetic energy 2.0 keV scattered by neutral helium ($Z = 2$) and by neutral iron ($Z = 26$), stating your assumptions.

19

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UQMII-11.04 · UQMII-11.09 **[15]**

A particle of mass m moves in one dimension in the potential $V(x) = \tfrac{1}{2}m\omega^2 x^2(1 - x/L)$, the model used for the phase of a current-biased Josephson junction. V has a minimum at $x = 0$ and a barrier beyond which, for $x > L$, it falls without limit, so the ground state of the well is metastable and escapes by tunnelling. For the numerical parts take $\omega/2\pi = 8.0$ GHz, with L such that the barrier height is $V_b = 3.0\hbar\omega$. A full treatment, matching the Gaussian ground state of the well onto the decaying WKB solution under the barrier, gives the escape rate $\Gamma = (\omega/2\pi)\sqrt{(864\pi V_b/\hbar\omega)}\,\exp(-36V_b/5\hbar\omega)$, which is quoted here. You may use $\int_0^1 u\sqrt{(1-u)}\,du = 4/15$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) SHOW THAT **[2]**

Show that the top of the barrier lies at $x = 2L/3$ and that $V_b = (2/27)m\omega^2L^2$.

(b) SKETCH **[2]**

Sketch $V(x)$ from $x = -L/3$ to $x = 4L/3$, marking V_b, the ground-state energy $\tfrac{1}{2}\hbar\omega$ and the classical turning points at that energy, and on the same axes sketch the magnitude of the metastable ground-state wavefunction.

(c) DERIVE **[3]**

Starting from the WKB tunnelling exponent $2S/\hbar = (2/\hbar)\int\sqrt{(2m(V - E))}\,dx$ taken across the barrier, derive its value for an energy $E \to 0$ at the bottom of the well, and express it in terms of $V_b/\hbar\omega$.

(d) CALCULATE **[2]**

Calculate, for $V_b = 3.0\hbar\omega$, the naive escape rate $\Gamma_0 = (\omega/2\pi)\exp(-2S/\hbar)$, which pictures the particle striking the barrier once per classical period, and the quoted rate Γ.

(e) EVALUATE **[3]**

Evaluated between its turning points at the actual ground-state energy $E = \tfrac{1}{2}\hbar\omega$, the barrier integral gives $2S/\hbar = 17.15$ for these parameters. Evaluate the naive estimate in the light of this: identify which of its approximations causes most of the discrepancy, and explain why the difficulty is worst for the lowest state.

(f) ESTIMATE **[3]**

Thermal activation over the barrier occurs at a rate of order $(\omega/2\pi)\exp(-V_b/k_BT)$. Estimate the temperature below which escape is dominated by tunnelling, stating your assumptions, and say what an experiment would observe below it.

MARK SCHEME

University Quantum Mechanics II · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 [1]

CORE COURSE

MULTIPLE CHOICE · RECALL · UQMII-01.02

A linear molecule attached by one end to a flat surface is modelled as a rigid rotor, of moment of inertia I about the attached end, that turns freely except that its axis cannot point into the surface. The polar angle θ of the axis, measured from the outward normal to the surface, is therefore confined to $0 \le \theta \le \pi/2$, and the surface acts as a hard wall: the wavefunction must vanish at $\theta = \pi/2$, where the axis would lie in the surface. The rotor's Hamiltonian is $H = L^2/(2I)$. For m = 0, with $x = \cos\theta$ and $\lambda = 2IE/\hbar^2$, the angular equation is Legendre's equation, and of its power-series solutions about x = 0 only the series of odd powers of x vanishes at x = 0, as the wall requires. Which values of λ are allowed?

KEY **C** **λ = l(l + 1) with l = 1, 3, 5, …, because the odd series must still terminate to stay finite at θ = 0, where the axis lies along the surface normal.**

C is correct. With the even series excluded, the solution is a series of odd powers alone, and unless it stops, its coefficients fall off only as 1/k at large k, so it diverges at x = +1 just as at x = −1. The pole θ = 0, where the molecule stands straight up, lies inside the allowed region, so the series must terminate, and a series of odd powers terminates only when λ = l(l + 1) with l odd. The allowed states are P_1, P_3, P_5 and so on, each of which vanishes at x = 0, with $E = \hbar^2 l(l+1)/(2I)$ for odd l only. D is the tempting error. On the full sphere, for any λ, one combination of the even and odd series stays finite at θ = 0 and diverges only at θ = π, so excluding θ = π seems to lift the restriction. But unless λ = l(l + 1) with l odd, that combination does not vanish at the wall, and the odd series the wall does allow has definite parity, so it diverges at both poles or at neither. A keeps the free rotor's spectrum, forgetting that the new boundary condition removes solutions: the even polynomials P_0, P_2, P_4 and so on are nonzero at x = 0, so half of the free rotor's m = 0 levels are lost. B mistakes a node for a symmetry. The wall allows only solutions that vanish at x = 0, which for polynomials of definite parity means the odd ones, with $P_l(-x) = -P_l(x)$; the even polynomials, symmetric about the equator, are nonzero there and are exactly the ones the wall excludes.

U1

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2 [1]

CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-02.09 · UQMII-07.07

A crude model of a valence electron outside a closed-shell core replaces the Coulomb potential by $V(r) = -k/r - \beta/r^2$, where $k = e^2/(4\pi\varepsilon_0)$ and β is a small positive constant representing the extra attraction felt by an electron that penetrates the core. Compared with hydrogen, which statement about the bound-state spectrum is correct?

KEY **B** **Levels of the same n but different l separate, with the lowest l pushed down most, while each keeps its (2l + 1)-fold degeneracy in m; the quantum Runge-Lenz vector is no longer conserved.**

B is correct. The potential is still central, so L^2 and L_z commute with H and the 2l + 1 states of each l stay degenerate. The extra term can be absorbed into the centrifugal term, $\hbar^2 l(l+1)/(2mr^2) - \beta/r^2 = \hbar^2 l'(l'+1)/(2mr^2)$, with $l - l' \approx (2m\beta/\hbar^2)/(2l+1)$, so the radial equation is still hydrogenic but with a non-integer l′, and $E = -Ry/(n_r + l' + 1)^2 = -Ry/(n - \delta_l)^2$: the quantum-defect form, with δ_l largest for s states. The extra symmetry of the pure 1/r potential, generated by the Runge-Lenz vector, is gone, and the l-degeneracy with it. A mistakes a change in the radial form of the potential for a loss of spherical symmetry; only a non-central term, such as an external field, can split m. C starts from the right observation, that the term adds to the centrifugal barrier, but how far it moves the effective l depends on l itself, so the shift differs from one l to the next. D uses the wrong expectation value: by the Feynman-Hellmann route of Unit 6, $\langle 1/r^2\rangle = 1/[n^3(l+\tfrac{1}{2})a_0^2]$ depends on l, so the first-order shifts $-\beta\langle 1/r^2\rangle$ already differ, and are largest for l = 0.

U1

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-03.01 · UQMII-03.03 **[1]**

In a nuclear magnetic resonance magnet whose field B points along +z, the spins of protons at rest precess at 400.0 MHz. A particle at rest whose magnetic moment is μ = γS has the Hamiltonian H = −γBS_z, and each spin component obeys the Heisenberg equation dS/dt = (i/ħ)[H, S]; γ = +2.675 × 10^{8} rad s^{-1} T^{-1} for the proton and γ = −1.761 × 10^{11} rad s^{-1} T^{-1} for the electron. A free electron is now held at rest in the same magnet. Which statement about the precession of its spin is correct?

KEY **D** **It precesses at 263.3 GHz, in the opposite sense to the proton spins, while S_z, which commutes with H, stays constant.**

The Heisenberg equation gives dS_x/dt = (i/ħ)(−γB)[S_z, S_x] = γBS_y and dS_y/dt = −γBS_x, while [H, S_z] = 0, so S_z is conserved and the transverse components rotate about z at the rate |γ|B, in a sense set by the sign of γ. The proton data fix B = 2π(400.0 MHz)/γ_p = 9.395 T, and the electron then precesses at |γ_e|B/2π = 400.0 MHz × (1.761 × 10^{11})/(2.675 × 10^{8}) = 263.3 GHz; because γ_e and γ_p have opposite signs, it turns the other way: D. A treats the Larmor frequency as a property of the field alone, but ω = |γ|B, and the electron's gyromagnetic ratio is 658 times the proton's, which is why electron-spin resonance in this magnet needs microwaves rather than radio waves. B has the right rate but drops the sign of γ: the electron's moment is antiparallel to its spin and the proton's parallel, so the torque turns the two spins in opposite senses. C is the half-angle trap. The two spinor components do acquire phases e^(±iγBt/2), but ⟨S⟩ is turned by their relative phase, γBt, so a vector operator precesses at the full rate |γ|B.

U2

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-08.01 · UQMII-08.06 **[1]**

In hydrogen the fine-structure shifts are $E_{fs} = -[E_n^2/(2mc^2)][4n/(j + ½) − 3]$, where $E_n = -13.6\ \text{eV}/n^2$, so they are smaller than the Bohr energies by a factor of order α^2 and depend on n and j only. The $2p_{3/2}$–$2p_{1/2}$ splitting is measured to be 10.97 GHz. What splitting does the formula then predict between the $3d_{5/2}$ and $3d_{3/2}$ levels?

KEY **A** **1.08 GHz**

Between two levels of the same n the formula gives a splitting $[E_n^2/(2mc^2)] \times 4n[1/(j_1 + ½) − 1/(j_2 + ½)]$, which falls as n^{-3} times the bracket in j. For 2p it is $(E_2^2/2mc^2) \times 8 \times (1 − ½) = 4E_2^2/(2mc^2)$, and for 3d it is $(E_3^2/2mc^2) \times 12 \times (½ − ⅓) = 2E_3^2/(2mc^2)$, so the ratio is $½ \times (E_3/E_2)^2 = ½ \times 16/81 = 8/81$, and the 3d splitting is 10.97 GHz × 8/81 = 1.08 GHz (A). B, 3.25 GHz, is the $3p_{3/2}$–$3p_{1/2}$ splitting: it treats every fine-structure doublet as a j = ½, 3/2 pair, when the d levels have j = 3/2 and 5/2. C scales the 2p splitting by $(2/3)^2$, as the Bohr energies scale, treating fine structure as a fixed fraction of $|E_n|$; the factor 4n makes it fall as n^{-3}, and the j values change the bracket as well. D drops the factor n from 4n/(j + ½), so it scales as n^{-4} and is too small by a factor 3/2. Because the energies depend on n and j only, $3d_{3/2}$ coincides with $3p_{3/2}$, so this 1.08 GHz is also the gap between that pair and the highest n = 3 level.

U3

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-09.01 · UQMII-09.02 **[1]**

A hydrogen atom in the $2p_{1/2}$ level is placed in a magnetic field of 0.10 T, which is weak compared with the 10.97 GHz fine-structure splitting of n = 2. By what frequency are its two m_j sublevels split? Take μ_B = 5.788 × 10^{-5} eV T^{-1} and h = 4.136 × 10^{-15} eV s.

KEY **C** **0.933 GHz**

In a weak field j and m_j stay good and the shift is g_J μ_B B m_j, with the Landé factor given by the projection theorem, g_J = 1 + [j(j + 1) + s(s + 1) − l(l + 1)]/[2j(j + 1)]. For l = 1 and j = ½ this is 1 + (¾ + ¾ − 2)/(3/2) = 2/3, so the m_j = ±½ sublevels are split by (2/3)μ_B B = (2/3)(5.788 × 10^{-5} eV T^{-1})(0.10 T) = 3.86 × 10^{-6} eV, or 0.933 GHz (C). A takes the spin g-factor of 2, which is right for $2s_{1/2}$ but not for a state whose spin is partly antiparallel to its orbital angular momentum. B takes the orbital g-factor of 1 and ignores the spin altogether. D uses 4/3, the factor for $2p_{3/2}$, reached by taking j = l + ½ or by a sign slip in the projection formula. The splitting is comparable with the 1.06 GHz Lamb shift, but that does not matter here: the Zeeman term commutes with L^2, so it cannot mix $2p_{1/2}$ with $2s_{1/2}$, and the splitting it must be weak against is the 10.97 GHz separation from $2p_{3/2}$.

U3

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-10.01 **[1]**

A system has exact ground and first excited energies $E_0 = -1.00$ eV and $E_1 = -0.25$ eV, with normalised eigenstates ψ_0 and ψ_1. A normalised trial state is $\varphi = \sqrt{(1-\varepsilon^2)}\,\psi_0 + \varepsilon\psi_1$ with $\varepsilon = 0.10$, so that 1% of its weight lies in the wrong state. What is $\langle\varphi|H|\varphi\rangle$?

KEY **B** **−0.9925 eV**

Orthogonality removes the cross terms, so $\langle H\rangle = (1-\varepsilon^2)E_0 + \varepsilon^2 E_1 = 0.99(-1.00\text{ eV}) + 0.01(-0.25\text{ eV}) = -0.9925$ eV (B). The error, $\varepsilon^2(E_1 - E_0) = 0.0075$ eV, is second order in the 10% admixture of amplitude, which is why a variational energy is far more accurate than the trial state that produced it. A weights the energies by the amplitudes rather than their squares, $\sqrt{0.99}\,E_0 + 0.10E_1 = -1.020$ eV. It lies below E_0, which the variational theorem forbids for any admissible trial state, so it should be rejected on sight. C also weights by amplitudes, as $(1-\varepsilon)E_0 + \varepsilon E_1$, and so makes the error first order: 0.075 eV, ten times too large. D supposes that a 1% admixture is too small to register. The energy is insensitive to it, but not blind, and the shift is always upwards.

U3

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7 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-11.06 · UQMII-11.07 **[1]**

A cold atom of mass m is held along x in a harmonic trap whose angular frequency ω satisfies ω/2π = 2.0 kHz. A thin repulsive light sheet is then switched on at the trap centre and is modelled as an impenetrable wall at x = 0, so that the atom moves in $V(x) = \tfrac{1}{2}m\omega^2x^2$ for $x > 0$ only. WKB quantisation is applied using the connection-formula phase at the smooth turning point and the condition ψ = 0 at the wall. What energy does it predict for the first excited state of this half-trap, expressed as the frequency E/h and measured from the bottom of the trap?

KEY **D** **7.0 kHz**

A smooth turning point contributes a phase of π/4 to the WKB condition and a hard wall, where ψ must vanish, contributes π/2, so $(1/\hbar)\int_0\hat{}a\ p\ dx = (n + \tfrac{3}{4})\pi$ with n = 0, 1, 2, ... For the half-trap the integral is a quarter of the area $2\pi E/\omega$ enclosed by the full oscillator's phase-space ellipse, $\pi E/(2\omega)$, so $E = (2n + 3/2)\hbar\omega$, and the first excited state, n = 1, has $E = 3.5\hbar\omega$, or $E/h = 3.5 \times 2.0$ kHz = 7.0 kHz (D). The result is exact: the allowed states are the odd states of the full oscillator, the ones that vanish at x = 0. A, 3.0 kHz, is tempting in two ways. It is the full trap's first excited state, $(n + \tfrac{1}{2})\hbar\omega$ with n = 1, found by ignoring the wall, and it is also the correct formula applied to the ground state, n = 0, instead of the first excited state. C, 6.0 kHz, applies the rule $(n + \tfrac{1}{2})\pi$, correct between two smooth turning points, to a region bounded by one smooth turning point and one wall, and so gives $(2n + 1)\hbar\omega$. B, 5.0 kHz, uses $(n + \tfrac{1}{4})\pi$, which treats the wall as contributing no phase at all; it gives $(2n + \tfrac{1}{2})\hbar\omega$, the even states of the full oscillator, which satisfy $\psi'(0) = 0$ rather than $\psi(0) = 0$. These phase corrections matter most for the lowest states, where they are a large fraction of the total.

U2

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-14.04 **[1]**

A two-level system is swept at a constant rate through an avoided crossing. The probability of a diabatic passage, in which the system stays in its initial diabatic state and so jumps the gap, is measured to be 0.20. According to the Landau-Zener formula, what is this probability when the sweep rate is doubled?

KEY **A** **0.45**

The Landau-Zener probability is $P = \exp(-2\pi|H_{12}|^2/(\hbar\alpha))$, where α is the rate at which the diabatic energies cross, so the exponent is inversely proportional to the rate. Doubling α halves the exponent, and P becomes $\sqrt{0.20} = 0.45$ (A). B doubles the probability, treating it as proportional to the rate, which would soon carry it past 1. C halves it, the answer of a candidate who expects a faster sweep to be more adiabatic when the opposite is true: a faster sweep gives the system less time to follow its instantaneous eigenstate. D squares the probability, which is what halving the sweep rate would do. The exponential dependence on 1/α is what makes a modest slowing of the sweep so effective at suppressing the jump.

U3

9

CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQMII-12.09 · UQMII-13.05 · UQMII-13.08 **[1]**

A hydrogen atom is prepared in the $2s_{1/2}$ state, isolated and in zero field. It lives for about 0.12 s, whereas 2p lives for 1.6 ns. Which process is responsible for its decay?

KEY **C** **Emission of two electric-dipole photons together, through virtual intermediate p states, sharing the 10.2 eV between them.**

C is correct: the dominant decay is two-photon emission, a second-order process through virtual p states, with a rate of about 8 s^{-1}; because the energy can be shared in any proportion, the photons form a continuum rather than a line. A is ruled out exactly, not merely slowed. The dipole operator is odd, 1s and 2s are both even, and $\Delta l = \pm 1$ is required, so the matrix element vanishes whatever the radial functions look like. B names an allowed transition, but its rate carries a factor ω^3, and at 1.06 GHz rather than the 2.47×10^{15} Hz of Lyman-α that factor is smaller by about 10^{19}, leaving a rate of order 10^{-9} s^{-1}. D is the subtle one. Parity and angular momentum do permit a magnetic-dipole transition, but its operator, $-(\mu_B/\hbar)(L + 2S)$, does not act on r, so between 2s and 1s it is multiplied by the overlap of two orthogonal radial functions and vanishes; relativistic corrections restore a rate of only about 2.5×10^{-6} s^{-1}. A state with no allowed one-photon decay is emptied by the fastest of the slower processes, and for hydrogen 2s that is two-photon emission.

U1

10

CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQMII-15.04 · UQMII-15.07 **[1]**

Slow electrons scatter from atoms of a noble gas, which are modelled as spherically symmetric attractive wells. At the energies of interest only the s wave is appreciably scattered, and at one energy its phase shift passes through $\delta_0 = \pi$. What is observed near that energy?

KEY **B** **A deep minimum in the total cross-section, because $(4\pi/k^2)\sin^2\delta_0$ vanishes, so the gas becomes nearly transparent to the electrons.**

The s-wave cross-section is $\sigma_0 = (4\pi/k^2)\sin^2\delta_0$, which vanishes when $\delta_0 = \pi$: a well that pulls the wave in by exactly half a wavelength leaves the outgoing wave indistinguishable from an unscattered one. With the higher partial waves small, the total cross-section has a deep minimum, the Ramsauer-Townsend effect (B). A describes $\delta_0 = \pi/2$, where $\sin^2\delta_0 = 1$ and the s wave scatters as strongly as unitarity allows. C misreads Levinson's theorem, which relates the phase shift at zero energy to the number of bound states; bound states have negative energy, and a phase shift passing through π at a positive energy creates none. D borrows a classical picture that fails twice over: phase shifts of 0 and π are physically identical for scattering, and even a genuine hard sphere at low energy scatters $4\pi R^2$, not πR^2.

U1

11

CORE COURSE

STRUCTURED · ROUTINE · UQMII-04.04 · UQMII-04.07 · UQMII-05.04 **[11]**

Two electrons, each held in one of a pair of neighbouring quantum dots, interact through the exchange Hamiltonian $H = (J/\hbar^2)S_1 \cdot S_2$, with $J = 2.0$ μeV. Their orbital states are frozen, so only the spins evolve. Write $|\uparrow\downarrow\rangle$ for spin 1 up and spin 2 down along z, and similarly for the other product states. Take $\hbar = 1.055 \times 10^{-34}$ J s and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) SHOW THAT · U2 **[2]**

Show that the eigenvalues of H are J/4 for each of the three triplet states and −3J/4 for the singlet.

1. $S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$, so $S_1 \cdot S_2 = (S^2 - S_1^2 - S_2^2)/2 = (\hbar^2/2)[s(s + 1) - 3/2]$ on states of total spin s
2. s = 1 (the three triplet states) gives $S_1 \cdot S_2 = \hbar^2/4$ and $E = J/4$; s = 0 (the singlet) gives $S_1 \cdot S_2 = -3\hbar^2/4$ and $E = -3J/4$, a triplet-singlet splitting of J

   Candidates who diagonalise a 4 × 4 matrix spend ten minutes on what the identity for S^2 does in one line. Those who drop the ½ from $S_1 \cdot S_2 = (S^2 - S_1^2 - S_2^2)/2$ find eigenvalues twice too large.

(b) DETERMINE · U2 **[3]**

At t = 0 the spins are prepared in $|\uparrow\downarrow\rangle$. By expressing this state in the coupled basis, determine the probability P(t) that a measurement at time t finds the spins in $|\downarrow\uparrow\rangle$.

1. $|\uparrow\downarrow\rangle = (|T_0\rangle + |S\rangle)/\sqrt{2}$, with $|T_0\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ and $|S\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$
2. each coupled state evolves with its own energy: $|\psi(t)\rangle = [e^{-iJt/4\hbar}|T_0\rangle + e^{3iJt/4\hbar}|S\rangle]/\sqrt{2}$, so $\langle\downarrow\uparrow|\psi(t)\rangle = [e^{-iJt/4\hbar} - e^{3iJt/4\hbar}]/2$
3. $P(t) = |1 - e^{iJt/\hbar}|^2/4 = \sin^2(Jt/2\hbar)$, oscillating between 0 and 1

The relative sign of $|\downarrow\uparrow\rangle$ in the singlet decides everything. A candidate who adds probabilities, a half from each coupled state, instead of amplitudes finds P constant at ½ and misses the swap entirely.

(c) CALCULATE · U3 **[1]**

Calculate the shortest time after which the two spins have completely exchanged their states.

1. P = 1 first when $Jt/2\hbar = \pi/2$, at $t = \pi\hbar/J = \pi \times 1.055 \times 10^{-34}\ \text{J s}/(2.0 \times 10^{-6} \times 1.602 \times 10^{-19}\ \text{J}) = 1.03 \times 10^{-9}$ s, about 1.0 ns

Using h instead of ħ gives 6.5 ns. The swap takes half a period of the triplet-singlet beat, whose angular frequency is J/ħ.

(d) DETERMINE · U2 **[3]**

Determine the reduced density matrix of spin 1 at time t and its purity $\text{Tr}\,\rho_1^2$, and hence the earliest time at which spin 1 is maximally mixed.

1. $|\psi(t)\rangle = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$ with $|a|^2 = \cos^2(Jt/2\hbar)$ and $|b|^2 = \sin^2(Jt/2\hbar)$; tracing over spin 2 removes the cross terms because they carry $\langle\downarrow|\uparrow\rangle = 0$ for spin 2, so $\rho_1 = \cos^2(Jt/2\hbar)|\uparrow\rangle\langle\uparrow| + \sin^2(Jt/2\hbar)|\downarrow\rangle\langle\downarrow|$
2. $\text{Tr}\,\rho_1^2 = \cos^4(Jt/2\hbar) + \sin^4(Jt/2\hbar) = 1 - \tfrac{1}{2}\sin^2(Jt/\hbar)$, which reaches its minimum of ½ when $Jt/\hbar = \pi/2$
3. $t = \pi\hbar/(2J) = 0.52$ ns, half the swap time: $\rho_1 = I/2$, its Bloch vector (0, 0, cos(Jt/ħ)) is at the centre of the ball, and the pair is in the maximally entangled pure state $(|\uparrow\downarrow\rangle - i|\downarrow\uparrow\rangle)/\sqrt{2}$, up to an overall phase, with entanglement entropy ln 2

The cross terms vanish because the two components carry orthogonal states of spin 2, not because the coefficients happen to be real. At the moment of maximal mixing the pair as a whole is in a pure state: a mixed part does not mean a mixed whole.

(e) EXPLAIN · U1 **[2]**

At that time an experimenter measures spin 1 alone, along any axis they choose. Explain why no such measurement can distinguish it from a spin drawn from an ensemble prepared half in $|\uparrow\rangle$ and half in $|\downarrow\rangle$, and what kind of measurement would reveal the difference.

1. every outcome probability for a measurement on spin 1 alone is $\text{Tr}(\rho_1\Pi)$ for the projector Π of that outcome, and both preparations have $\rho_1 = I/2$, so every axis gives ½ and ½: preparations with the same density operator cannot be told apart by any measurement on that system
2. the difference lies only in correlations between the two spins, which need joint measurements: the entangled state has $\langle\sigma_x \otimes \sigma_y\rangle = +1$, whereas a mixture of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ has zero transverse correlations (a measurement in the Bell basis would also distinguish them)

The answer that the spin is secretly up or down, and simply not yet known, is the error this part catches. The randomness of spin 1 comes from its entanglement with spin 2, and no measurement on one side can reveal it.

12

CORE COURSE

STRUCTURED · DEMANDING · UQMII-06.05 · UQMII-06.06 · UQMII-07 **[11]**

An electron in a cubic InAs nanocrystal is modelled as a particle of charge $-e$ and effective mass $m^* = 0.023m_e$ in a cubical infinite well of side $L = 20$ nm centred on the origin, $-L/2 < x, y, z < L/2$. Spin is ignored, since its Zeeman energy is the same for every orbital state. The unperturbed states are $\psi_{n_1n_2n_3} = u_{n_1}(x)u_{n_2}(y)u_{n_3}(z)$, with energies $(n_1^2 + n_2^2 + n_3^2)\varepsilon$, where $\varepsilon = \pi^2\hbar^2/(2m^*L^2)$, $u_1(x) = \sqrt{(2/L)}\cos(\pi x/L)$ and $u_2(x) = \sqrt{(2/L)}\sin(2\pi x/L)$. A uniform magnetic field $B = 1.5$ T is applied along z. In the symmetric gauge, and to first order in B, it adds $H' = (\mu^*B/\hbar)L_z$, where $\mu^* = e\hbar/(2m^*)$ is the effective magneton and $L_z = -i\hbar(x\,\partial/\partial y - y\,\partial/\partial x)$; the term in B^2 is neglected. You may use $\int u_1(x)\,x\,u_2(x)\,dx = 16L/(9\pi^2)$ and $\int u_1(du_2/dx)\,dx = 8/(3L) = -\int u_2(du_1/dx)\,dx$, each integral running from $-L/2$ to $L/2$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C and 1 eV $= 1.602 \times 10^{-19}$ J.

(a) CALCULATE · U3 **[2]**

Calculate ε in meV, and the energies and degeneracies of the ground level and of the first excited level.

1. $\varepsilon = \pi^2\hbar^2/(2m^*L^2) = \pi^2(1.055 \times 10^{-34}\text{ J s})^2/(2 \times 0.023 \times 9.109 \times 10^{-31}\text{ kg} \times (20 \times 10^{-9}\text{ m})^2) = 6.55 \times 10^{-21}\text{ J} = 40.9\text{ meV}$
2. the ground level ψ_{111} lies at $3\varepsilon = 123$ meV and is non-degenerate; the first excited level lies at $6\varepsilon = 245$ meV and is threefold degenerate, spanned by ψ_{211}, ψ_{121} and ψ_{112}

   Omitting the effective mass gives $\varepsilon = 0.94$ meV, over forty times too small. The threefold degeneracy of the first excited level, on which the rest of the question turns, should be noticed here.

(b) CALCULATE · U3 **[2]**

Calculate μ^*B in meV, and its ratio to the spacing 3ε that separates the first excited level from the levels on either side of it.

1. $\mu^*B = e\hbar B/(2m^*) = (1.602 \times 10^{-19}\text{ C})(1.055 \times 10^{-34}\text{ J s})(1.5\text{ T})/(2 \times 0.023 \times 9.109 \times 10^{-31}\text{ kg}) = 6.05 \times 10^{-22}\text{ J} = 3.78\text{ meV}$
2. $\mu^*B/3\varepsilon = 3.78\text{ meV}/123\text{ meV} = 0.031 \ll 1$, so admixtures from outside the level are at most a few per cent and second-order shifts at most about $(\mu^*B)^2/3\varepsilon \approx 0.1$ meV: first-order theory within the degenerate level is adequate

   The free-electron Bohr magneton gives $\mu_B B = 0.087$ meV, too small by the factor $m_e/m^* \approx 43$; the small effective mass is why orbital magnetic effects are so large in InAs. The ratio should be taken to the gap, not to the level energy.

(c) DETERMINE · U2 **[3]**

Using the integrals given, determine the matrix of H' in the first excited level, its eigenvalues in meV, and the good states.

1. the diagonal elements vanish because each ψ is real, and ψ_{112} is uncoupled because L_z leaves the z factor alone and $\langle u_2|u_1\rangle = 0$; the one independent element is $\langle\psi_{211}|L_z|\psi_{121}\rangle = -i\hbar[\langle u_2|x|u_1\rangle\langle u_1|u_2'\rangle - \langle u_2|u_1'\rangle\langle u_1|y|u_2\rangle] = -i\hbar \times 2 \times (16L/9\pi^2)(8/3L) = -i\hbar(256/27\pi^2) = -0.961i\hbar$
2. in the basis $(\psi_{211}, \psi_{121}, \psi_{112})$, $H' = 0.961\mu^*B\,[[0, -i, 0], [i, 0, 0], [0, 0, 0]]$, with eigenvalues $+0.961\mu^*B = +3.63$ meV, 0 and -3.63 meV
3. the good states are $(\psi_{211} + i\psi_{121})/\sqrt{2}$, raised by 3.63 meV, $(\psi_{211} - i\psi_{121})/\sqrt{2}$, lowered by 3.63 meV, and ψ_{112}, unshifted, so the level at 245 meV splits into three levels spanning 7.26 meV

   The sign of the second term is the trap: $\int u_2u_1'dx = -8/(3L)$, so the two contributions add rather than cancel. A candidate who finds the element zero, or who applies the non-degenerate formula to ψ_{211} alone and finds no shift, concludes wrongly that the level does not split.

(d) EXPLAIN · U1 **[2]**

Explain how the rotation through 90° about the z-axis identifies these good states without any diagonalisation.

1. the rotation R, which sends ψ(x, y, z) to ψ(y, −x, z), maps the cube onto itself and so commutes with H_0, and it commutes with L_z, the generator of rotations about z, and so with H′; because u_1 is even and u_2 odd, it takes ψ_{211} to ψ_{121}, ψ_{121} to $-\psi_{211}$ and ψ_{112} to itself
2. within the level its eigenstates are $(\psi_{211} \pm i\psi_{121})/\sqrt{2}$, with eigenvalues $\mp i$, and ψ_{112}, with eigenvalue 1; the three eigenvalues are distinct, so H′ has no element between these states and they are the good states, the analogues of m = +1, −1 and 0

   *The symmetry used must belong to the cube as well as to the field. The walls break full rotational symmetry about z, so L_z does not commute with H_0 and m is not an exact quantum number; a candidate who labels the states by exact m = ±1 cannot explain why the splitting is 0.961μ*B rather than μ*B.*

(e) SHOW THAT · U2 **[2]**

The field is turned, at the same strength, to lie along the body diagonal $\hat{n} = (1, 1, 1)/\sqrt{3}$, so that $H' = (\mu^*B/\hbar)\hat{n}\cdot L$. Show that the first excited level splits in exactly the same way, into shifts of +3.63 meV, 0 and −3.63 meV.

1. the cube is unchanged by the cyclic relabelling x → y → z, so L_x couples only ψ_{121} and ψ_{112}, and L_y only ψ_{112} and ψ_{211}, with the same magnitude as L_z; in the basis (ψ_{211}, ψ_{121}, ψ_{112}) every element is $(L_k)_{ij} = -i\hbar c\varepsilon_{kij}$, with $c = 256/27\pi^2 = 0.961$
2. these are c times the matrices of an l = 1 angular momentum in the Cartesian basis, so $\hat{n}\cdot L$ has the characteristic equation $\lambda(\lambda^2 - c^2\hbar^2) = 0$ and the eigenvalues 0 and $\pm c\hbar$ for every unit vector $\hat{n}$: the shifts are 0 and $\pm 0.961\mu^*B = \pm 3.63$ meV whatever the direction of B

   The expected error is to suppose that a cube, being less symmetric than a sphere, must split differently for different field directions. At first order within a threefold level of a cube the splitting is isotropic; the anisotropy appears only at higher order in B.

13

CORE COURSE

STRUCTURED · DEMANDING · UQMII-05.07 · UQMII-09.06 · UQMII-13.05 **[12]**

A hydrogen atom in the n = 2 shell is placed in a uniform static electric field F along z; hyperfine structure is ignored. Without the field, $2s_{1/2}$ lies above $2p_{1/2}$ by the Lamb shift, L = h × 1057.8 MHz, and $2p_{3/2}$ lies h × 10.97 GHz above $2p_{1/2}$. The perturbation is H′ = eFz. In the uncoupled basis $\langle 2s|z|2p, m_l = 0\rangle = -3a_0$, with $a_0 = 5.29 \times 10^{-11}$ m, and in the convention used here $|2p_{1/2}, m_j = +\tfrac{1}{2}\rangle = \sqrt{(2/3)}|m_l = 1\rangle|{\downarrow}\rangle - \sqrt{(1/3)}|m_l = 0\rangle|{\uparrow}\rangle$ and $|2s_{1/2}, m_j = +\tfrac{1}{2}\rangle = |2s\rangle|{\uparrow}\rangle$. The 2p lifetime is 1.60 ns, and an isolated 2s atom lives for 0.122 s. Take $h = 6.626 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) SHOW THAT · U2 **[2]**

Show that H′ couples $|2s_{1/2}, +\tfrac{1}{2}\rangle$ to $|2p_{1/2}, +\tfrac{1}{2}\rangle$ and to no other $2p_{1/2}$ state, with a matrix element of magnitude $\sqrt{3}\, ea_0F$.

1. z does not act on spin and, being proportional to rY_1^0, cannot change m_l, so it conserves m_j: $|2s_{1/2}, +\tfrac{1}{2}\rangle$ couples only to $m_j = +\tfrac{1}{2}$, and within $|2p_{1/2}, +\tfrac{1}{2}\rangle$ only the $|m_l = 0\rangle|{\uparrow}\rangle$ component contributes
2. $\langle 2s_{1/2}, +\tfrac{1}{2}|eFz|2p_{1/2}, +\tfrac{1}{2}\rangle = eF \times (-\sqrt{(1/3)}) \times (-3a_0) = \sqrt{3}\, ea_0F$

   Candidates who include the m_l = 1 component have forgotten that z cannot change m_l. The √3 is the Clebsch-Gordan weight √(1/3) multiplying the $3a_0$ of the uncoupled element.

(b) EXPLAIN · U1 **[2]**

Explain why the $2s_{1/2}$ level has no linear Stark shift in a weak field, although the n = 2 level of hydrogen is often said to show a linear Stark effect of $\pm 3ea_0F$.

1. the Lamb shift separates $2s_{1/2}$ from $2p_{1/2}$, so in a weak field the two $2s_{1/2}$ states ($m_j = \pm\frac{1}{2}$) are degenerate only with each other and both have definite (even) parity; every first-order matrix element between them, $\langle 2s_{1/2}|eFz|2s_{1/2}\rangle$, vanishes because z is odd, so the leading shift is second order and, dominated by $2p_{1/2}$, about $3e^2a_0^2F^2/L$
2. the linear effect $\pm 3ea_0F$ assumes 2s and $2p_0$ exactly degenerate, so that the field mixes them at first order; with the Lamb shift present the shift becomes linear only once $\sqrt{3}\ ea_0F$ is large compared with L, and reaches $\pm 3ea_0F$ only when the field also overwhelms the 10.97 GHz fine structure

The textbook linear Stark effect belongs to a hydrogen atom without fine structure or Lamb shift. A state of definite parity has no first-order shift in any field too weak to overcome the splitting that isolates it.

(c) DETERMINE · U2 **[3]**

Neglecting $2p_{3/2}$, determine the energies of the two $m_j = +\frac{1}{2}$ states as functions of F, measured from the midpoint between $2s_{1/2}$ and $2p_{1/2}$, and the field at which the coupling equals half the Lamb shift.

1. in the basis $\{|2s_{1/2}, +\frac{1}{2}\rangle, |2p_{1/2}, +\frac{1}{2}\rangle\}$ the Hamiltonian is $[[L/2, V], [V, -L/2]]$, with $V = \sqrt{3}\ ea_0F$
2. $E\pm = \pm\sqrt{(L^2/4 + 3e^2a_0^2F^2)}$: for $\sqrt{3}\ ea_0F \ll L$ the 2s-like level rises quadratically, by $3e^2a_0^2F^2/L$, and for $\sqrt{3}\ ea_0F \gg L$ the shifts become linear, $\pm\sqrt{3}\ ea_0F$
3. $V = L/2$ at $F = L/(2\sqrt{3}\ ea_0) = (6.626 \times 10^{-34} \times 1057.8 \times 10^6\ \text{J})/(2\sqrt{3} \times 1.602 \times 10^{-19}\ \text{C} \times 5.29 \times 10^{-11}\ \text{m}) = 2.39 \times 10^4\ \text{V m}^{-1}$, about 240 V cm^{-1}

Non-degenerate perturbation theory used at every field gives a quadratic shift that grows without limit. The 2 × 2 solution shows the crossover from quadratic to linear, near 240 V cm^{-1}, which is the scale the word 'weak' refers to.

(d) CALCULATE · U3 **[3]**

The 2s-like state decays through the $2p_{1/2}$ component that the field mixes into it. Calculate its lifetime in a field of 2.0 V cm^{-1}.

1. with $F = 200\ \text{V m}^{-1}$, $V/h = \sqrt{3}\ ea_0F/h = \sqrt{3} \times 1.602 \times 10^{-19} \times 5.29 \times 10^{-11} \times 200/(6.626 \times 10^{-34})$ Hz = 4.43 MHz
2. the admixed 2p probability is $(V/L)^2 = (4.43/1057.8)^2 = 1.75 \times 10^{-5}$, so the decay rate is $1.75 \times 10^{-5}/(1.60 \times 10^{-9}\ \text{s}) = 1.10 \times 10^4\ \text{s}^{-1}$, beside which the field-free rate $1/(0.122\ \text{s}) = 8.2\ \text{s}^{-1}$ is negligible
3. $\tau = 1/(1.10 \times 10^4\ \text{s}^{-1}) = 91\ \mu\text{s}$, about 1300 times shorter than the field-free lifetime: a field of a few V cm^{-1} quenches the metastable state

The admixture is a probability, $(V/L)^2$, not an amplitude. A candidate who multiplies the 2p rate by V/L finds 0.38 µs, some 240 times too short; one who mixes angular frequency for L with ordinary frequency for V is out by $(2\pi)^2$.

(e) JUSTIFY · U4 **[2]**

Justify the neglect in (d) of the $2p_{3/2}$ level and of the natural width of the 2p state.

1. $|2p_{3/2}, +\frac{1}{2}\rangle = \sqrt{(1/3)}|m_l = 1\rangle|\downarrow\rangle + \sqrt{(2/3)}|m_l = 0\rangle|\uparrow\rangle$, orthogonal to the $2p_{1/2}$ state, is coupled by $\sqrt{6}\ ea_0F$, twice as strongly in square, but it lies $10.97 - 1.06 = 9.91$ GHz from $2s_{1/2}$, so its admixture is $2 \times (1.058/9.91)^2 \approx 0.023$ of the $2p_{1/2}$ one: it adds about 2% to the rate
2. the 2p natural width is $\Gamma/2\pi = 1/(2\pi \times 1.60\ \text{ns}) = 99.5$ MHz; including it replaces L^2 by $L^2 + (\hbar\Gamma/2)^2$, a fractional change of $(49.7/1057.8)^2 \approx 0.2\%$, so neither neglect changes the 91 µs by more than a few per cent

A justification must size what it neglects. 'The $2p_{3/2}$ level is far away' earns nothing until the larger matrix element has been set against the larger detuning.

14

CORE COURSE

STRUCTURED · DEMANDING · UQMII-10.01 · UQMII-10.02 · UQMII-10.06 **[12]**

A heavy quark and antiquark bound in a meson are modelled non-relativistically as one particle of reduced mass μ, with μc² = 0.75 GeV, in the linear confining potential V(r) = σr, with σ = 0.90 GeV fm⁻¹. Only l = 0 is considered. The exact ground-state energy is $E_0 = 2.3381\varepsilon_0$, where $\varepsilon_0 = (\hbar^2\sigma^2/2\mu)^{(1/3)}$ and −2.3381 is the first zero of the Airy function Ai; for any s state of this potential it can also be shown that $|\psi(0)|^2 = \mu\sigma/(2\pi\hbar^2)$ exactly. Take ħc = 0.1973 GeV fm, and $\int_0^\infty r^n e^{(-\beta r)}\,dr = n!/\beta^{n+1}$.

(a) SHOW THAT · U2 **[2]**

Show that for the normalised trial function $\psi_\beta(r) = \sqrt{(\beta^3/\pi)}\, e^{(-\beta r)}$, $\langle H\rangle = \hbar^2\beta^2/(2\mu) + 3\sigma/(2\beta)$.

1. $\langle T\rangle = (\hbar^2/2\mu)\int|\nabla\psi|^2\,d^3r$, and $|\nabla\psi|^2 = \beta^2|\psi|^2$ for this function, so $\langle T\rangle = \hbar^2\beta^2/(2\mu)$ (equivalently, from the full radial Laplacian with the integrals given)
2. $\langle r\rangle = (\beta^3/\pi) \times 4\pi\int_0^\infty r^3 e^{(-2\beta r)}\,dr = 4\beta^3 \times 3!/(2\beta)^4 = 3/(2\beta)$, so $\langle V\rangle = 3\sigma/(2\beta)$

Candidates who use $-(\hbar^2/2\mu)d^2\psi/dr^2$ alone, dropping the $(2/r)d\psi/dr$ term of the three-dimensional Laplacian, find a negative kinetic energy, which should stop them at once.

(b) DETERMINE · U2 **[3]**

Determine the value of β that minimises ⟨H⟩, and show that the resulting bound is $E_{exp} = 3(3/4)^{(2/3)}\varepsilon_0 = 2.476\varepsilon_0$.

1. $d\langle H\rangle/d\beta = \hbar^2\beta/\mu - 3\sigma/(2\beta^2) = 0$ gives $\beta^3 = 3\mu\sigma/(2\hbar^2)$
2. at the minimum $3\sigma/(2\beta) = \hbar^2\beta^2/\mu$, twice the kinetic term (the virial theorem for a linear potential), so $\langle H\rangle = 3\hbar^2\beta^2/(2\mu)$
3. substituting $\beta^2 = (3\mu\sigma/2\hbar^2)^{(2/3)}$ gives $E_{exp} = 3(\hbar^2/2\mu)^{(1/3)}(3\sigma/4)^{(2/3)} = 3(3/4)^{(2/3)}\varepsilon_0 = 2.476\varepsilon_0$

Checking $\langle V\rangle = 2\langle T\rangle$ at the minimum catches an arithmetic slip before the bound is quoted. A candidate who substitutes a wrong β into ⟨H⟩ still obtains an upper bound, only a poorer one, so the check matters.

(c) CALCULATE · U3 **[2]**

Calculate ε_0, the exponential bound and the exact ground-state energy in GeV, and the percentage by which the bound exceeds the exact value.

1. $\varepsilon_0 = [(\hbar c)^2\sigma^2/(2\mu c^2)]^{(1/3)} = [(0.1973\ \text{GeV fm})^2 \times (0.90\ \text{GeV fm}^{-1})^2/(1.50\ \text{GeV})]^{(1/3)} = 0.276$ GeV
2. $E_{exp} = 2.476 \times 0.276$ GeV $= 0.683$ GeV against $E_0 = 2.3381 \times 0.276$ GeV $= 0.645$ GeV: the bound is 5.9% high, and above the exact value, as it must be

A bound below 0.645 GeV would signal an error: the theorem allows a trial energy only above the exact one. It says nothing about how far above, which is why the exact value is needed to judge the trial function.

(d) EXPLAIN · U1 **[2]**

A Gaussian trial function $e^{(-\alpha r^2)}$, optimised in the same way, gives $2.3448\varepsilon_0$, within 0.3% of the exact value, although an exponential is the exact ground state of hydrogen. Explain why the Gaussian does so much better in this potential.

1. for a potential that is finite at r = 0, the s-wave equation gives $u = r\psi = c_1 r + c_3 r^3 + \ldots$ with no r^2 term, so ψ has zero slope at the origin; the exponential's cusp, $\psi \approx 1 - \beta r$, is right only where a 1/r singularity balances it, and here it costs kinetic energy for nothing, whereas the Gaussian is smooth at the origin
2. the energy is decided by the shape of ψ where $|\psi|^2 r^2$ is large; the exact function decays as $\exp[-(2/3)(r/l)^{(3/2)}]$ with $l = (\hbar^2/2\mu\sigma)^{(1/3)}$, between the two trials, and the Gaussian's smooth, rounded peak matches the bulk of the exact function far better than the exponential's cusp

The tempting answer is that a Gaussian is 'more like an oscillator'. The decisive difference is at r = 0: a cusp belongs to a 1/r potential, and paying for one where the potential is smooth wastes kinetic energy.

(e) EVALUATE · U3 **[3]**

The meson's decay rate to a lepton pair is proportional to $|\psi(0)|^2$. The optimal Gaussian trial, $\psi = (2\alpha/\pi)^{(3/4)}\, e^{(-\alpha r^2)}$, has $\alpha^{(3/2)} = (\mu\sigma/3\hbar^2)\sqrt{(2/\pi)}$. A student proposes using it to predict $|\psi(0)|^2$, arguing that an energy within 0.3% of the exact value shows the trial state to be accurate. Evaluate this proposal.

1. $|\psi(0)|^2 = (2\alpha/\pi)^{(3/2)} = (2/\pi)^{(3/2)}(\mu\sigma/3\hbar^2)\sqrt{(2/\pi)} = 4\mu\sigma/(3\pi^2\hbar^2)$
2. against the exact $\mu\sigma/(2\pi\hbar^2)$ the ratio is $8/(3\pi) = 0.85$, so the Gaussian predicts a decay rate 15% too low; the exponential trial, with $\beta^3 = 3\mu\sigma/(2\hbar^2)$, gives $\beta^3/\pi = 3\mu\sigma/(2\pi\hbar^2)$, three times too high
3. the proposal should be rejected: $\langle H\rangle$ is stationary under variations of ψ, so a first-order error in ψ gives only a second-order error in E, while $|\psi(0)|^2$ has no such protection and is wrong at first order; an accurate energy certifies an upper bound on E, not the wavefunction, least of all its value at one point, where the weight $r^2|\psi|^2$ contributes nothing to E

   This is the sentence that separates the answers: a variational energy certifies an upper bound and nothing else. A 0.3% energy can accompany a 15% error in a local property of the state.

15

CORE
COURSE

STRUCTURED · DEMANDING · UQMII-13.03 · UQMII-13.06 · UQMII-13.07 **[11]**

For spontaneous emission by an electron from an upper state $|b\rangle$ to a lower state $|a\rangle$ at angular frequency ω, Einstein's detailed-balance argument with the Planck spectrum gives $A = \omega^3|\langle a|er|b\rangle|^2/(3\pi\varepsilon_0\hbar c^3)$. The absorption oscillator strength from a non-degenerate state $|a\rangle$ to a level b is $f = (2m_e\omega/3\hbar)\Sigma|\langle b|r|a\rangle|^2$, summed over the degenerate states of b, and for one electron the Thomas-Reiche-Kuhn sum rule gives $\Sigma f = 1$ over all levels b. As a model, take an electron bound in the isotropic three-dimensional harmonic potential $\tfrac{1}{2}m_e\omega^2r^2$, with states $|n_x, n_y, n_z\rangle$ and $\langle 0|x|1\rangle = \sqrt{(\hbar/2m_e\omega)}$ for each Cartesian direction. Take $e = 1.602 \times 10^{-19}$ C, $m_e = 9.109 \times 10^{-31}$ kg, $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, $c = 2.998 \times 10^8$ m s^{-1}, $k_B = 1.381 \times 10^{-23}$ J K^{-1} and 1 u $= 1.661 \times 10^{-27}$ kg.

(a) EXPLAIN · U1 **[2]**

Explain why an atom treated by perturbation theory in a classical field would, in an excited state with no applied field, never decay, and how Einstein's argument nevertheless fixes A.

1. with no applied field the perturbation $er \cdot E$ is zero, and the excited state is an eigenstate of the atomic Hamiltonian, so it is stationary: the semiclassical theory has no spontaneous emission
2. in equilibrium with thermal radiation, absorption at the rate $B\rho(\omega)$ must balance stimulated emission plus spontaneous emission, $A + B\rho(\omega)$, between Boltzmann-populated levels; with Planck's $\rho(\omega)$ this forces $A = (\hbar\omega^3/\pi^2c^3)B$, and B is computed from the golden rule in a classical field (the quantum-electrodynamic derivation, which gives the same A, is quoted)

   Candidates often say the excited state decays because it is unstable. In this theory it is an eigenstate and nothing makes it decay; the marks are for seeing that A is borrowed from equilibrium with Planck's law.

(b) SHOW THAT · U2 **[2]**

Show that for the oscillator the rate from $|1, 0, 0\rangle$ to $|0, 0, 0\rangle$ is $A = e^2\omega^2/(6\pi\varepsilon_0m_ec^3)$, and comment on the absence of $\hbar$.

1. only the x component connects the two states: $|\langle 0,0,0|r|1,0,0\rangle|^2 = |\langle 0|x|1\rangle|^2 = \hbar/(2m_e\omega)$, the y and z components vanishing because they would have to change n_y or n_z
2. $A = \omega^3e^2(\hbar/2m_e\omega)/(3\pi\varepsilon_0\hbar c^3) = e^2\omega^2/(6\pi\varepsilon_0m_ec^3)$; $\hbar$ cancels, and the result is exactly the classical radiation-damping rate of a charged oscillator, as the correspondence principle requires

   Candidates who include y and z components, or take $|\langle 0|x|1\rangle|^2 = \hbar/(m_e\omega)$ without the 2, double the rate. The disappearance of $\hbar$ is the check: a harmonically bound charge radiates as it does classically.

(c) DETERMINE · U2 **[2]**

Determine the absorption oscillator strength from $|0, 0, 0\rangle$ to the first excited level, and state what the Thomas-Reiche-Kuhn sum rule then implies for every other transition from the ground state.

1. the first excited level has three states, $|1,0,0\rangle$, $|0,1,0\rangle$ and $|0,0,1\rangle$, each with $|\langle b|r|0,0,0\rangle|^2 = \hbar/(2m_e\omega)$, so $f = (2m_e\omega/3\hbar) \times 3\hbar/(2m_e\omega) = 1$
2. this one transition exhausts the sum rule, so every other transition from the ground state has $f = 0$, consistent with the dipole operator changing a single quantum number by one

   A sum rule exhausted by one line is a strong statement. Candidates who expect the oscillator to absorb at 2ω as well have forgotten what the dipole operator can do in one step.

(d) CALCULATE · U3 **[3]**

The sodium resonance lines near 589.0 nm (3s to 3p) have statistical weights in the same 1 : 3 ratio as the oscillator's, so for them $A = f \times e^2\omega^2/(6\pi\varepsilon_0 m_e c^3)$. The measured 3p lifetime is 16.2 ns. Calculate the oscillator's lifetime at 589.0 nm, and hence the oscillator strength of the sodium resonance transition.

1. $\omega = 2\pi c/\lambda = 2\pi \times 2.998 \times 10^8$ m s^{-1}/$(589.0 \times 10^{-9}$ m$) = 3.198 \times 10^{15}$ rad s^{-1}
2. $A = e^2\omega^2/(6\pi\varepsilon_0 m_e c^3) = 6.41 \times 10^7$ s^{-1}, so the oscillator lifetime is 15.6 ns
3. $f = A_{Na}/A_{osc} = 15.6$ ns$/16.2$ ns $= 0.96$: the single 3s–3p transition carries about 96% of the one-electron sum, the rest going to 3s–np with $n \geq 4$ and to the continuum

   The ratio of the two lifetimes is the oscillator strength. A value above 1 would contradict the sum rule and signals a slip in ω, most often a wavelength left in nanometres or a frequency used in place of an angular frequency.

(e) COMPARE · U3 **[2]**

Compare the natural linewidth of the sodium line with its Doppler width in a vapour at 500 K, for which the full width at half maximum is $\Delta\nu_D = (\nu/c)\sqrt{(8k_B T \ln 2/M)}$, with $M = 22.99$ u.

1. natural width $\Delta\nu = 1/(2\pi\tau) = 1/(2\pi \times 16.2 \times 10^{-9}$ s$) = 9.8$ MHz, a Lorentzian full width
2. $\Delta\nu_D = [1/(589.0 \times 10^{-9}$ m$)] \times \sqrt{(8 \times 1.381 \times 10^{-23}$ J K$^{-1} \times 500$ K $\times \ln 2/(22.99 \times 1.661 \times 10^{-27}$ kg$)) = (1.698 \times 10^6$ m$^{-1})(1.00 \times 10^3$ m s$^{-1}) = 1.70$ GHz, about 170 times wider, so the vapour line is Doppler-dominated and the natural width shows only in the far wings or in Doppler-free or cold-atom measurements

   Candidates who quote $1/\tau$ as the width in hertz are out by 2π. The comparison is the reason the natural width of an allowed optical line is seen only once the Doppler width has been removed.

16

CORE COURSE

STRUCTURED · DISCRIMINATING · UQMII-14.02 · UQMII-14.05 · UQMII-14.06 · UQMII-14.07 **[11]**

A superconducting qubit is driven close to resonance. In the frame rotating with the drive its Hamiltonian is $H(\varphi) = (\hbar/2)[\Delta\sigma_z + \Omega(\cos\varphi\,\sigma_x + \sin\varphi\,\sigma_y)]$, where the detuning is $\Delta/2\pi = 3.0$ MHz, the Rabi frequency is $\Omega/2\pi = 4.0$ MHz, and φ is the phase of the drive, which can be ramped. Equivalently $H = (\hbar\Omega_{eff}/2)\,\hat{n}\cdot\sigma$, with $\Omega_{eff} = \sqrt{(\Delta^2 + \Omega^2)}$ and $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. In the experiment φ is ramped at a constant rate from 0 to 2π in a time $T = 4.0$ µs, so that $\hat{n}$ traces a circle on a cone about the z-axis.

(a) STATE · U1 **[2]**

State the eigenvalues of H, and the angle θ between $\hat{n}$ and the z-axis.

1. $E\pm = \pm\hbar\Omega_{eff}/2 = \pm(\hbar/2)\sqrt{(\Delta^2 + \Omega^2)}$, that is $\pm h \times 2.5$ MHz, a gap of $h \times 5.0$ MHz; the two levels meet only at $\Delta = \Omega = 0$
2. $\cos\theta = \Delta/\Omega_{eff} = 3/5$ ($\sin\theta = 4/5$), so $\theta = 53.1°$

   The drive is part of the Hamiltonian in this frame, so the eigenvalues are $\pm\hbar\Omega_{eff}/2$, not $\pm\hbar\Delta/2$. A candidate who forgets it places $\hat{n}$ along z and finds no geometric phase at all.

(b) SHOW THAT · U2 **[2]**

Show that the adiabatic condition for the ramp becomes $T \gg \pi\Omega/(\Delta^2 + \Omega^2)$, and evaluate the right-hand side.

1. $dH/dt = (2\pi/T)\partial H/\partial\varphi = (2\pi/T)(\hbar\Omega/2)(-\sin\varphi\,\sigma_x + \cos\varphi\,\sigma_y)$; the unit vector $(-\sin\varphi, \cos\varphi, 0)$ is perpendicular to $\hat{n}$, so its matrix element between the eigenstates along $\pm\hat{n}$ has magnitude 1, and $|\langle -|dH/dt|+\rangle| = \pi\hbar\Omega/T$
2. the condition $\hbar|\langle -|dH/dt|+\rangle| \ll (E_+ - E_-)^2$ becomes $\pi\hbar^2\Omega/T \ll \hbar^2(\Delta^2 + \Omega^2)$, so $T \gg \pi\Omega/(\Delta^2 + \Omega^2) = \pi(2\pi \times 4.0 \times 10^6\ \mathrm{s}^{-1})/(2\pi \times 5.0 \times 10^6\ \mathrm{s}^{-1})^2 = 80$ ns, and the 4.0 µs ramp is fifty times longer

The rate of change must be compared with the square of the gap divided by $\hbar$, not with the gap. Candidates who compare $2\pi/T$ with Ω_{eff} reach a condition of the right order but cannot show where the factor Ω/Ω_{eff} comes from.

(c) CALCULATE · U3 **[3]**

Calculate the geometric phase acquired in one circuit by each of the two eigenstates, and the difference between their dynamic phases.

1. $\hat{n}$ sweeps a cone of half-angle θ, enclosing a solid angle $2\pi(1 - \cos\theta) = 2\pi \times 2/5 = 0.8\pi$ sr
2. the geometric phase is minus half the solid angle, -0.4π rad ($-72°$), for the state along $+\hat{n}$ (the upper level) and $+0.4\pi$ rad for the state along $-\hat{n}$, so the two differ by 0.8π
3. the dynamic phases differ by $\Omega_{eff}\,T = 2\pi \times 5.0$ MHz $\times$ 4.0 µs $= 40\pi$ rad, twenty full turns, proportional to both T and Ω_{eff}: a 1% error in Ω_{eff} shifts it by 0.4π, as much as the geometric phase itself

The solid angle of a cone is $2\pi(1 - \cos\theta)$, not $\pi\theta^2$. A candidate who notes that 40π is zero modulo 2π misses the point: it is the drift of the dynamic phase with Ω_{eff} and T that has to be removed before the geometric phase can be seen.

(d) PREDICT · U1 **[2]**

In a spin-echo version, the qubit starts in an equal superposition of the two eigenstates at $\varphi = 0$, is taken once round the circuit, has its two eigenstates exchanged by a π pulse, and is then taken round the circuit in the opposite sense in the same time T. Predict the relative phase between the two components at the end, and how it changes if T is doubled or if $\Delta/2\pi$ is raised to 6.0 MHz with Ω unchanged.

1. each component spends time T in each eigenstate, so the dynamic phases cancel, while exchanging the eigenstates and reversing the sense of the circuit each reverse the sign of a component's geometric phase, so the geometric phases add: the relative phase is $2 \times 0.8\pi = 1.6\pi$ rad (288°, equivalently -0.4π), unchanged when T is doubled as long as the ramp stays adiabatic
2. with $\Delta/2\pi = 6.0$ MHz, $\cos\theta = 6/\sqrt{52} = 0.832$ and the relative phase becomes $4\pi(1 - 0.832) = 0.67\pi$ rad (121°): it depends only on the geometry of the circuit

Candidates who expect the echo to cancel both phases have not seen that the two reversals, of the eigenstate labels and of the circuit, together leave the geometric phase adding. Independence of T is the experimental signature that the phase is geometric.

(e) EXPLAIN · U1 **[2]**

The same circuit also bounds the rest of the sphere, a cap of solid angle 4π minus the one used in (c). Explain why that cap gives the same phase factor, and what this has to do with the degeneracy of H at $\Delta = \Omega = 0$.

1. seen from the other cap the circuit runs in the opposite sense, so for the $+\hat{n}$ state $\gamma = +\tfrac{1}{2}(4\pi - 0.8\pi) = 2\pi - 0.4\pi$, which equals -0.4π modulo 2π: e^(iγ) is the same whichever surface is used
2. the Berry curvature is the field of a monopole at the degeneracy, the origin of the space of $(\Omega\cos\varphi, \Omega\sin\varphi, \Delta)$, where the two eigenvalues meet; its total flux for a spin-half is 2π in magnitude, a multiple of 2π, which is why the phase of a closed circuit, the flux through it, is unambiguous

Surface-dependent Berry phases are the error this part is set to catch. The phase is defined modulo 2π only because the flux of the monopole at the degeneracy is quantised.

17

CORE COURSE

DATA ANALYSIS · DEMANDING · UQMII-09.07 · UQMII-09.08 · UQMII-09.09 **[10]**

A hydrogen maser oscillates on the transition from $F = 1, m_F = 0$ to $F = 0$ in the hyperfine structure of the ground state, whose zero-field frequency is ν_{hfs}. A small static field B along z is applied by a solenoid around the storage bulb, but the ambient field is not fully cancelled. The field inside the bulb is measured by the Zeeman frequency f_Z, the low-frequency transition between neighbouring $F = 1$ sublevels, which in these weak fields is $f_Z = \mu_B B/h$. For each solenoid current the maser's output is compared with a reference oscillator, giving a frequency offset $\delta\nu$ that also contains a constant but unknown contribution from the reference and from collisions with the bulb wall. With $g_e = 2$ and the proton's magnetic moment neglected, the ground-state Hamiltonian is $H = A\,\mathbf{S}_e \cdot \mathbf{S}_p/\hbar^2 + (2\mu_B B/\hbar)S_{ez}$, where $A = h\nu_{hfs}$. Take $\mu_B = 9.274 \times 10^{-24}$ J T^{-1} and $h = 6.626 \times 10^{-34}$ J s.

(a) SHOW THAT · U2 **[3]**

Working in the $m_F = 0$ subspace spanned by $|{\uparrow\Downarrow}\rangle$ and $|{\downarrow\Uparrow}\rangle$ (electron spin first), show that the frequency of the $F = 1, m_F = 0$ to $F = 0$ transition is $\nu_c = \sqrt{(\nu_{hfs}^2 + 4f_Z^2)} \approx \nu_{hfs} + 2f_Z^2/\nu_{hfs}$.

1. $\mathbf{S}_e \cdot \mathbf{S}_p = S_{ez}S_{pz} + \tfrac{1}{2}(S_{e+}S_{p-} + S_{e-}S_{p+})$ has diagonal elements $-\hbar^2/4$ and off-diagonal elements $\hbar^2/2$ in this subspace, while the Zeeman term adds $+\mu_B B$ to $|{\uparrow\Downarrow}\rangle$ and $-\mu_B B$ to $|{\downarrow\Uparrow}\rangle$
2. the matrix $[[-A/4 + \mu_B B, A/2], [A/2, -A/4 - \mu_B B]]$ has eigenvalues $-A/4 \pm \sqrt{(A^2/4 + \mu_B^2 B^2)}$, which at $B = 0$ are $A/4$ ($F = 1$) and $-3A/4$ ($F = 0$)
3. $h\nu_c = 2\sqrt{(A^2/4 + \mu_B^2 B^2)}$, so $\nu_c = \sqrt{(\nu_{hfs}^2 + 4f_Z^2)} \approx \nu_{hfs} + 2f_Z^2/\nu_{hfs}$ when $f_Z \ll \nu_{hfs}$: the clock transition has no term linear in B

Candidates who keep only the diagonal elements find ν_c independent of B; those who use the weak-field linear result find a shift proportional to $m_F = 0$, which is zero, and miss the quadratic term the maser has to control.

(b) DESCRIBE · U4 **[2]**

Describe how the readings should be plotted to test this model and to extract ν_{hfs}, and explain why the solenoid current should not be used as the measure of the field.

1. plot $\delta\nu$ against f_Z^2: the model predicts a straight line $\delta\nu = c + (2/\nu_{hfs})f_Z^2$, the unknown constant offset c appearing only in the intercept and the gradient being $2/\nu_{hfs}$
2. B is not proportional to I, because the uncancelled ambient field adds to the solenoid's (here $f_Z = 0.10$ kHz $+ 0.30$ kHz mA^{-1} × I, a residual of about 7 nT), and the solenoid's calibration inside the bulb is not known; f_Z measures the field the atoms actually experience

A plot against I^2 gives a gradient in mHz mA^{-2}, which cannot yield ν_{hfs} without an independent calibration of the solenoid, and the residual field makes it curve slightly. The Zeeman frequency is the atoms' own magnetometer.

(c) DETERMINE · U4 **[3]**

Determine the gradient of your graph and hence a value of ν_{hfs}, and comment on its agreement with the accepted 1420.406 MHz.

1. $f_Z^2 = 0.160$, 0.490, 1.00, 1.69, 2.56 and 3.61 kHz^2, and the six points lie on a straight line with intercept $c \approx 3.50$ mHz
2. gradient = 1.41 mHz kHz^{-2} (accept 1.39 to 1.42 mHz kHz^{-2}), which is 1.41×10^{-9} Hz^{-1}
3. $\nu_{hfs} = 2/\text{gradient} = 2/(1.41 \times 10^{-9}\ Hz^{-1}) = 1.42 \times 10^{9}$ Hz (accept 1.41 to 1.44 GHz), agreeing with 1420.406 MHz within the half-per-cent uncertainty of the gradient, which confirms the quadratic Zeeman model; the neglected proton moment would change the gradient by only about 0.6%

Candidates who keep f_Z in kilohertz and $\delta\nu$ in millihertz must carry the factor of 10^{-9} through; an answer near 1.4 MHz or 1.4 THz signals a lost 10^3. That a millihertz shift of the maser returns the gigahertz hyperfine frequency is a strong test of the model.

(d) ESTIMATE · U4 **[2]**

The maser is normally operated at $f_Z = 0.80$ kHz. Estimate how constant f_Z, and hence the field, must be kept if field fluctuations are to change the maser frequency by less than 1 part in 10^{15}, stating your assumptions.

1. $d\nu_c/df_Z = 4f_Z/\nu_{hfs} = 4 \times 800\ Hz/(1.4204 \times 10^{9}\ Hz) = 2.25 \times 10^{-6}$, and the tolerance $10^{-15} \times 1.42 \times 10^{9}$ Hz $= 1.4 \times 10^{-6}$ Hz allows a change in f_Z of about 0.63 Hz
2. that is a fractional stability of $\delta f_Z/f_Z \approx 8 \times 10^{-4}$: the field of about 57 nT must be held to about 0.05 nT, assuming the quadratic law of (a) holds and every other frequency shift stays constant

Candidates who require the whole quadratic shift, rather than its change, to stay below 10^{-15} conclude that the field must be almost zero. It is the fluctuation that matters, and the derivative $4f_Z/\nu_{hfs}$ converts a field fluctuation into a frequency fluctuation.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMII-12.06 · UQMII-15.02 · UQMII-15.05 · UQMII-15.08 · UQMII-15.09 **[15]**

A particle of mass m and energy $E = \hbar^2k^2/2m$ is scattered elastically by a potential V(r) that vanishes at large r. To apply Fermi's golden rule, the system is placed in a cubic box of side L with periodic boundary conditions, so that the plane-wave states $\langle r|k\rangle = e^{(ik\cdot r)}/L^{(3/2)}$ are normalised and the incident state carries a probability current density $\hbar k/(mL^3)$. The golden rule gives the rate $\Gamma = (2\pi/\hbar)|\langle f|V|i\rangle|^2\rho(E)$ into a group of final states whose density per unit energy is $\rho(E)$. Write $\tilde{V}(q) = \int e^{(-iq\cdot r)}V(r)\,d^3r$. Take $m_e = 9.109 \times 10^{-31}$ kg, $e = 1.602 \times 10^{-19}$ C, $c = 2.998 \times 10^{8}$ m s^{-1} and $\alpha = 1/137.0$.

(a) DERIVE · U2 **[3]**

Derive, from the periodic boundary conditions, the density of final states per unit energy $\rho(E)$ for plane waves whose wavevectors point into a small solid angle $d\Omega$.

1. periodic boundary conditions allow $k = (2\pi/L)(n_x, n_y, n_z)$ with integer n_x, n_y and n_z, one state for each volume $(2\pi/L)^3 = (2\pi)^3/L^3$ of k-space
2. the states with wavevectors in a shell k to k + dk inside $d\Omega$ number $dN = L^3k^2\,dk\,d\Omega/(2\pi)^3$
3. with $E = \hbar^2k^2/2m$, $dE = \hbar^2k\,dk/m$, so $\rho(E) = dN/dE = L^3mk\,d\Omega/(8\pi^3\hbar^2)$

Candidates who count states on the whole sphere of radius k lose the $d\Omega$, and those who use $2\pi/L$ rather than $(2\pi/L)^3$ for the k-space volume per state lose a factor of L^2. Either leaves the cross-section depending on the box.

(b) DERIVE · U2 **[3]**

Using the golden rule, derive the differential cross-section dσ/dΩ for elastic scattering in terms of $\tilde{V}(q)$, where $q = k_f - k_i$, and confirm that it does not depend on L.

1. $\langle k_f|V|k_i\rangle = L^{-3}\int e^{-i(k_f - k_i)\cdot r}V(r)\, d^3r = \tilde{V}(q)/L^3$, with $|k_f| = |k_i| = k$ by energy conservation
2. the rate into dΩ is $d\Gamma = (2\pi/\hbar)(|\tilde{V}|^2/L^6)(L^3 mk\, d\Omega/8\pi^3\hbar^2) = mk|\tilde{V}(q)|^2\, d\Omega/(4\pi^2\hbar^3L^3)$
3. dividing by the incident current density $\hbar k/(mL^3)$ gives $d\sigma/d\Omega = (m/2\pi\hbar^2)^2|\tilde{V}(q)|^2$, in which L^3 has cancelled between the matrix element, the density of states and the flux

The box must cancel. A candidate whose cross-section still contains L has mishandled one of the three places L enters: the normalisation of the states, the density of states, or the flux.

(c) SHOW THAT · U2 **[2]**

For the screened Coulomb potential $V(r) = (g/r)e^{-r/a}$, show that $\tilde{V}(q) = 4\pi g/(q^2 + 1/a^2)$.

1. for a central potential, integrating over the angle between q and r with $\int_{-1}^{1} e^{-iqru}\, du = 2\sin(qr)/(qr)$ gives $\tilde{V}(q) = (4\pi/q)\int_0^\infty rV(r)\sin(qr)\, dr$
2. $\tilde{V}(q) = (4\pi g/q)\int_0^\infty e^{-r/a}\sin(qr)\, dr = (4\pi g/q) \times q/(q^2 + 1/a^2) = 4\pi g/(q^2 + 1/a^2)$

Candidates who expand $e^{-iq\cdot r}$ in partial waves spend a page on what the angular integral does in a line. $\int_0^\infty e^{-\lambda r}\sin(qr)\, dr = q/(q^2 + \lambda^2)$ is the whole of the radial work.

(d) DETERMINE · U2 **[2]**

Take the limit $a \to \infty$ with $g = Z_1Z_2e^2/(4\pi\varepsilon_0)$. Determine dσ/dΩ as a function of the scattering angle θ, and compare it with Rutherford's classical result, $d\sigma/d\Omega = g^2/(16E^2\sin^4(\theta/2))$.

1. for elastic scattering through θ, $q = |k_f - k_i| = 2k\sin(\theta/2)$, so $d\sigma/d\Omega = (m/2\pi\hbar^2)^2(4\pi g)^2/q^4 = 4m^2g^2/(\hbar^4q^4) = m^2g^2/(4\hbar^4k^4\sin^4(\theta/2))$
2. with $\hbar^2k^2 = 2mE$ this is $g^2/(16E^2\sin^4(\theta/2))$, identical to Rutherford's result; the total cross-section diverges at small θ, which a finite screening length a removes

That the first Born approximation reproduces Rutherford's formula exactly is a special property of the Coulomb potential, not evidence that the approximation is good. The divergence of the total cross-section without screening is why a finite a was needed.

(e) EXPLAIN · U1 **[2]**

The first Born amplitude, $f(\theta) = -(m/2\pi\hbar^2)\tilde{V}(q)$, is real for this potential, so Im f(0) = 0, yet the total cross-section it gives is positive. Explain why this does not contradict the optical theorem, $\sigma = (4\pi/k)\,\mathrm{Im}\, f(0)$.

1. the Born cross-section is of second order in V, so the optical theorem relates it to the imaginary part of the forward amplitude at second order; the first-order amplitude is real for a real potential, and Im f(0) first appears in the second Born term
2. the theorem holds order by order in V: the first Born approximation conserves probability only to first order, and its failure to satisfy the optical theorem on its own is a statement about the order of the approximation, not a violation of unitarity

Candidates who conclude that the optical theorem fails in the Born approximation have compared quantities of different order in V. The real amplitude is consistent with a positive cross-section once each side is taken at the same order.

(f) ESTIMATE · U4 **[3]**

For a Coulomb-like potential of strength $g = Ze^2/(4\pi\varepsilon_0)$, the first Born approximation is reliable when $g/(\hbar v) = Z\alpha c/v \ll 1$, where v is the speed of the incident particle. Estimate whether the Born cross-section can be trusted for electrons of kinetic energy 2.0 keV scattered by neutral helium (Z = 2) and by neutral iron (Z = 26), stating your assumptions.

1. $v = \sqrt{(2E/m_e)} = \sqrt{(2 \times 2.0 \times 10^3 \times 1.602 \times 10^{-19}\ \text{J}/9.109 \times 10^{-31}\ \text{kg})} = 2.65 \times 10^7\ \text{m s}^{-1}$, so $v/c = 0.088$ and non-relativistic kinematics are adequate
2. $Z\alpha c/v = Z \times (1/137.0)/0.0885 = 0.0825Z$, which is 0.17 for helium and 2.1 for iron
3. the Born cross-section is adequate for helium, with corrections of order $Z\alpha c/v$, about 17%, but fails for iron, where partial waves or an exact treatment are needed; the estimate assumes that the scattering that matters happens close to the nucleus, where screening is weak, and ignores exchange with, and excitation of, the atomic electrons

The criterion compares the strength of the potential with $\hbar v$, not the range with the wavelength; high energy helps only because v grows. Iron fails at 2 keV because Z enters linearly.

19

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UQMII-11.04 · UQMII-11.09 **[15]**

A particle of mass m moves in one dimension in the potential $V(x) = \tfrac{1}{2}m\omega^2x^2(1 - x/L)$, the model used for the phase of a current-biased Josephson junction. V has a minimum at x = 0 and a barrier beyond which, for x > L, it falls without limit, so the ground state of the well is metastable and escapes by tunnelling. For the numerical parts take $\omega/2\pi = 8.0$ GHz, with L such that the barrier height is $V_b = 3.0\hbar\omega$. A full treatment, matching the Gaussian ground state of the well onto the decaying WKB solution under the barrier, gives the escape rate $\Gamma = (\omega/2\pi)\sqrt{(864\pi V_b/\hbar\omega)}\ \exp(-36V_b/5\hbar\omega)$, which is quoted here. You may use $\int_0^1 u\sqrt{(1-u)}\,du = 4/15$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) SHOW THAT · U2 **[2]**

Show that the top of the barrier lies at x = 2L/3 and that $V_b = (2/27)m\omega^2L^2$.

1. $dV/dx = m\omega^2(x - 3x^2/2L) = m\omega^2x(1 - 3x/2L)$, which vanishes at x = 0, the minimum, and at x = 2L/3, the maximum
2. $V(2L/3) = \tfrac{1}{2}m\omega^2(4L^2/9)(1/3) = (2/27)m\omega^2L^2$

Candidates who look for the barrier at the zeros of V confuse x = L, where V returns to zero, with the top. The barrier height grows as L^2, which is why the escape rate is so sensitive to the parameters that set L.

(b) SKETCH · U1 **[2]**

Sketch V(x) from x = −L/3 to x = 4L/3, marking V_b, the ground-state energy $\tfrac{1}{2}\hbar\omega$ and the classical turning points at that energy, and on the same axes sketch the magnitude of the metastable ground-state wavefunction.

1. V: a parabolic minimum at x = 0 with V rising for x < 0, a maximum V_b at x = 2L/3, back to zero at x = L and falling without limit beyond; the line $E = \tfrac{1}{2}\hbar\omega = V_b/6$ meets V at two points inside the well, near $x \approx -0.15L$ and $x \approx 0.17L$, and once more on the outer slope just inside x = L (near 0.97L)
2. $|\psi|$: a Gaussian-like hump centred in the well, decaying roughly exponentially through the barrier by several orders of magnitude, then emerging beyond the outer turning point as a small outgoing oscillatory wave that does not decay, with a wavelength that shortens as V falls

The sketch must show that the wavefunction beyond the barrier does not decay: an outgoing wave carries probability away, which is what makes the state metastable rather than bound.

(c) DERIVE · U2 **[3]**

Starting from the WKB tunnelling exponent $2S/\hbar = (2/\hbar)\int\sqrt{(2m(V - E))}\,dx$ taken across the barrier, derive its value for an energy $E \to 0$ at the bottom of the well, and express it in terms of $V_b/\hbar\omega$.

1. at $E \to 0$ the turning points bounding the barrier are $x = 0$ and $x = L$, where $V = 0$, and between them $\sqrt{(2mV)} = m\omega x\sqrt{(1 - x/L)}$
2. $\int_0^L m\omega x\sqrt{(1 - x/L)}\,dx = m\omega L^2\int_0^1 u\sqrt{(1 - u)}\,du = (4/15)m\omega L^2$, so $2S/\hbar = (8/15)m\omega L^2/\hbar$
3. from (a), $m\omega^2L^2 = (27/2)V_b$, so $m\omega L^2/\hbar = (27/2)V_b/\hbar\omega$ and $2S/\hbar = (8/15)(27/2)V_b/\hbar\omega = 36V_b/(5\hbar\omega) = 7.2V_b/\hbar\omega$, the exponent of the quoted result

Candidates who integrate from $x = 0$ to $2L/3$ treat the top of the barrier as a turning point. The WKB integral runs between the two points where $V = E$, which at $E = 0$ are $x = 0$ and $x = L$.

(d) CALCULATE · U3 **[2]**

Calculate, for $V_b = 3.0\hbar\omega$, the naive escape rate $\Gamma_0 = (\omega/2\pi)\exp(-2S/\hbar)$, which pictures the particle striking the barrier once per classical period, and the quoted rate Γ.

1. $2S/\hbar = 7.2 \times 3.0 = 21.6$ and $e^{-21.6} = 4.2 \times 10^{-10}$, so $\Gamma_0 = 8.0 \times 10^9\ s^{-1} \times 4.2 \times 10^{-10} = 3.3\ s^{-1}$
2. $\sqrt{(864\pi \times 3.0)} = 90$, so $\Gamma = 90 \times 3.3\ s^{-1} = 3.0 \times 10^2\ s^{-1}$, a lifetime of 3.3 ms: the naive estimate is too low by a factor of about 90

An exponent of 21.6 gives a factor of 4×10^{-10}, so a candidate who writes $e^{-7.2}$ has dropped $V_b/\hbar\omega = 3.0$. The factor of 90 in the prefactor is equivalent to changing the exponent by 4.5, a large correction for a method whose exponent is right.

(e) EVALUATE · U1 **[3]**

Evaluated between its turning points at the actual ground-state energy $E = \tfrac{1}{2}\hbar\omega$, the barrier integral gives $2S/\hbar = 17.15$ for these parameters. Evaluate the naive estimate in the light of this: identify which of its approximations causes most of the discrepancy, and explain why the difficulty is worst for the lowest state.

1. $(\omega/2\pi)e^{-17.15} = 8.0 \times 10^9\ s^{-1} \times 3.6 \times 10^{-8} = 2.9 \times 10^2\ s^{-1}$, within about 5% of the quoted $3.0 \times 10^2\ s^{-1}$
2. so most of the factor of 90 comes from setting $E = 0$ in the exponent: $E_0 = \tfrac{1}{2}\hbar\omega = V_b/6$ is not small compared with V_b, and lowering the exponent by 4.45 multiplies the rate by about 86; the $\sqrt{(V_b/\hbar\omega)}$ in the quoted prefactor is this energy dependence of the exponent, expanded about $E = 0$
3. near the bottom of the well the classically allowed region is only about one zero-point width across, so neither the WKB wavefunction nor the classical attempt frequency is valid there, and the slope of the exponent, $d(2S/\hbar)/dE$, grows logarithmically as E falls towards the bottom of the well, so the exponent cannot simply be taken at $E = 0$; for highly excited states the Gamow form with the true energy works, but the ground state needs its Gaussian matched onto the WKB tail

The 'failure of the prefactor' for the lowest state is the energy dependence of the exponent near the bottom of the well. Candidates who blame the attempt frequency alone miss that a shift of $\tfrac{1}{2}\hbar\omega$ in E changes the exponent by more than 4.

(f) ESTIMATE · U4 **[3]**

Thermal activation over the barrier occurs at a rate of order $(\omega/2\pi)\exp(-V_b/k_BT)$. Estimate the temperature below which escape is dominated by tunnelling, stating your assumptions, and say what an experiment would observe below it.

1. comparing exponents, and assuming the prefactors are of comparable size, tunnelling wins when $V_b/k_BT > 7.2V_b/\hbar\omega$, that is below $k_BT^* = \hbar\omega/7.2$
2. $\hbar\omega = 1.055 \times 10^{-34}\ \text{J s} \times 2\pi \times 8.0 \times 10^{9}\ \text{s}^{-1} = 5.30 \times 10^{-24}\ \text{J}$, so $\hbar\omega/k_B = 0.384\ \text{K}$ and $T^* \approx 53\ \text{mK}$ (a fuller treatment gives $\hbar\omega/(2\pi k_B) = 61\ \text{mK}$, the same order)
3. below T^* the escape rate stops falling with temperature and levels off at the tunnelling rate, the signature of macroscopic quantum tunnelling; at 53 mK, $k_BT \approx 0.14\hbar\omega$, so the excited levels of the well are scarcely populated, justifying the ground-state treatment, and dissipation is neglected throughout

An estimate without its assumptions earns two of the three marks. Comparing exponents ignores the prefactors, which move the crossover only logarithmically.

University Advanced Quantum Mechanics · Midterm Examination

The first eight units of Advanced Quantum Mechanics on one paper: time reversal and Kramers degeneracy, identical particles scattering with spin, the Klein–Gordon equation and its exact Coulomb levels, Dirac's free spinors and their boosts, the exact Dirac–Coulomb spectrum, and the Klein step, Zitterbewegung and hole theory. The extended derivation takes the Dirac equation to its non-relativistic limit and derives g = 2, the spin–orbit term with its Thomas half and the Darwin term, which Quantum Mechanics II quoted.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**74**	**14**	**8**

SECTION — **UAQM · Midterm · University Advanced Quantum Mechanics — Midterm examination**

IN THE FORMAT OF — A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation

DEMAND — **Recall 3 · Routine 6 · Demanding 3 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UAQM-01 · UAQM-02 · UAQM-03 · UAQM-04 · UAQM-05 · UAQM-06 · UAQM-07 · UAQM-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Advanced Quantum Mechanics · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 **[1]**

For a system of total angular momentum J, the time-reversal operator satisfies Θ² = (−1)^(2J), whatever phase is chosen for it. Options A to C place an ion at a crystal site with no point symmetry at all, so the crystal field removes every degeneracy that symmetry does not protect; option D is a free atom. In which case does time-reversal invariance guarantee that every level of the ground multiplet stays at least twofold degenerate?

A Ho^{3+}, whose ground multiplet has J = 8, in the crystal field alone

B Er^{3+}, whose ground multiplet has J = 15/2, in the crystal field and a uniform static magnetic field of 0.10 T

C Er^{3+}, whose ground multiplet has J = 15/2, in the crystal field and a uniform static electric field of 1.0×10^{7} V/m

D A free hydrogen atom in its 1s level, the proton's spin included, in a uniform static electric field of 1.0×10^{7} V/m

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-02.06 **[1]**

Two electrons scatter in their centre-of-mass frame through a spin-independent interaction whose direct amplitude is f(θ). Beam and target are unpolarised, and a detector at θ = 90° counts an electron whichever of the two it is. In units of $|f(90°)|^2$, what differential cross-section does the detector measure?

A 0

B 1

C 2

D 4

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-03.07 · UAQM-03.08 **[1]**

For a spinless particle bound to a point nucleus of charge Ze, the exact Klein–Gordon energies are E = mc²[1 + (Zα)²/(n − l − ½ + s)²]^(−1/2), with s = √((l + ½)² − (Zα)²) and α = 1/137.04. Above about what Z does the formula first fail, giving a complex energy for the 1s state?

A 69

B 137

C 172

D 206

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-04.07 · UAQM-04.08 **[1]**

A free Dirac particle has H = cα·p + βmc², where α_i and β are the 4 × 4 Dirac matrices, Σ = diag(σ, σ) is the spin matrix and L = r × p. Which statement is correct?

A [H, L] = 0, because a free particle experiences no torque

B [H, Σ_z] = 0, because H contains no magnetic field to turn the spin

C [H, β] = 0, so the upper and lower components of ψ evolve independently

D [H, Σ·p] = 0, so helicity is conserved, although Σ_z on its own is not

Answer

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-05.05 · UAQM-05.06 **[1]**

Under parity a Dirac field transforms as ψ(r, t) → γ⁰ψ(−r, t), with ψ̄ = ψ†γ⁰ and γ⁵ = iγ⁰γ¹γ²γ³. Which bilinear is unchanged by every proper Lorentz transformation but changes sign under parity?

A ψ̄ψ

B ψ̄γ⁰ψ

C ψ̄γ⁵ψ

D ψ̄γ⁰γ⁵ψ

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-06.03 · UAQM-06.04 **[1]**

An electron is held in a Penning trap whose magnetic field is 4.70 T. Its cyclotron frequency is $f_c = eB/(2\pi m)$, with $e/m = 1.75882 \times 10^{11}$ C kg^{-1}. In the exact Dirac Landau levels, where $g = 2$, the level with n quanta of cyclotron motion and spin up along B coincides with the level with n + 1 quanta and spin down. Taking $g = 2(1 + a)$, with $a = 1.15965 \times 10^{-3}$, and neglecting relativistic shifts and the trap's electric field, at what frequency must a drive be applied to connect these two levels?

A 152.6 MHz

B 305.1 MHz

C 131.72 GHz

D 76.3 MHz

Answer ☐

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-07.06 · UAQM-07.07 **[1]**

In the exact Dirac–Coulomb spectrum of a hydrogen-like ion with a point nucleus and $Z < 137$, which statement is correct?

A 2S_(1/2) and 2P_(1/2) coincide only to order $(Z\alpha)^4$; the exact energies separate them at order $(Z\alpha)^6$, and that separation is the Lamb shift.

B 3P_(3/2) and 3D_(3/2) coincide exactly, because the energies depend only on n and on $|\kappa| = j + \tfrac{1}{2}$.

C 3D_(3/2) and 3D_(5/2) coincide exactly, because the energies depend only on n and l.

D 2S_(1/2), with $\kappa = -1$, and 2P_(1/2), with $\kappa = +1$, are separated, because the energies depend on the sign of κ.

Answer ☐

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-08.02 · UAQM-08.06 **[1]**

An electron of total energy $E = 2mc^2$, so kinetic energy mc^2, meets an electrostatic step on which its potential energy rises by V_0. Beyond the step its momentum $\hbar q$ obeys $c^2\hbar^2q^2 = (E - V_0)^2 - m^2c^4$. For which step heights is the transmitted wave oscillatory rather than evanescent?

A $V_0 < mc^2$ only: any higher step is penetrated only evanescently

B $V_0 < 2mc^2$ only: a step below the total energy can be crossed

C $V_0 < mc^2$ and $V_0 > 2mc^2$

D $V_0 < mc^2$ and $V_0 > 3mc^2$

Answer ☐

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9 CORE COURSE

STRUCTURED · ROUTINE · UAQM-01.02 · UAQM-01.06 · UAQM-01.08 · UAQM-08.07 **[8]**

For a spin-½ particle, the time-reversal operator in the basis of S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ is $\Theta = \exp(-i\pi S_y/\hbar)K = -i\sigma_y K$, where K takes the complex conjugate of every component in that basis. Θ is antiunitary: $\Theta(a|u\rangle + b|v\rangle) = a^*\Theta|u\rangle + b^*\Theta|v\rangle$ and $\langle\Theta u|\Theta v\rangle = \langle v|u\rangle$. The Pauli matrices are $\sigma_x = [[0, 1], [1, 0]]$, $\sigma_y = [[0, -i], [i, 0]]$ and $\sigma_z = [[1, 0], [0, -1]]$, and $S = (\hbar/2)\sigma$.

(a) DETERMINE **[2]**

Determine $\Theta|\uparrow\rangle$ and $\Theta|\downarrow\rangle$, and hence Θ^2. Show that replacing Θ by $e^{(i\delta)}\Theta$, for any real δ, leaves Θ^2 unchanged.

(b) SHOW THAT **[1]**

Show that $\Theta\sigma\Theta^{-1} = -\sigma$, so that $\Theta S\Theta^{-1} = -S$.

(c) JUSTIFY **[2]**

A particle, or an atom, whose only degeneracy is the $2j + 1$ values of m in a rotation-invariant theory is said to have a permanent electric dipole moment d if $\langle j, m|d|j, m\rangle \neq 0$. Using the Wigner–Eckart theorem within the multiplet, justify the claim that such a moment requires both parity and time-reversal invariance to fail.

(d) EXPLAIN **[1]**

The $n = 2$ level of hydrogen shows a linear Stark effect, a shift proportional to the applied field, as though the atom had a permanent dipole moment. Explain why this does not contradict (c).

(e) SHOW THAT **[2]**

For a Dirac spinor in the Dirac representation, $\gamma^i = [[0, \sigma_i], [-\sigma_i, 0]]$ for $i = 1, 2, 3$ in 2×2 blocks, and time reversal acts as $T = i\gamma^1\gamma^3 K$. Show that $T^2 = -1$.

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10 CORE COURSE

STRUCTURED · DEMANDING · UAQM-03.01 · UAQM-03.03 · UAQM-03.04 · UAQM-03.06 · UAQM-03.07 **[11]**

A spinless particle of mass m and charge $-e$ is bound by a point nucleus of charge Ze, treated as the fixed source of the static potential energy $V(r) = -Z\alpha\hbar c/r$. Covariant notation uses $x^\mu = (ct, r)$, the metric $g = \mathrm{diag}(1, -1, -1, -1)$ and $p^\mu = i\hbar\partial^\mu$, so that $p^0 = (i\hbar/c)\partial/\partial t$ and $p = -i\hbar\nabla$; minimal coupling replaces the energy E by $E - V$, and a stationary state has $\psi \propto e^{(-iEt/\hbar)}$. Schrödinger's hydrogen problem, $-(\hbar^2/2M)u'' + [\hbar^2\lambda(\lambda + 1)/(2Mr^2) - k/r]u = \varepsilon u$, has bound states $\varepsilon = -Mk^2/[2\hbar^2(n_r + \lambda + 1)^2]$, $n_r = 0, 1, 2, \ldots$, for any real $\lambda > -½$, taking the solution regular at the origin, $u \propto r^{(\lambda+1)}$. Take $\alpha = 1/137.036$.

(a) DERIVE **[2]**

Derive the free Klein–Gordon equation from $p_\mu p^\mu = m^2c^2$, and show that with the Coulomb potential included a stationary state obeys $[(E - V)^2 - m^2c^4]\psi = -\hbar^2c^2\nabla^2\psi$.

(b) DETERMINE **[3]**

With $\psi = [u(r)/r]Y_l^m(\theta, \varphi)$, the equation becomes $-\hbar^2c^2u'' + [\hbar^2c^2l(l + 1) - (Z\alpha\hbar c)^2]u/r^2 - 2EZ\alpha\hbar c\, u/r = (E^2 - m^2c^4)u$. By mapping it onto Schrödinger's problem, determine the exact bound-state energies, writing $n = n_r + l + 1$.

(c) SHOW THAT **[2]**

Show that, to order $(Z\alpha)^4$, $E = mc^2 - mc^2(Z\alpha)^2/(2n^2) - [mc^2(Z\alpha)^4/(2n^4)][n/(l + \tfrac{1}{2}) - \tfrac{3}{4}]$.

(d) CALCULATE **[2]**

The charged-pion mass is determined from the X-rays of the 5g → 4f transition in pionic nitrogen, a π^- bound to ^{14}N ($Z = 7$), where the strong interaction is negligible for these orbits. Treat the pion as moving in the static field of the nucleus with its mass replaced by the reduced mass μ, using $m_\pi c^2 = 139.570$ MeV and a nuclear rest energy of 13040.20 MeV. Calculate the transition energy to order $(Z\alpha)^2$, and the order-$(Z\alpha)^4$ correction to it, in eV.

(e) SKETCH **[2]**

On one energy-level diagram, sketch the $n = 2$ levels of a spinless particle of the electron's mass bound to a proton, from (c) with $Z = 1$, beside the $n = 2$ levels of hydrogen given by the Dirac equation, $E = mc^2 - mc^2\alpha^2/(2n^2) - [mc^2\alpha^4/(2n^4)][n/(j + \tfrac{1}{2}) - \tfrac{3}{4}]$. Mark each level's shift from the unperturbed $n = 2$ level in units of $mc^2\alpha^4/128$, and label the hydrogen levels with the terms of the fine-structure Hamiltonian that move them away from the Klein–Gordon levels.

11 CORE COURSE

STRUCTURED · DEMANDING · UAQM-04.01 · UAQM-04.06 · UAQM-05.04 · UAQM-08.01 **[10]**

In the Dirac representation the free Dirac Hamiltonian is $H = c\alpha\cdot p + \beta mc^2$, with $\beta = [[I, 0], [0, -I]]$ and $\alpha_i = [[0, \sigma_i], [\sigma_i, 0]]$ in 2×2 blocks, and $\bar{\psi} = \psi^\dagger\beta$. An electron moves along $+z$ with momentum $p = (3/4)mc$. Positive-energy spinors are normalised to $u^\dagger u = 2E/c$, and the matrix that carries a rest spinor into the spinor of an electron moving along $+z$ with Lorentz factor γ is $S = \cosh(\eta/2) + \alpha_z \sinh(\eta/2)$, where $\cosh\eta = \gamma$. Take $mc^2 = 0.511$ MeV, $\hbar = 1.055 \times 10^{-34}$ J s, $\hbar c = 197.3$ MeV fm and 1 MeV $= 1.602 \times 10^{-13}$ J.

(a) DERIVE **[3]**

From the Dirac equation $Hu = Eu$, derive the positive-energy spinor $u(p, \uparrow)$, with spin up along z, in the form $N(1, 0, b, 0)$, finding E, b and N.

(b) SHOW THAT **[2]**

Show that S applied to the rest spinor $\sqrt{(2mc)}(1, 0, 0, 0)$ gives the spinor of (a), and that S is not unitary, in a way consistent with $\bar{u}u$ staying at $2mc$ while $u^\dagger u$ grows to $2E/c$.

(c) CALCULATE **[2]**

Calculate $\langle c\alpha_z\rangle = u^\dagger c\alpha_z u/u^\dagger u$ for this spinor, and compare it with the group velocity $\partial E/\partial p$.

(d) EXPLAIN **[1]**

Every eigenvalue of $c\alpha_z$ is $+c$ or $-c$. Explain how this is consistent with (c).

(e) ESTIMATE **[2]**

A wave packet built from both positive- and negative-energy solutions near this momentum shows Zitterbewegung. Estimate the angular frequency and the amplitude of the trembling, stating your assumptions, and say what happens to it in a packet built from positive-energy solutions alone.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UAQM-07.02 · UAQM-07.04 · UAQM-07.05 · UAQM-07.06 · UAQM-07.08 · UAQM-07.09 **[11]**

An electron of mass m is bound to a point nucleus of charge Ze, with Z < 137, so that V(r) = −Zαℏc/r. In a central field the Dirac Hamiltonian commutes with J, with the Dirac parity operator and with K = β(Σ·L + ℏ), whose eigenvalue is written −κℏ, so that κ = −(j + ½) for j = l + ½ and κ = +(j + ½) for j = l − ½, l being the orbital angular momentum of the upper component. Writing the upper component as [G(r)/r] times a spinor harmonic of orbital angular momentum l, and the lower as i[F(r)/r] times one of orbital angular momentum l′, the radial equations are ℏc(dG/dr + κG/r) = (E + mc² − V)F and ℏc(dF/dr − κF/r) = −(E − mc² − V)G. Take α = 1/137.036 and mc² = 511.0 keV.

(a) STATE **[2]**

State κ for each state of the n = 2 shell, 2S_(1/2), 2P_(1/2) and 2P_(3/2), together with l′ and the parity of each state.

(b) DERIVE **[3]**

For a solution with no radial nodes, try G = Ar^γ e^(−λr) and F = Br^γ e^(−λr), with λ > 0. Derive from the radial equations γ, the ratio B/A, and λ in terms of E.

(c) SHOW THAT **[2]**

Show that E = mc²γ/|κ|, and that such a nodeless solution exists only for κ < 0.

(d) CALCULATE **[2]**

For hydrogen-like lead, Pb^{81+} (Z = 82), calculate the 1s binding energy mc² − E, comparing it with the value to order (Zα)⁴, mc²[(Zα)²/2 + (Zα)⁴/8], and the fraction of the 1s probability carried by the small component.

(e) PREDICT **[2]**

Predict how the 1s solution behaves as Z approaches 137 and beyond, for a point nucleus, and say what a nucleus of finite size changes.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UAQM-02.05 · UAQM-02.06 · UAQM-02.07 · UAQM-02.08 **[10]**

Identical nuclei, each in its ground state of spin s, scatter in their centre-of-mass frame through the Coulomb interaction alone, which does not act on spin; every run is far enough below the Coulomb barrier that the nuclear force can be neglected. A detector of angular half-width 0.50° centred on θ = 90° counts N_{90} nuclei, and N_R is the number it would have counted in the same run if $|f(\theta)|^2$ and $|f(\pi - \theta)|^2$ added without interference, computed from the Rutherford cross-section and the measured beam and target. For the exact Coulomb amplitude f(θ), the interference term is proportional to $\cos[\eta \ln \tan^2(\theta/2)]$, where $\eta = Z^2\alpha/\beta$, βc is the relative speed, $\beta^2 = 2E_{cm}/(\mu c^2)$ and μ is the reduced mass. Take $\alpha = 1/137.036$ and $1\ u = 931.5\ \mathrm{MeV}/c^2$, treat N_{90} as a Poisson count and N_R as exact.

Counts near 90° for six pairs of identical nuclei

Pair	Z	E_{cm} (MeV)	N_{90} (counts)	N_R (counts)
$^{12}C + {}^{12}C$	6	3.00	9958	5000
$^{13}C + {}^{13}C$	6	3.00	4035	8000
$^{14}N + {}^{14}N$	7	3.50	12046	9000
$^{10}B + {}^{10}B$	5	2.50	13650	12000
$^{11}B + {}^{11}B$	5	2.50	7462	10000
$^{17}O + {}^{17}O$	8	4.00	8371	10000

(a) SHOW THAT **[2]**

The $(2s + 1)^2$ spin states of the pair divide into $(s + 1)(2s + 1)$ that are symmetric under exchange of the two spins and $s(2s + 1)$ that are antisymmetric. Show that for an unpolarised beam and target $d\sigma/d\Omega = |f(\theta)|^2 + |f(\pi - \theta)|^2 + [2(-1)^{(2s)}/(2s + 1)]\mathrm{Re}[f(\theta)f^*(\pi - \theta)]$, and hence that $R = N_{90}/N_R$ should equal $1 + (-1)^{(2s)}/(2s + 1)$.

(b) CALCULATE **[2]**

Calculate R and its standard uncertainty for each pair.

(c) DETERMINE **[3]**

Determine the spin of each nucleus, and the significance with which the nearest alternative value is excluded. Say which assignments are secure at three standard deviations.

(d) EVALUATE **[2]**

Near 90°, $\ln\tan^2(\theta/2) \approx 2(\theta - 90°)$, with the angle in radians, so the detector averages $\cos[2\eta(\theta - 90°)]$ over its acceptance. Evaluate whether this averaging biases any spin assignment, using the pair with the largest η.

(e) EXPLAIN **[1]**

Explain why the measurement is made at 90° rather than at another angle.

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14 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-06.01 · UAQM-06.02 · UAQM-06.05 · UAQM-06.06 · UAQM-06.07 · UAQM-06.08 · UAQM-06.09 **[16]**

An electron of charge $q = -e$ and mass m moves in static potentials with Dirac Hamiltonian $H = c\alpha\cdot\pi + \beta mc^2 + V(r)$, where $\pi = p - qA$, $p = -i\hbar\nabla$ and V is the electrostatic potential energy. In the Dirac representation write $\psi = (\varphi, \chi)$, with φ the large and χ the small two-component spinor, and $E = mc^2 + E'$. You may use $(\sigma\cdot a)(\sigma\cdot b) = a\cdot b + i\sigma\cdot(a \times b)$ for vector operators a and b that commute with σ, and $[p, f(r)] = -i\hbar\nabla f$. Take $\alpha = 1/137.036$ and $mc^2 = 0.5110$ MeV.

(a) DERIVE **[3]**

Derive the two coupled equations for φ and χ. For $|E' - V| \ll mc^2$, eliminate χ to leading order and derive the Pauli Hamiltonian, showing that its spin term is $-(q/m)S\cdot B$, so that $g = 2$.

(b) ESTIMATE **[2]**

Estimate the ratio $|\chi|/|\varphi|$ and the fraction of the probability carried by χ in the ground state of hydrogen, stating your assumptions.

(c) EXPLAIN **[2]**

Now set $A = 0$. Keeping the next term in χ gives $E'\varphi = [p^2/(2m) + V - (1/(4m^2c^2))(\sigma\cdot p)(E' - V)(\sigma\cdot p)]\varphi$. Explain why this cannot be read as a Schrödinger equation for φ with a Hermitian Hamiltonian at order $(v/c)^2$.

(d) SHOW THAT **[4]**

Show that $\Phi = [1 + p^2/(8m^2c^2)]\varphi$ is normalised to order $(v/c)^2$, and that it obeys $E'\Phi = H_{FW}\Phi$ with $H_{FW} = p^2/(2m) + V - p^4/(8m^3c^2) + (\hbar/(4m^2c^2))\sigma\cdot(\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$.

(e) DETERMINE **[3]**

For a central potential $V(r)$, and then for $V = -Z\alpha\hbar c/r$, determine the spin–orbit term in the form $\xi(r)\mathbf{S}\cdot\mathbf{L}$ and the Darwin term, and evaluate the Darwin shift of the 1s level of hydrogen in eV, given $|\psi_{100}(0)|^2 = 1/(\pi a_0^3)$ with $a_0 = \hbar/(\alpha mc)$.

(f) COMPARE **[2]**

In a semiclassical picture the electron, moving with velocity $\mathbf{v}$ through the electric field $\mathcal{E} = \nabla V/e$, sees in its rest frame the magnetic field $\mathbf{B}' = -\mathbf{v} \times \mathcal{E}/c^2$. Compare the spin–orbit coefficient that $-\boldsymbol{\mu}\cdot\mathbf{B}'$ gives, with $\boldsymbol{\mu} = -g(e/2m)\mathbf{S}$, with the one in (e), account for the difference, and say what changes if $g = 2(1 + a)$.

MARK SCHEME

University Advanced Quantum Mechanics · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 **[1]**

For a system of total angular momentum J, the time-reversal operator satisfies Θ² = (−1)^(2J), whatever phase is chosen for it. Options A to C place an ion at a crystal site with no point symmetry at all, so the crystal field removes every degeneracy that symmetry does not protect; option D is a free atom. In which case does time-reversal invariance guarantee that every level of the ground multiplet stays at least twofold degenerate?

KEY **C** **Er^{3+}, whose ground multiplet has J = 15/2, in the crystal field and a uniform static electric field of 1.0 × 10^{7} V/m**

A takes Kramers' theorem to protect every ion. Holmium's J = 8 is an integer, so Θ² = +1, and a crystal field with no point symmetry can split the 17 states of the multiplet into 17 singlets, each mapped into itself by Θ; whatever pairing a non-Kramers ion shows comes from the symmetry of its site, which this site lacks. B has the right ion and the wrong field: a magnetic field is odd under time reversal, so a Hamiltonian containing −μ·B no longer commutes with Θ, and each Kramers doublet splits linearly in B, which is how such doublets are seen in electron paramagnetic resonance. D counts the electron's spin alone, but the atom also contains the proton: its total angular momentum F is 0 or 1, both integers, so Θ² = +1, and the F = 0 ground level is already a non-degenerate singlet. C is correct: J = 15/2 gives Θ² = −1, and an electric field is even under T, so H still commutes with Θ and each state is orthogonal to its time-reversed partner at the same energy. The field can shift the eight Kramers doublets but never split them, however strong it is.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-02.06 **[1]**

Two electrons scatter in their centre-of-mass frame through a spin-independent interaction whose direct amplitude is f(θ). Beam and target are unpolarised, and a detector at θ = 90° counts an electron whichever of the two it is. In units of |f(90°)|², what differential cross-section does the detector measure?

KEY **B** **1**

A is the result for beam and target fully polarised parallel: the pair is then in the spin triplet, whose spatial amplitude f(θ) − f(π − θ) vanishes at 90°; but an unpolarised pair is in the singlet a quarter of the time. C is what two distinguishable particles would give, |f(θ)|² + |f(π − θ)|² = 2|f|² at 90°, the classical count of either particle reaching the detector with no interference term. D is the result for identical spinless bosons, |f(θ) + f(π − θ)|² = 4|f|², and would hold for electrons only if they were always in the singlet. B is correct: averaging over spins weights the singlet, whose spatial amplitude is symmetric, by 1/4 and the triplet, antisymmetric, by 3/4, so dσ/dΩ = (1/4)|f + f′|² + (3/4)|f − f′|² with f′ = f(π − θ). At 90°, f′ = f, giving (1/4) × 4|f|² + 0 = |f|², half the distinguishable-particle value.

U2

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3 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-03.07 · UAQM-03.08 **[1]**

For a spinless particle bound to a point nucleus of charge Ze, the exact Klein–Gordon energies are E = mc²[1 + (Zα)²/(n − l − ½ + s)²]^(−1/2), with s = √((l + ½)² − (Zα)²) and α = 1/137.04. Above about what Z does the formula first fail, giving a complex energy for the 1s state?

KEY **A** **69**

B is the limit of the Dirac equation's point-nucleus 1s state, where √(1 − (Zα)²) becomes imaginary at Zα = 1. The Klein–Gordon limit is lower because its effective centrifugal coefficient, l(l + 1) − (Zα)², turns attractive enough to produce fall to the centre once (Zα)² exceeds (l + ½)², not κ². C is the charge at which, for a nucleus of finite size, the Dirac 1s level reaches −mc² and an empty level would be filled by pair creation, a different question from Unit 8. D is the limit for the 2p states, Zα = 3/2, since s stays real up to Zα = l + ½. A is correct: s is real only while Zα < l + ½, and for l = 0 that means Zα < ½, Z < 68.5; beyond it the solutions oscillate infinitely often near r = 0 and the point-Coulomb problem has no ground state.

U1

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-04.07 · UAQM-04.08 **[1]**

A free Dirac particle has H = cα·p + βmc², where α_i and β are the 4 × 4 Dirac matrices, Σ = diag(σ, σ) is the spin matrix and L = r × p. Which statement is correct?

KEY **D** **[H, Σ·p] = 0, so helicity is conserved, although Σ_z on its own is not**

A carries a classical argument into a Hamiltonian it does not fit: [H, L] = c[α·p, L] = −iħc(α × p), which is not zero, so orbital angular momentum alone is not conserved. B fails in the same way in the spin sector: [α_i, Σ_j] = 2iε_ijkα_k gives [H, Σ] = 2ic(α × p), and the two commutators cancel only in J = L + (ħ/2)Σ, which is why the equation itself describes a particle of spin ½. C overlooks that β anticommutes with every α_i, so [H, β] = −2cβ(α·p); the kinetic term continually mixes upper and lower components, which is how a moving electron acquires its small component. D is correct: the same relation gives [α·p, Σ·p] = 2iε_ijk p_i p_j α_k = 0 by antisymmetry, and Σ·p commutes with β, so helicity is conserved, whereas Σ_z alone changes whenever p has a component transverse to z.

U1

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5 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-05.05 · UAQM-05.06 **[1]**

Under parity a Dirac field transforms as ψ(r, t) → γ⁰ψ(−r, t), with ψ̄ = ψ†γ⁰ and γ⁵ = iγ⁰γ¹γ²γ³. Which bilinear is unchanged by every proper Lorentz transformation but changes sign under parity?

KEY **C** **ψ̄γ⁵ψ**

Under parity ψ → γ⁰ψ and ψ̄ → ψ̄γ⁰, so a bilinear ψ̄Γψ becomes ψ̄γ⁰Γγ⁰ψ at the reflected point. A is a Lorentz scalar, but γ⁰ · 1 · γ⁰ = 1 makes it even under parity: it is the true scalar. B is even under parity too, since γ⁰γ⁰γ⁰ = γ⁰, and it is not a scalar at all but the time component of the vector current, the density ψ†ψ, which a boost mixes with the spatial current. D does change sign, because γ⁰(γ⁰γ⁵)γ⁰ = −γ⁰γ⁵, but it is the time component of the axial-vector current, and a boost mixes it with ψ̄γ^iγ⁵ψ; it behaves as a pseudoscalar only under rotations. C is correct: γ⁵ anticommutes with every γ^μ and so commutes with every generator σ^(μν), giving ψ̄γ⁵ψ → ψ̄S⁻¹γ⁵Sψ = ψ̄γ⁵ψ under a proper transformation, while γ⁰γ⁵γ⁰ = −γ⁵ makes it odd under parity: a pseudoscalar.

U1

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-06.03 · UAQM-06.04 **[1]**

An electron is held in a Penning trap whose magnetic field is 4.70 T. Its cyclotron frequency is $f_c = eB/(2\pi m)$, with $e/m = 1.75882 \times 10^{11}$ C kg^{-1}. In the exact Dirac Landau levels, where $g = 2$, the level with n quanta of cyclotron motion and spin up along B coincides with the level with n + 1 quanta and spin down. Taking $g = 2(1 + a)$, with $a = 1.15965 \times 10^{-3}$, and neglecting relativistic shifts and the trap's electric field, at what frequency must a drive be applied to connect these two levels?

KEY **A** **152.6 MHz**

B doubles the anomaly. The spin-flip energy of the electron is $(g/2)\hbar\omega_c$, and going from n quanta to n + 1 removes $\hbar\omega_c$ of it, leaving $(g/2 - 1)\hbar\omega_c = a\hbar\omega_c$, not $(g - 2)\hbar\omega_c$. C is the spin-flip frequency itself, $f_s = (g/2)f_c = (1 + a)f_c = 131.72$ GHz, which connects n with spin up to n with spin down; the degenerate pair also differs by a cyclotron quantum, and that quantum cancels all of f_s except the anomaly. D takes the shift of one level for the separation of the pair: with $g = 2(1 + a)$ the spin-up level rises by $a\hbar\omega_c/2$ and the spin-down level falls by as much, so they separate by $a\hbar\omega_c$. A is correct: $f_c = (1.75882 \times 10^{11}$ C kg$^{-1} \times 4.70$ T$)/(2\pi) = 131.56$ GHz, and $f_a = af_c = 152.6$ MHz. Because the trap measures f_a and f_c together, it yields $a = f_a/f_c$ directly, which is why $g - 2$ is measured rather than g.

U3

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7 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-07.06 · UAQM-07.07 **[1]**

In the exact Dirac–Coulomb spectrum of a hydrogen-like ion with a point nucleus and $Z < 137$, which statement is correct?

KEY **B** **3P_(3/2) and 3D_(3/2) coincide exactly, because the energies depend only on n and on |κ| = j + ½.**

A mistakes a radiative correction for a relativistic one. The exact energies, $E = mc^2[1 + (Z\alpha)^2/(n_r + \gamma)^2]^{(-1/2)}$ with $\gamma = \sqrt{(\kappa^2 - (Z\alpha)^2)}$, contain κ only through κ^2, so the degeneracy survives to all orders in Zα, and the Lamb shift needs the quantised radiation field. C carries the Schrödinger degeneracy over the wrong way: in Dirac's spectrum levels of the same l and different j split, which is the fine structure, here with 3D_(5/2) above 3D_(3/2). D states the opposite of the theorem: $\kappa = -1$ and $\kappa = +1$ give the same κ^2 and, for $n = 2$, the same $n_r = 1$, so the two states coincide. B is correct: 3P_(3/2) has $\kappa = -2$ and 3D_(3/2) has $\kappa = +2$, both with $n_r = 1$, so they share one energy although they have opposite parity.

U1

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-08.02 · UAQM-08.06 **[1]**

An electron of total energy $E = 2mc^2$, so kinetic energy mc^2, meets an electrostatic step on which its potential energy rises by V_0. Beyond the step its momentum $\hbar q$ obeys $c^2\hbar^2q^2 = (E - V_0)^2 - m^2c^4$. For which step heights is the transmitted wave oscillatory rather than evanescent?

KEY **D** **$V_0 < mc^2$ and $V_0 > 3mc^2$**

A is the Schrödinger answer: a step lower than the kinetic energy is crossed and a higher one only penetrated. For the Dirac equation that holds only up to $V_0 = E + mc^2$. B compares the step with the total energy instead of the kinetic energy, and misses the gap in which $|E - V_0| < mc^2$: for $mc^2 < V_0 < 3mc^2$, q is imaginary. C uses the threshold $2mc^2$, which is the Klein threshold for an electron at rest, $E = mc^2$; for this electron the step must exceed $E + mc^2 = 3mc^2$. D is correct: q is real when $|E - V_0| > mc^2$, that is for $V_0 < E - mc^2 = mc^2$, the ordinary transmitted electron, or for $V_0 > E + mc^2 = 3mc^2$, the Klein regime, in which $E - V_0$ lies below $-mc^2$ and hole theory reads the transmitted wave as positrons created at the step.

U2

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9 CORE COURSE

STRUCTURED · ROUTINE · UAQM-01.02 · UAQM-01.06 · UAQM-01.08 · UAQM-08.07 **[8]**

For a spin-½ particle, the time-reversal operator in the basis of S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ is $\Theta = \exp(-i\pi S_y/\hbar)K = -i\sigma_y K$, where K takes the complex conjugate of every component in that basis. Θ is antiunitary: $\Theta(a|u\rangle + b|v\rangle) = a^*\Theta|u\rangle + b^*\Theta|v\rangle$ and $\langle\Theta u|\Theta v\rangle = \langle v|u\rangle$. The Pauli matrices are σ_x = [[0, 1], [1, 0]], σ_y = [[0, −i], [i, 0]] and σ_z = [[1, 0], [0, −1]], and S = (ħ/2)σ.

(a) DETERMINE · U2 **[2]**

Determine $\Theta|\uparrow\rangle$ and $\Theta|\downarrow\rangle$, and hence Θ^2. Show that replacing Θ by e^(iδ)Θ, for any real δ, leaves Θ^2 unchanged.

1. $-i\sigma_y = [[0, -1], [1, 0]]$ is real, so $\Theta|\uparrow\rangle = |\downarrow\rangle$ and $\Theta|\downarrow\rangle = -|\uparrow\rangle$; then $\Theta^2|\uparrow\rangle = \Theta|\downarrow\rangle = -|\uparrow\rangle$ and $\Theta^2|\downarrow\rangle = -\Theta|\uparrow\rangle = -|\downarrow\rangle$, so $\Theta^2 = -1$
2. (e^(iδ)Θ)(e^(iδ)Θ) = e^(iδ)e^(−iδ)$\Theta^2 = \Theta^2$, because the antilinear Θ conjugates the phase it passes through: the sign of Θ^2 belongs to the spin, not to a convention

Candidates who treat Θ as linear find $(e^{i\delta}\Theta)^2 = e^{2i\delta}\Theta^2$, conclude that the sign of Θ^2 can be chosen at will, and so lose the whole basis of Kramers' theorem.

(b) SHOW THAT · U2 **[1]**

Show that $\Theta\sigma\Theta^{-1} = -\sigma$, so that $\Theta S\Theta^{-1} = -S$.

1. with $\Theta^{-1} = K(i\sigma_y)$, $\Theta\sigma\Theta^{-1} = \sigma_y\sigma^*\sigma_y$: σ_x and σ_z are real and anticommute with σ_y, giving $\sigma_y\sigma_x\sigma_y = -\sigma_x$ and $\sigma_y\sigma_z\sigma_y = -\sigma_z$, while $\sigma_y^* = -\sigma_y$ gives $\sigma_y(-\sigma_y)\sigma_y = -\sigma_y$

(c) JUSTIFY · U1 **[2]**

A particle, or an atom, whose only degeneracy is the 2j + 1 values of m in a rotation-invariant theory is said to have a permanent electric dipole moment d if $\langle j, m|d|j, m\rangle \neq 0$. Using the Wigner–Eckart theorem within the multiplet, justify the claim that such a moment requires both parity and time-reversal invariance to fail.

1. within one multiplet the Wigner–Eckart theorem makes every vector operator proportional to J: $\langle j, m'|d|j, m\rangle = k\langle j, m'|J|j, m\rangle$, with k a single real constant because d and J are Hermitian, so a permanent moment means $k \neq 0$
2. d is odd under P and even under T, while J is even under P and odd under T; if P is a symmetry it maps the non-degenerate multiplet into itself, and $PdP^{-1} = -d$ with $PJP^{-1} = J$ forces $k = -k$, and if T is a symmetry, $\Theta d\Theta^{-1} = d$ with $\Theta J\Theta^{-1} = -J$ forces $k = -k$ again, so $k \neq 0$ needs both to fail

Answers that show only that d is odd under parity have proved that P must fail and have said nothing about T. The argument needs the proportionality to J, which carries the T-odd signature into the expectation value.

(d) EXPLAIN · U1 **[1]**

The n = 2 level of hydrogen shows a linear Stark effect, a shift proportional to the applied field, as though the atom had a permanent dipole moment. Explain why this does not contradict (c).

1. the linear effect needs states of opposite parity at one energy, 2s and 2p, so the level is not the non-degenerate multiplet that (c) assumes; the field mixes them into states that carry a dipole, and once the field is weak beside the 2S_(1/2)–2P_(1/2) splitting the shift becomes quadratic, as it must for a non-degenerate level

(e) SHOW THAT · U2 **[2]**

For a Dirac spinor in the Dirac representation, γ^i = [[0, σ_i], [−σ_i, 0]] for i = 1, 2, 3 in 2 × 2 blocks, and time reversal acts as $T = i\gamma^1\gamma^3 K$. Show that $T^2 = -1$.

1. γ^1 and γ^3 contain only σ_x and σ_z, so they are real, and $K(i\gamma^1\gamma^3) = (-i\gamma^1\gamma^3)K$; hence $T^2 = (i\gamma^1\gamma^3)(-i\gamma^1\gamma^3)K^2 = \gamma^1\gamma^3\gamma^1\gamma^3$
2. $\gamma^3\gamma^1 = -\gamma^1\gamma^3$ and $(\gamma^i)^2 = g^{ii} = -1$, so $\gamma^1\gamma^3\gamma^1\gamma^3 = -(\gamma^1)^2(\gamma^3)^2 = -(-1)(-1) = -1$: the four-component spinor, like the two-component one in (a), has $T^2 = -1$, and a Dirac particle's levels in any T-invariant field are Kramers doublets

Forgetting that K acts on the i in front of the second factor gives $T^2 = i^2\gamma^1\gamma^3\gamma^1\gamma^3 = +1$, the wrong sign for a spin-½ particle.

10 CORE COURSE

STRUCTURED · DEMANDING · UAQM-03.01 · UAQM-03.03 · UAQM-03.04 · UAQM-03.06 · UAQM-03.07 **[11]**

A spinless particle of mass m and charge −e is bound by a point nucleus of charge Ze, treated as the fixed source of the static potential energy V(r) = −Zαℏc/r. Covariant notation uses x^μ = (ct, r), the metric g = diag(1, −1, −1, −1) and p^μ = iℏ∂^μ, so that $p^0 = (i\hbar/c)\partial/\partial t$ and p = −iℏ∇; minimal coupling replaces the energy E by E − V, and a stationary state has ψ ∝ e^(−iEt/ℏ). Schrödinger's hydrogen problem, $-(\hbar^2/2M)u'' + [\hbar^2\lambda(\lambda+1)/(2Mr^2) - k/r]u = \varepsilon u$, has bound states $\varepsilon = -Mk^2/[2\hbar^2(n_r + \lambda + 1)^2]$, $n_r = 0, 1, 2, \ldots$, for any real $\lambda > -\tfrac{1}{2}$, taking the solution regular at the origin, u ∝ r^(λ+1). Take α = 1/137.036.

(a) DERIVE · U2 **[2]**

Derive the free Klein–Gordon equation from $p_\mu p^\mu = m^2c^2$, and show that with the Coulomb potential included a stationary state obeys $[(E - V)^2 - m^2c^4]\psi = -\hbar^2c^2\nabla^2\psi$.

1. $p_\mu p^\mu = (p^0)^2 - \mathbf{p}\cdot\mathbf{p}$, with $p^0 = (i\hbar/c)\partial/\partial t$ and $\mathbf{p} = -i\hbar\nabla$, gives $-(\hbar^2/c^2)\partial^2\psi/\partial t^2 + \hbar^2\nabla^2\psi = m^2c^2\psi$, that is $(1/c^2)\partial^2\psi/\partial t^2 - \nabla^2\psi + (mc/\hbar)^2\psi = 0$, Lorentz invariant because $p_\mu p^\mu$ is a scalar
2. for ψ ∝ e^(−iEt/ℏ), cp^0 acting on ψ gives E; minimal coupling replaces it by E − V, so $(E - V)^2\psi - c^2p^2\psi = m^2c^4\psi$, that is $[(E - V)^2 - m^2c^4]\psi = -\hbar^2c^2\nabla^2\psi$

   The sign of the metric decides the sign of the ∇^2 term. Writing $(p^0)^2 + p^2$ builds a Euclidean invariant, and the equation that follows has no plane-wave solutions with real energy above mc^2.

(b) DETERMINE · U2 **[3]**

With $\psi = [u(r)/r]Y_l^m(\theta, \varphi)$, the equation becomes $-\hbar^2c^2u'' + [\hbar^2c^2l(l+1) - (Z\alpha\hbar c)^2]u/r^2 - 2EZ\alpha\hbar c\, u/r = (E^2 - m^2c^4)u$. By mapping it onto Schrödinger's problem, determine the exact bound-state energies, writing $n = n_r + l + 1$.

1. dividing by 2E gives Schrödinger's form with M = E/c², k = Zαℏc, $\varepsilon = (E^2 - m^2c^4)/(2E)$ and $\lambda(\lambda+1) = l(l+1) - (Z\alpha)^2$, so $\lambda = -\tfrac{1}{2} + \sqrt{((l+\tfrac{1}{2})^2 - (Z\alpha)^2)}$, the root that tends to l as Zα → 0
2. then $(E^2 - m^2c^4)/(2E) = -(E/c^2)(Z\alpha\hbar c)^2/(2\hbar^2N^2) = -E(Z\alpha)^2/(2N^2)$, with $N = n_r + \lambda + 1 = n - l - \tfrac{1}{2} + \sqrt{((l+\tfrac{1}{2})^2 - (Z\alpha)^2)}$
3. so $E^2[1 + (Z\alpha)^2/N^2] = m^2c^4$ and $E = mc^2[1 + (Z\alpha)^2/N^2]^{(-1/2)}$, which depends on l through N, unlike the Schrödinger levels

   The mapping fails for candidates who identify M with m: the effective mass, E/c^2, and the effective energy, $(E^2 - m^2c^4)/(2E)$, both contain E, and only when both are written that way does E^2 factor out. Taking the other root of $\lambda(\lambda+1)$ gives a solution that is singular at the origin and does not reduce to l.

(c) SHOW THAT · U2 **[2]**

Show that, to order $(Z\alpha)^4$, $E = mc^2 - mc^2(Z\alpha)^2/(2n^2) - [mc^2(Z\alpha)^4/(2n^4)][n/(l+\tfrac{1}{2}) - \tfrac{3}{4}]$.

1. $N = n - \delta$ with $\delta = (l+\tfrac{1}{2}) - \sqrt{((l+\tfrac{1}{2})^2 - (Z\alpha)^2)} \approx (Z\alpha)^2/(2l+1)$, so $(Z\alpha)^2/N^2 \approx (Z\alpha)^2/n^2 + 2(Z\alpha)^4/[n^3(2l+1)]$
2. with $(1+x)^{(-1/2)} \approx 1 - x/2 + 3x^2/8$ and $x^2 \approx (Z\alpha)^4/n^4$, $E \approx mc^2[1 - (Z\alpha)^2/(2n^2) - (Z\alpha)^4/(n^3(2l+1)) + 3(Z\alpha)^4/(8n^4)]$, and $(Z\alpha)^4/(n^3(2l+1)) = [(Z\alpha)^4/(2n^4)]n/(l+\tfrac{1}{2})$, which gives the result

   The $(Z\alpha)^4$ term has two sources, the shift δ in N and the second-order term of the inverse square root. Dropping $3x^2/8$ loses the $-\tfrac{3}{4}$, and the correction then no longer equals the expectation value of $-p^4/(8m^3c^2)$, the check that ties the exact result to its non-relativistic limit.

(d) CALCULATE · U3 **[2]**

The charged-pion mass is determined from the X-rays of the 5g → 4f transition in pionic nitrogen, a π^- bound to ^{14}N (Z = 7), where the strong interaction is negligible for these orbits. Treat the pion as moving in the static field of the nucleus with its mass replaced by the reduced mass μ, using $m_\pi c^2 = 139.570$ MeV and a nuclear rest energy of 13040.20 MeV. Calculate the transition energy to order $(Z\alpha)^2$, and the order-$(Z\alpha)^4$ correction to it, in eV.

1. $\mu c^2 = (139.570 \times 13040.20/13179.77)$ MeV = 138.092 MeV and $Z\alpha = 0.051081$, so to order $(Z\alpha)^2$ the line is $[\mu c^2 (Z\alpha)^2/2](1/16 - 1/25) = 4053.66$ eV
2. the $(Z\alpha)^4$ terms are −0.272 eV for 5g (n = 5, l = 4) and −0.721 eV for 4f (n = 4, l = 3), so the correction is +0.450 eV and the line lies at 4054.11 eV; at 1.1×10^{-4} of the energy, a pion mass quoted to a few parts per million must include it

   Using m_π in place of μ moves the line by about 1.1%, 43 eV, a hundred times the relativistic correction the part asks for. The 4f term dominates the correction because it falls as $1/n^4$ and is larger at smaller l.

(e) SKETCH · U1 **[2]**

On one energy-level diagram, sketch the n = 2 levels of a spinless particle of the electron's mass bound to a proton, from (c) with Z = 1, beside the n = 2 levels of hydrogen given by the Dirac equation, $E = mc^2 - mc^2\alpha^2/(2n^2) - [mc^2\alpha^4/(2n^4)][n/(j + ½) - ¾]$. Mark each level's shift from the unperturbed n = 2 level in units of $mc^2\alpha^4/128$, and label the hydrogen levels with the terms of the fine-structure Hamiltonian that move them away from the Klein–Gordon levels.

1. Klein–Gordon: 2s at −13 units and 2p at −7/3 units, 2p above 2s by 32/3 units, $mc^2\alpha^4/12$ or 29.2 GHz; hydrogen: $2S_{1/2}$ and $2P_{1/2}$ drawn as one level at −5 units and $2P_{3/2}$ at −1 unit, every level below the unperturbed one
2. the differences labelled: the Darwin term raises 2s by 8 units to $2S_{1/2}$; spin–orbit coupling raises 2p by 4/3 units to $2P_{3/2}$ and lowers it by 8/3 units to $2P_{1/2}$; the Klein–Gordon levels carry only the $-p^4/(8m^3c^2)$ term, so their pattern is right only for a spinless particle, as in a pionic atom

   The expected wrong sketch puts the Klein–Gordon 2s and 2p levels together, carrying over Schrödinger's degeneracy in l. The exact Klein–Gordon levels depend on l through N; it is the Dirac equation, through the spin–orbit and Darwin terms, that restores a degeneracy, now in j.

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CORE COURSE

STRUCTURED · DEMANDING · UAQM-04.01 · UAQM-04.06 · UAQM-05.04 · UAQM-08.01 **[10]**

In the Dirac representation the free Dirac Hamiltonian is $H = c\alpha\cdot p + \beta mc^2$, with $\beta = [[I, 0], [0, -I]]$ and $\alpha_i = [[0, \sigma_i], [\sigma_i, 0]]$ in 2 × 2 blocks, and $\bar\psi = \psi^\dagger\beta$. An electron moves along +z with momentum $p = (3/4)mc$. Positive-energy spinors are normalised to $u^\dagger u = 2E/c$, and the matrix that carries a rest spinor into the spinor of an electron moving along +z with Lorentz factor γ is $S = \cosh(\eta/2) + \alpha_z \sinh(\eta/2)$, where $\cosh\eta = \gamma$. Take $mc^2 = 0.511$ MeV, $\hbar = 1.055 \times 10^{-34}$ J s, $\hbar c = 197.3$ MeV fm and 1 MeV = 1.602×10^{-13} J.

(a) DERIVE · U2 **[3]**

From the Dirac equation $Hu = Eu$, derive the positive-energy spinor $u(p, \uparrow)$, with spin up along z, in the form N(1, 0, b, 0), finding E, b and N.

1. the anticommutation relations make $H^2 = (p^2c^2 + m^2c^4)I$, so the positive energy is $E = mc^2\sqrt{(1 + 9/16)} = (5/4)mc^2$
2. writing $u = (\varphi, \chi)$, the lower block of $Hu = Eu$ reads $c\sigma\cdot p\,\varphi - mc^2\chi = E\chi$, so $\chi = [cp\sigma_z/(E + mc^2)]\varphi$, and with $\varphi = (1, 0)$, $b = (3/4)/(9/4) = 1/3$
3. $N^2(1 + 1/9) = 2E/c = (5/2)mc$, so $N^2 = (9/4)mc$ and $u = \sqrt{(mc)}(3/2, 0, 1/2, 0)$

   The slip to catch is the sign in the lower block: writing $E - mc^2$ in the denominator gives b = 3, a spinor dominated by its lower component, which is the pattern of a negative-energy solution, not of an electron at 0.6c.

(b) SHOW THAT · U2 **[2]**

Show that S applied to the rest spinor $\sqrt{(2mc)}(1, 0, 0, 0)$ gives the spinor of (a), and that S is not unitary, in a way consistent with $\bar{u}u$ staying at $2mc$ while $u\dagger u$ grows to $2E/c$.

1. $\gamma = 5/4$, so $\cosh(\eta/2) = \sqrt{((\gamma + 1)/2)} = 3/(2\sqrt{2})$ and $\sinh(\eta/2) = \sqrt{((\gamma - 1)/2)} = 1/(2\sqrt{2})$; since $\alpha_z(1, 0, 0, 0) = (0, 0, 1, 0)$, $S\sqrt{(2mc)}(1, 0, 0, 0) = \sqrt{(2mc)}(3/(2\sqrt{2}), 0, 1/(2\sqrt{2}), 0) = \sqrt{(mc)}(3/2, 0, 1/2, 0)$, the spinor of (a)
2. S is Hermitian, so $S\dagger S = S^2 = \cosh\eta + \alpha_z \sinh\eta \neq I$, and $u\dagger u = 2mc \cosh\eta = (5/2)mc = 2E/c$ because α_z has no diagonal element in the rest spinor; but $\beta S\beta = S^{-1}$, so $S\dagger\beta S = \beta$ and $\bar{u}u = 2mc$ is unchanged

   A boost matrix that preserved $u\dagger u$ would leave the density the same in every frame, though $\psi\dagger\psi$ is the time component of a four-vector and must grow by γ. Candidates who assume S unitary, as the rotation matrices are, cannot reach $2E/c$.

(c) CALCULATE · U3 **[2]**

Calculate $\langle c\alpha_z\rangle = u\dagger c\alpha_z u/u\dagger u$ for this spinor, and compare it with the group velocity $\partial E/\partial p$.

1. $\alpha_z u = \sqrt{(mc)}(1/2, 0, 3/2, 0)$, so $u\dagger\alpha_z u = mc(3/4 + 3/4) = (3/2)mc$ and $\langle c\alpha_z\rangle = [(3/2)/(5/2)]c = 0.6c$
2. $\partial E/\partial p = pc^2/E = [(3/4)/(5/4)]c = 0.6c$, the same: in a positive-energy plane wave the velocity operator $c\alpha = (i/\hbar)[H, r]$ has the group velocity as its expectation value

   Candidates who take the velocity to be p/m get $0.75c$, the non-relativistic answer, and miss that $c\alpha$ already contains the relativistic kinematics.

(d) EXPLAIN · U1 **[1]**

Every eigenvalue of $c\alpha_z$ is $+c$ or $-c$. Explain how this is consistent with (c).

1. $\alpha_z^2 = I$ gives eigenvalues ± 1, but α_z does not commute with H, since it anticommutes with β, so its eigenstates are superpositions of positive- and negative-energy solutions; the electron of (c) is not in an α_z eigenstate, and $0.6c$ is the average of $c\alpha_z$ over a state of one sign of energy

   The weak answer says the electron moves at c and averages to $0.6c$ by reversing direction, as if the eigenvalues were velocities a positive-energy electron could have. The eigenstates of α_z are not states of one sign of energy, and the resolution needs the negative-energy components of Unit 8.

(e) ESTIMATE · U4 **[2]**

A wave packet built from both positive- and negative-energy solutions near this momentum shows Zitterbewegung. Estimate the angular frequency and the amplitude of the trembling, stating your assumptions, and say what happens to it in a packet built from positive-energy solutions alone.

1. assuming the packet is narrow in momentum about p, the interference between components at $+E$ and $-E$ oscillates at $\omega = 2E/\hbar = 2 \times 0.639 \times 1.602 \times 10^{-13}$ J$/(1.055 \times 10^{-34}$ J s$) = 1.9 \times 10^{21}$ s^{-1}, with amplitude of order $\hbar c/(2E) = 197.3$ MeV fm$/1.28$ MeV $\approx 1.5 \times 10^{-13}$ m, close to $\hbar/(2mc) = 1.9 \times 10^{-13}$ m, a fraction of the Compton wavelength
2. the trembling is an interference between the two signs of energy, so it vanishes identically in a packet of positive energy only, where $\langle r\rangle$ moves uniformly at the group velocity of (c); confining a packet within about $\hbar/(mc)$ forces in negative-energy components, and the trembling with them

   A frequency of $E/\hbar$, the rate at which one plane wave's phase turns, misses that the effect is a beat between $+E$ and $-E$. Candidates who describe the trembling as an observable motion of every free electron forget that it exists only in packets containing both signs of energy.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UAQM-07.02 · UAQM-07.04 · UAQM-07.05 · UAQM-07.06 · UAQM-07.08 · UAQM-07.09 **[11]**

An electron of mass m is bound to a point nucleus of charge Ze, with Z < 137, so that V(r) = −Zαℏc/r. In a central field the Dirac Hamiltonian commutes with J, with the Dirac parity operator and with K = β(Σ·L + ℏ), whose eigenvalue is written −κℏ, so that κ = −(j + ½) for j = l + ½ and κ = +(j + ½) for j = l − ½, l being the orbital angular momentum of the upper component. Writing the upper component as [G(r)/r] times a spinor harmonic of orbital angular momentum l, and the lower as i[F(r)/r] times one of orbital angular momentum l′, the radial equations are ℏc(dG/dr + κG/r) = (E + mc² − V)F and ℏc(dF/dr − κF/r) = −(E − mc² − V)G. Take α = 1/137.036 and mc² = 511.0 keV.

(a) STATE · U1 **[2]**

State κ for each state of the n = 2 shell, 2S_(1/2), 2P_(1/2) and 2P_(3/2), together with l′ and the parity of each state.

1. 2S_(1/2): κ = −1; 2P_(1/2): κ = +1; 2P_(3/2): κ = −2
2. l′ = 2j − l: 1 for 2S_(1/2), 0 for 2P_(1/2) and 2 for 2P_(3/2); the parity is (−1)^l of the upper component, even for 2S_(1/2) and odd for both 2P states, the β in the Dirac parity operator compensating the opposite (−1)^(l′) of the lower component

   K's eigenvalue is −κℏ, so the states with j = l + ½, 1s and 2P_(3/2) among them, have negative κ; candidates who drop the minus sign assign the wrong states to the nodeless solutions of (c). Reading the parity from l′ instead makes 2S_(1/2) odd and 2P_(1/2) even, which would forbid 2P_(1/2) → 1S_(1/2) as an electric-dipole transition, although it is one of the two components of Lyman-α.

(b) DERIVE · U2 **[3]**

For a solution with no radial nodes, try G = Ar^γ e^(−λr) and F = Br^γ e^(−λr), with λ > 0. Derive from the radial equations γ, the ratio B/A, and λ in terms of E.

1. the 1/r terms of the first equation give ℏc(γ + κ)A = ZαℏcB, so B/A = (γ + κ)/(Zα); those of the second give (γ − κ)B = −ZαA, so B/A = −Zα/(γ − κ)
2. equating the two ratios, γ² − κ² = −(Zα)², so γ = √(κ² − (Zα)²), the positive root, since the negative one gives G and F that diverge at the origin, as r^(−|κ|) when Zα is small, and cannot be normalised; for |κ| = 1, γ is less than 1
3. the r-independent terms give −ℏcλA = (E + mc²)B and −ℏcλB = −(E − mc²)A; multiplying them, (ℏcλ)² = m²c⁴ − E², so λ = √(m²c⁴ − E²)/(ℏc), real for a bound state with |E| < mc²

   Candidates who match powers of r without separating the 1/r terms from the constant terms obtain one tangled relation between E and γ. The structure to see is that the singular terms fix γ independently of E, as the centrifugal term fixes the power at the origin in Schrödinger's problem.

(c) SHOW THAT · U2 **[2]**

Show that E = mc²γ/|κ|, and that such a nodeless solution exists only for κ < 0.

1. from the constant terms B/A = −ℏcλ/(E + mc²) = −√((mc² − E)/(mc² + E)), which is negative; with B/A = (γ + κ)/(Zα) this needs γ + κ < 0, impossible for κ > 0 because γ > 0, so κ = −|κ| and only states with j = l + ½, 1S_(1/2), 2P_(3/2), 3D_(5/2) and so on, can be nodeless
2. squaring, (|κ| − γ)²/(Zα)² = (mc² − E)/(mc² + E); since (Zα)² = (|κ| − γ)(|κ| + γ) the left side is (|κ| − γ)/(|κ| + γ), so E/mc² = γ/|κ| = √(1 − (Zα)²/κ²)

   Both lines are needed. A candidate who squares at once loses the sign of B/A, and with it the reason that no 2S_(1/2) or 2P_(1/2) state is a single term: those states need a radial node, as the n_r = 1 that the general formula gives them shows.

(d) CALCULATE · U3 **[2]**

For hydrogen-like lead, Pb^{81+} ($Z = 82$), calculate the 1s binding energy $mc^2 - E$, comparing it with the value to order $(Z\alpha)^4$, $mc^2[(Z\alpha)^2/2 + (Z\alpha)^4/8]$, and the fraction of the 1s probability carried by the small component.

1. $Z\alpha = 0.59838$ and $\gamma = \sqrt{(1 - 0.35806)} = 0.80121$, so $mc^2 - E = 511.0$ keV $\times$ 0.19879 = 101.6 keV, against 99.7 keV to order $(Z\alpha)^4$ and 91.5 keV non-relativistically
2. G and F share the radial factor, so the small component carries $B^2/(A^2 + B^2) = (1 - \gamma)/2 = 0.099$, about 10% of the probability, against $\alpha^2/4 = 1.3 \times 10^{-5}$ in hydrogen

   The order-$(Z\alpha)^4$ expansion misses 1.9 keV, 2% of the binding, at $Z = 82$. Candidates who then estimate the small-component share as $(Z\alpha)^2/4 = 0.090$ underestimate it by a tenth for the same reason.

(e) PREDICT · U1 **[2]**

Predict how the 1s solution behaves as Z approaches 137 and beyond, for a point nucleus, and say what a nucleus of finite size changes.

1. as $Z\alpha \to 1$, $\gamma \to 0$ and $E = mc^2\gamma \to 0$, so the 1s level reaches zero energy, not $-mc^2$, while the density near the origin, $\propto r^{(2\gamma - 2)}$, grows ever more singular; for $Z\alpha > 1$, γ is imaginary, the solutions oscillate as $r^{(\pm i|\gamma|)}$ with no limit at the origin, and the point-nucleus problem has no ground state, the Dirac analogue of the Klein–Gordon fall to the centre at $Z\alpha = ½$
2. a finite nucleus removes the singularity, so the 1s level continues smoothly past $Z = 137$ and reaches the top of the negative-energy continuum, $E = -mc^2$, near $Z \approx 172$; beyond that the one-particle theory fails, and an empty 1s level would be filled by spontaneous pair creation

   The common prediction is that the 1s level reaches $-mc^2$ at $Z = 137$. The point-nucleus formula ends at $E = 0$ there; only because a finite nucleus keeps the solution regular can the level go on falling, and it takes about 35 more units of charge to reach $-mc^2$, which is why $Z \approx 172$, not 137, is the threshold for spontaneous positron emission.

13

CORE COURSE

DATA ANALYSIS · DEMANDING · UAQM-02.05 · UAQM-02.06 · UAQM-02.07 · UAQM-02.08 **[10]**

Identical nuclei, each in its ground state of spin s, scatter in their centre-of-mass frame through the Coulomb interaction alone, which does not act on spin; every run is far enough below the Coulomb barrier that the nuclear force can be neglected. A detector of angular half-width 0.50° centred on $\theta = 90°$ counts N_{90} nuclei, and N_R is the number it would have counted in the same run if $|f(\theta)|^2$ and $|f(\pi - \theta)|^2$ added without interference, computed from the Rutherford cross-section and the measured beam and target. For the exact Coulomb amplitude $f(\theta)$, the interference term is proportional to $\cos[\eta \ln \tan^2(\theta/2)]$, where $\eta = Z^2\alpha/\beta$, βc is the relative speed, $\beta^2 = 2E_{cm}/(\mu c^2)$ and μ is the reduced mass. Take $\alpha = 1/137.036$ and 1 u = 931.5 MeV/c^2, treat N_{90} as a Poisson count and N_R as exact.

(a) SHOW THAT · U2 **[2]**

The $(2s + 1)^2$ spin states of the pair divide into $(s + 1)(2s + 1)$ that are symmetric under exchange of the two spins and $s(2s + 1)$ that are antisymmetric. Show that for an unpolarised beam and target $d\sigma/d\Omega = |f(\theta)|^2 + |f(\pi - \theta)|^2 + [2(-1)^{(2s)}/(2s + 1)]\mathrm{Re}[f(\theta)f^*(\pi - \theta)]$, and hence that $R = N_{90}/N_R$ should equal $1 + (-1)^{(2s)}/(2s + 1)$.

1. the whole state must be symmetric for integer s and antisymmetric for half-integer s, so for bosons the symmetric spin states, a fraction $(s + 1)/(2s + 1)$, go with $|f + f'|^2$ and the antisymmetric fraction $s/(2s + 1)$ with $|f - f'|^2$, where $f' = f(\pi - \theta)$, and for fermions the two fractions are exchanged; the cross terms $\pm 2\mathrm{Re}(ff'^*)$ then combine with weight $[(s + 1) - s]/(2s + 1) = 1/(2s + 1)$ and sign $(-1)^{(2s)}$
2. at $\theta = 90°$, $f(\pi - \theta) = f(\theta)$, phase included, so $\mathrm{Re}(ff'^*) = |f|^2$ and $d\sigma/d\Omega = 2|f|^2[1 + (-1)^{(2s)}/(2s + 1)]$, while the non-interfering sum is $2|f|^2$, giving $R = 1 + (-1)^{(2s)}/(2s + 1)$

   Candidates who weight the symmetric and antisymmetric spin spaces equally lose the $1/(2s + 1)$. The checks are $s = ½$, where the weights must reduce to the 1/4 and 3/4 of two electrons, and $s = 0$, where R must be 2.

(b) CALCULATE · U4 **[2]**

Calculate R and its standard uncertainty for each pair.

1. R = N_{90}/N_R: 1.9916 for ^{12}C, 0.5044 for ^{13}C, 1.3384 for ^{14}N, 1.1375 for ^{10}B, 0.7462 for ^{11}B and 0.8371 for ^{17}O
2. $\sigma_R = \sqrt{N_{90}}/N_R$: 0.0200, 0.0079, 0.0122, 0.0097, 0.0086 and 0.0091 respectively

   $\sigma_R = \sqrt{N_{90}}/N_R = \sqrt{(R/N_R)}$, so it grows with R and falls with the non-interfering count N_R: ^{12}C, with the largest R and the smallest N_R, has the largest σ_R, while ^{13}C, with the fewest counts N_{90}, has the smallest. Candidates who quote one uncertainty for the whole table, or rank the rows by N_{90} alone, cannot judge the high-spin rows in (c).

(c) DETERMINE · U4 **[3]**

Determine the spin of each nucleus, and the significance with which the nearest alternative value is excluded. Say which assignments are secure at three standard deviations.

1. R > 1 marks bosons and R < 1 fermions, and $|R - 1| = 1/(2s + 1)$: ^{12}C gives 1.992 ± 0.020, consistent with s = 0, with s = 1 (R = 1.333) excluded at 33σ, and so a check on the normalisation of N_R; ^{13}C gives 2s + 1 = 2.0, s = ½, with s = 3/2 (R = 0.750) excluded at 31σ
2. ^{14}N: 2s + 1 = 2.95, s = 1, with s = 2 (R = 1.200) excluded at 11σ; ^{11}B: 2s + 1 = 3.94, s = 3/2, with s = 5/2 (R = 0.833) excluded at 10σ; ^{17}O: 2s + 1 = 6.1, s = 5/2, with s = 7/2 (R = 0.875) excluded at 4.1σ and s = 3/2 at 9.5σ
3. ^{10}B: 2s + 1 = 7.3, so s = 3 (R = 1.1429, 0.6σ away) is favoured and s = 2 is excluded at 6.4σ, but s = 4 (R = 1.1111) is only 2.7σ away, so this assignment is not secure at 3σ; separating 3 from 4 at 5σ needs $\sigma_R \approx 0.0318/5 = 0.0064$, about 2.3 times as many counts

   The spacing between successive spins, $1/(2s + 1) - 1/(2s + 3)$, falls roughly as $1/(2s^2)$, so a precision that settles s = 1 by a wide margin cannot settle s = 3. Candidates who assign each spin from the nearest value without testing the alternatives cannot say which assignments the data support.

(d) EVALUATE · U4 **[2]**

Near 90°, $\ln \tan^2(\theta/2) \approx 2(\theta - 90°)$, with the angle in radians, so the detector averages $\cos[2\eta(\theta - 90°)]$ over its acceptance. Evaluate whether this averaging biases any spin assignment, using the pair with the largest η.

1. the largest η is for ^{17}O + ^{17}O: μc^2 = 8.5 × 931.5 MeV = 7918 MeV, $\beta = \sqrt{(8.00/7918)} = 0.0318$ and $\eta = 64/(137.036 \times 0.0318) = 14.7$; averaging over ±0.50° = ±0.00873 rad multiplies the interference term by $\sin x/x$ with $x = 2\eta \times 0.00873 = 0.256$, that is by 0.989
2. the expected R for s = 5/2 moves from 0.8333 to 0.8352, a shift of 0.0018, that is 0.2σ and about 4% of the 0.042 gap to the s = 7/2 value; the fractional loss of interference, $1 - \sin x/x \approx x^2/6$, grows as η^2, so every other pair is moved by under 2% of the gap to its neighbouring spin and by at most 0.22σ, and the averaging biases no assignment; a detector three times wider would move ^{17}O by 1.7σ and begin to pull the high-spin values towards 1

   Candidates often dismiss the acceptance without a number, or test it on the pair with the smallest η. The fractional loss grows as η^2, and $\eta = Z^2\alpha/\beta$ is dominated by Z^2, since β hardly differs between these runs, which is why the highest-Z pair is the one to check. Comparing raw shifts in R across pairs misleads: a spin-0 pair has the largest interference term to lose, but also the widest gap to the next spin.

(e) EXPLAIN · U1 **[1]**

Explain why the measurement is made at 90° rather than at another angle.

1 only at 90° does f(π − θ) equal f(θ) exactly, so the Coulomb phases cancel and the interference term has its full size whatever η and E_cm are; at any other angle it carries cos[η ln tan²(θ/2)], which near 90° oscillates with a period of 180°/η, between 12° and 32° for these pairs, and the spin would be entangled with η

Answers that say 90° is where the cross-section is largest miss the point: for fermions it is a minimum. What matters is that the interference phase vanishes there.

14

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-06.01 · UAQM-06.02 · UAQM-06.05 · UAQM-06.06 · UAQM-06.07 · UAQM-06.08 · UAQM-06.09 **[16]**

An electron of charge q = −e and mass m moves in static potentials with Dirac Hamiltonian H = cα·π + βmc² + V(r), where π = p − qA, p = −iℏ∇ and V is the electrostatic potential energy. In the Dirac representation write ψ = (φ, χ), with φ the large and χ the small two-component spinor, and E = mc² + E′. You may use (σ·a)(σ·b) = a·b + iσ·(a × b) for vector operators a and b that commute with σ, and [p, f(r)] = −iℏ∇f. Take α = 1/137.036 and mc² = 0.5110 MeV.

(a) DERIVE · U2 **[3]**

Derive the two coupled equations for φ and χ. For |E′ − V| ≪ mc², eliminate χ to leading order and derive the Pauli Hamiltonian, showing that its spin term is −(q/m)S·B, so that g = 2.

1 the blocks of Hψ = Eψ are (E′ − V)φ = cσ·πχ and (E′ + 2mc² − V)χ = cσ·πφ, so χ = [c/(2mc² + E′ − V)]σ·πφ ≈ [σ·π/(2mc)]φ, smaller than φ by about v/(2c)

2 substituting, E′φ = [(σ·π)²/(2m) + V]φ, and (σ·π)² = π² + iσ·(π × π), the cross product surviving because the components of π do not commute

3 [π_x, π_y] = iqℏ(∂A_y/∂x − ∂A_x/∂y) = iqℏB_z and cyclically, so π × π = iqℏB and (σ·π)² = π² − qℏσ·B; with S = (ℏ/2)σ, H_P = π²/(2m) − (q/m)S·B + V, a moment (q/m)S = g(q/2m)S with g = 2, fixed by minimal coupling rather than put in by hand

The error to catch is treating π like p and writing (σ·π)² = π². That is right for a free particle and loses the whole magnetic moment, which lives entirely in the non-commuting components of p − qA.

(b) ESTIMATE · U4 **[2]**

Estimate the ratio |χ|/|φ| and the fraction of the probability carried by χ in the ground state of hydrogen, stating your assumptions.

1 assuming the typical momentum is ℏ/a₀ = αmc, so that v/c ≈ α, |χ|/|φ| ≈ p/(2mc) ≈ α/2 = 3.6 × 10⁻³

2 the probability in χ is then about (α/2)² = 1.3 × 10⁻⁵, in agreement with the exact Dirac 1s value (1 − √(1 − α²))/2 = 1.33 × 10⁻⁵; the norm of φ alone therefore falls short of 1 by about 10⁻⁵, the same relative order as the fine structure, which is why the next order needs care

Estimates that quote v/c ≈ α without saying where it comes from have not stated an assumption, and those that take |χ|/|φ| ≈ α rather than α/2 overestimate the share by a factor of four.

(c) EXPLAIN · U1 **[2]**

Now set A = 0. Keeping the next term in χ gives $E'\varphi = [p^2/(2m) + V - (1/(4m^2c^2))(\sigma\cdot p)(E' - V)(\sigma\cdot p)]\varphi$. Explain why this cannot be read as a Schrödinger equation for φ with a Hermitian Hamiltonian at order $(v/c)^2$.

1. the bracket contains the eigenvalue E′, so each state obeys a different operator and the equation is not an eigenvalue problem for one fixed Hamiltonian; removing E′ with the lowest-order relation $(E' - V)\varphi \approx [p^2/(2m)]\varphi$ means commuting $E' - V$ through σ·p, and the commutator leaves the term $(i\hbar/(4m^2c^2))\nabla V\cdot p$, which is not Hermitian, although $(\sigma\cdot p)V(\sigma\cdot p)$ itself is
2. the reason is that probability is shared: Dirac eigenstates satisfy $\int(\varphi_n\dagger\varphi_m + \chi_n\dagger\chi_m)\, d^3r = \delta_{nm}$, so the large components of different states are neither normalised nor orthogonal, $\int\varphi_n\dagger\varphi_n\, d^3r \approx 1 - \langle p^2\rangle/(4m^2c^2)$, and no Hermitian operator can have them as its eigenfunctions until φ is renormalised

   Candidates notice the stray E′ and replace it by $p^2/(2m)$, which hides the problem rather than solving it. Calling $(\sigma\cdot p)V(\sigma\cdot p)$ non-Hermitian is wrong, since it is its own adjoint: the non-Hermiticity is a symptom of using a wavefunction that is not normalised, and the cure has to act on φ itself.

(d) SHOW THAT · U2 **[4]**

Show that $\Phi = [1 + p^2/(8m^2c^2)]\varphi$ is normalised to order $(v/c)^2$, and that it obeys $E'\Phi = H_{FW}\Phi$ with $H_{FW} = p^2/(2m) + V - p^4/(8m^3c^2) + (\hbar/(4m^2c^2))\sigma\cdot(\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$.

1. $\int\chi\dagger\chi\, d^3r \approx \int\varphi\dagger[p^2/(4m^2c^2)]\varphi\, d^3r$, so $\int(\varphi\dagger\varphi + \chi\dagger\chi)\, d^3r = \int\varphi\dagger[1 + p^2/(4m^2c^2)]\varphi\, d^3r = \int\Phi\dagger\Phi\, d^3r$ to this order, since $[1 + p^2/(8m^2c^2)]^2 \approx 1 + p^2/(4m^2c^2)$ and p^2 is Hermitian
2. substituting $\varphi = [1 - p^2/(8m^2c^2)]\Phi$ and multiplying on the left by $1 - p^2/(8m^2c^2)$, the left side becomes $E'[1 - p^2/(4m^2c^2)]\Phi$, whose $E'p^2$ term cancels the $-E'p^2/(4m^2c^2)$ inside $(\sigma\cdot p)(E' - V)(\sigma\cdot p)$, leaving $E'\Phi = [p^2/(2m) + V - p^4/(8m^3c^2) - (1/(8m^2c^2))(p^2V + Vp^2) + (1/(4m^2c^2))(\sigma\cdot p)V(\sigma\cdot p)]\Phi$
3. $(\sigma\cdot p)V = V\sigma\cdot p - i\hbar\sigma\cdot\nabla V$, so $(\sigma\cdot p)V(\sigma\cdot p) = Vp^2 - i\hbar\nabla V\cdot p + \hbar\sigma\cdot(\nabla V \times p)$ by the given identity, and $p^2V = Vp^2 - \hbar^2\nabla^2 V - 2i\hbar\nabla V\cdot p$, so $p^2V + Vp^2 = 2Vp^2 - \hbar^2\nabla^2 V - 2i\hbar\nabla V\cdot p$
4. combining, the Vp^2 terms cancel and so do the two $\nabla V\cdot p$ terms, neither Hermitian on its own, $(1/(4m^2c^2))(-i\hbar\nabla V\cdot p) - (1/(8m^2c^2))(-2i\hbar\nabla V\cdot p) = 0$, leaving $(\hbar/(4m^2c^2))\sigma\cdot(\nabla V \times p) + (\hbar^2/(8m^2c^2))\nabla^2 V$: H_{FW}, Hermitian, with every term traced to a step

   Most attempts fail at the bookkeeping of the second line: the E′ inside the bracket must be cancelled by the renormalisation, not replaced by $p^2/(2m)$, and p^2 and V must be kept in the symmetric order the anticommutator gives. Because the result is printed, the last mark needs the two $-i\hbar\nabla V\cdot p$ terms shown cancelling.

(e) DETERMINE · U2 **[3]**

For a central potential V(r), and then for $V = -Z\alpha\hbar c/r$, determine the spin–orbit term in the form $\xi(r)S\cdot L$ and the Darwin term, and evaluate the Darwin shift of the 1s level of hydrogen in eV, given $|\psi_{100}(0)|^2 = 1/(\pi a_0^3)$ with $a_0 = \hbar/(\alpha mc)$.

1. $\nabla V = (dV/dr)r/r$, so $\sigma\cdot(\nabla V \times p) = (1/r)(dV/dr)\sigma\cdot L$, and with $\sigma = 2S/\hbar$ the spin–orbit term is $[1/(2m^2c^2)](1/r)(dV/dr)S\cdot L$
2. for $V = -Z\alpha\hbar c/r$, $\nabla^2 V = -Z\alpha\hbar c\nabla^2(1/r) = 4\pi Z\alpha\hbar c\delta^3(r)$, so the Darwin term is $[\pi\hbar^2/(2m^2c^2)]Z\alpha\hbar c\,\delta^3(r)$, which acts only on s states
3. for hydrogen 1s, $\langle H_D\rangle = [\pi\hbar^2\alpha\hbar c/(2m^2c^2)]/(\pi a_0^3) = \alpha^4 mc^2/2 = 7.25 \times 10^{-4}$ eV

   Leaving the result as $\nabla^2 V$ hides the physics: the Coulomb Laplacian is a delta function, so the Darwin term shifts only s levels, which is how it takes the place of the spin–orbit term at $l = 0$. A sign slip in $\nabla^2(1/r) = -4\pi\delta^3(r)$ gives a Darwin term that lowers the s levels.

(f) COMPARE · U1 **[2]**

In a semiclassical picture the electron, moving with velocity v through the electric field $\mathcal{E} = \nabla V/e$, sees in its rest frame the magnetic field $B' = -v \times \mathcal{E}/c^2$. Compare the spin–orbit coefficient that $-\mu \cdot B'$ gives, with $\mu = -g(e/2m)S$, with the one in (e), account for the difference, and say what changes if $g = 2(1 + a)$.

1 $B' = [1/(emc^2)](1/r)(dV/dr)L$, so $-\mu \cdot B' = [g/(2m^2c^2)](1/r)(dV/dr)S \cdot L$, which for $g = 2$ has coefficient $1/(m^2c^2)$, twice the Dirac result

2 the rest frame of an accelerated electron rotates relative to the laboratory, the Thomas precession, which adds $-[1/(2m^2c^2)](1/r)(dV/dr)S \cdot L$; the total, $[(g - 1)/(2m^2c^2)](1/r)(dV/dr)S \cdot L$, equals the Dirac result for $g = 2$, where it emerges with no frame argument at all, and becomes $(1 + 2a)/(2m^2c^2)$ for an anomalous moment, since the kinematic part does not depend on g

The factor of two is where answers split: some insert the Thomas half without justification, others declare the rest-frame argument simply wrong. The point is that the reduction, done in the laboratory frame, contains both the magnetic precession and the kinematic one, and only the magnetic one scales with g.

University Advanced Quantum Mechanics · Final Examination

The whole of Advanced Quantum Mechanics, weighted towards the second half: quantised fields and spin-statistics, second quantisation, the radiation field and spontaneous emission, Bethe's Lamb shift, open systems and the Bose gas. Symmetries, identical-particle scattering, the Klein-Gordon and Dirac equations, g = 2 and exact Dirac hydrogen are sampled from the first half.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**114**	**18**	**10**

SECTION — **UAQM · Final · University Advanced Quantum Mechanics — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 1 · Routine 8 · Demanding 7 · Discriminating 2**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UAQM-01 · UAQM-02 · UAQM-03 · UAQM-04 · UAQM-05 · UAQM-06 · UAQM-07 · UAQM-08 · UAQM-09 · UAQM-10 · UAQM-11 · UAQM-12 · UAQM-13 · UAQM-14 · UAQM-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Advanced Quantum Mechanics · Final Examination

Answer all questions. 114 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 **[1]**

A metallic grain a few nanometres across holds an odd number N of conduction electrons. Its shape is irregular, its impurities are random, and strong spin-orbit scattering mixes spin with orbital motion, so that no spatial symmetry and no component of the total spin is conserved. No magnetic field is applied, so every term in the grain's Hamiltonian H is even under time reversal, and for N spin-half particles the time-reversal operator satisfies $\Theta^2 = (-1)^N$. Which statement about the energy levels of the grain is correct?

A Every level is non-degenerate: once spin-orbit scattering has mixed the spin states, only a spatial symmetry could force two states to share an energy.

B Every level is at least twofold degenerate: for an eigenstate $|\psi\rangle$, $\Theta|\psi\rangle$ has the same energy, and $\Theta^2 = -1$ makes it orthogonal to $|\psi\rangle$.

C The levels are twofold degenerate only while the spin-orbit scattering is weak, because the pairs are the spin-up and spin-down states of each orbital, which strong spin-orbit coupling mixes.

D Every level is twofold degenerate, and stays so when one more electron is added, because each electron's states still come in time-reversed pairs.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UAQM-02.05 · UAQM-02.08 **[1]**

The elastic scattering of ^{16}O by ^{16}O is followed at a fixed centre-of-mass angle $\theta = 60°$ while the centre-of-mass energy is stepped down from 8.0 MeV to 3.0 MeV, below the Coulomb barrier of about 10 MeV; at this angle the nuclei never come closer than about 17 fm, so only the Coulomb interaction acts. ^{16}O nuclei are identical spin-0 bosons. The Coulomb amplitude the pair would have if the nuclei were distinguishable is, up to a phase that does not depend on θ, $f(\theta) = -[\eta/(2k\sin^2(\theta/2))]\exp[-i\eta\ln\sin^2(\theta/2)]$, where k is the relative wavenumber and the Sommerfeld parameter $\eta = Z^2\alpha/\beta$, with βc the relative speed, rises from 10.1 to 16.5 over the range. Let R be the measured differential cross-section divided by $|f(60°)|^2$, the Rutherford value for the projectile alone. How does R behave over the range?

A R stays at $1 + \tan^4 30° = 10/9$ at every energy: spin-0 nuclei have no spin states to average over, and the Coulomb phase cancels when each amplitude is squared.

B R oscillates between 0 and 4, because the amplitudes of identical bosons always add, so they can interfere completely.

C R oscillates between 4/9 and 16/9, passing through more than one full cycle as $\eta\ln 3$ changes by about 7 rad.

D R oscillates between 10/9 and 16/9, because the interference of identical bosons is always constructive, so the cross-section never falls below the sum of the two Rutherford terms.

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-03.06 · UAQM-03.08 **[1]**

A negative kaon, K^-, is a spinless particle of charge $-e$ that can be captured into atomic orbits. Consider a kaon bound to a lead nucleus ($Z = 82$), treated as a point charge, with the strong interaction ignored. Its Klein-Gordon radial equation contains the centrifugal term $\hbar^2c^2l(l + 1)/r^2$ together with an attractive term $-(Z\alpha\hbar c)^2/r^2$ that comes from squaring the Coulomb energy, so that near the origin the radial function $R(r)$ behaves as r^s, with $s(s + 1) = l(l + 1) - (Z\alpha)^2$. Take $\alpha = 1/137.04$. Which statement about the kaon's s, p and d orbitals ($l = 0$, 1 and 2) is correct?

A Only the s orbitals fail: for $l = 0$, s is complex and the solutions oscillate without limit near the origin, a fall to the centre, while the p and d orbitals have $s \approx 0.88$ and $s \approx 1.93$.

B None fails, because $Z\alpha = 0.60$ is below 1; the s orbitals merely acquire a weak singularity, $R \propto r^s$ with $s \approx -0.36$.

C Every orbital fails, because the attractive $-(Z\alpha)^2/r^2$ term outweighs the Coulomb term near the origin whatever l is.

D Only the s orbitals fail, and the p and d orbitals keep $s = 1$ and $s = 2$ exactly, as in the non-relativistic atom, because $(Z\alpha)^2$ is negligible beside $l(l + 1)$.

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-05.05 · UAQM-05.06 **[1]**

In Fermi's theory of beta decay and its generalisations, $n \to p\, e^-\, \bar{\nu}$ arises from contact terms in the Lagrangian density, each the product of a bilinear in the proton and neutron fields and a bilinear in the electron and neutrino fields, with $\bar{\psi} = \psi\dagger\gamma^0$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. Under parity each Dirac field becomes $\gamma^0\psi(t, -x)$ times an intrinsic phase, and the phases are chosen so that Fermi's vector term $G_V(\bar{\psi}_p\gamma^\mu\psi_n)(\bar{\psi}_e\gamma_\mu\psi_\nu)$ conserves parity. Which further term, with a real constant G, would make the interaction violate parity?

A $G(\bar{\psi}_p\psi_n)(\bar{\psi}_e\psi_\nu)$

B $G(\bar{\psi}_p\sigma^{\mu\nu}\psi_n)(\bar{\psi}_e\sigma_{\mu\nu}\psi_\nu)$

C $G(\bar{\psi}_p\gamma^\mu\gamma^5\psi_n)(\bar{\psi}_e\gamma_\mu\gamma^5\psi_\nu)$

D $G(\bar{\psi}_p\gamma^\mu\psi_n)(\bar{\psi}_e\gamma_\mu\gamma^5\psi_\nu)$

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-08.02 · UAQM-08.06 **[1]**

In a uniform static electric field of strength $\mathcal{E}$, the potential energy of an electron changes steadily along the field, so that a positive-energy electron state at one point has the same total energy as a negative-energy state a distance $d = 2mc^2/(e\mathcal{E})$ further along: Sauter's smooth version of the Klein step. In hole theory an electron tunnelling from the filled negative-energy states across this gap leaves a hole behind, so the field creates electron-positron pairs. Take $mc^2 = 0.511$ MeV, the reduced Compton wavelength $\hbar/mc = 3.86 \times 10^{-13}$ m and $e = 1.602 \times 10^{-19}$ C. At about what field strength does pair creation stop being exponentially suppressed?

A About $m^2c^3/(e\hbar) \approx 1.3 \times 10^{18}$ V m^{-1}, at which the gap d has shrunk to two reduced Compton wavelengths, so tunnelling across it is no longer suppressed.

B About 1.9×10^{16} V m^{-1}, at which the potential energy changes by $2mc^2$ across one Bohr radius, the size of the atoms that would supply the electrons.

C Any field at all, because Klein's step needs only a total rise of more than $2mc^2$, which a uniform field supplies over a long enough distance, whatever its gradient.

D About 3.6×10^{20} V m^{-1}, at which the potential energy changes by $2mc^2$ across one classical electron radius, 2.82 fm, the length at which classical electrodynamics fails.

Answer

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6 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-09.06 · UAQM-09.08 · UAQM-09.09 **[1]**

A free complex scalar field and a free Dirac field are each quantised in two ways: with commutators and with anticommutators of their mode operators. Two requirements are imposed: a Hamiltonian bounded below on a space of states of positive norm, and local observables that commute at spacelike separation. Which statement is correct?

A The Dirac field with commutators keeps a positive energy but fails microcausality, while the scalar field with anticommutators has an energy unbounded below.

B The Dirac field with commutators has an energy unbounded below, or states of negative norm if creators and annihilators are relabelled, while the scalar field with anticommutators fails microcausality; only the scalar with commutators and the Dirac field with anticommutators pass both tests.

C Both wrong quantisations fail only the positivity requirement, since microcausality holds for any choice because the vacuum amplitude $D(x - y)$ vanishes outside the light cone.

D The scalar field with anticommutators is excluded only because it allows at most one quantum in each mode, which contradicts the classical limit of the field.

Answer

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-10.01 · UAQM-10.03 **[1]**

Three fermion orbitals are ordered 1, 2, 3, and occupation-number states are defined by $|n_1\ n_2\ n_3\rangle = (c_1\dagger)^{n_1}(c_2\dagger)^{n_2}(c_3\dagger)^{n_3}|0\rangle$, with $\{c_i, c_j\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$. The one-body operator $T = t(c_3\dagger c_1 + c_1\dagger c_3)$ moves a fermion between orbitals 1 and 3. What is $T|1\ 1\ 0\rangle$?

A $+t|0\ 1\ 1\rangle$

B 0, because orbital 2 lies between orbitals 1 and 3 and is occupied

C $-t|1\ 0\ 1\rangle$

D $-t|0\ 1\ 1\rangle$

Answer

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-11.07 **[1]**

A single mode of the quantised field is prepared first in the number state $|n = 4\rangle$ and then in the coherent state $|\alpha\rangle$ with $|\alpha|^2 = 4$. Its normalised second-order correlation at zero delay is $g^{(2)}(0) = \langle a\dagger a\dagger aa\rangle/\langle a\dagger a\rangle^2$. What are the two values of $g^{(2)}(0)$, number state first?

A 1 and 1

B 0.75 and 2

C 0.75 and 1

D 0 and 1

Answer

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-12.08 **[1]**

A quantum dot emitting at a free-space wavelength $\lambda = 930$ nm sits at the field maximum of a GaAs micropillar cavity of refractive index $n = 3.5$, quality factor $Q = 6000$ and mode volume $V = 1.5(\lambda/n)^3$. Its transition is resonant with the cavity mode and its dipole is parallel to the cavity field. The Purcell factor, the ratio of its emission rate into the cavity mode to its spontaneous emission rate in bulk GaAs, is $F_P = (3/4\pi^2)(\lambda/n)^3(Q/V)$. What is F_P?

A 304

B 1.30×10^4

C 4.00×10^3

D It cannot be found without the dot's transition dipole moment, which sets both rates.

Answer

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-15.02 · UAQM-15.03 · UAQM-15.04 **[1]**

In Hartree-Fock theory the energy per electron of the uniform electron gas, in rydbergs, is E/N = 2.21/r_s² − 0.916/r_s, where r_s is the radius, in Bohr radii, of the sphere that contains one electron, and 1 Ry = 13.61 eV. The conduction electrons of lithium have r_s = 3.25. What is E/N at lithium's density, and how does that density compare with the density at which the Hartree-Fock E/N is lowest?

A E/N = −0.073 Ry (−0.99 eV), and lithium's electron gas is less dense than at the Hartree-Fock minimum.

B E/N = −0.282 Ry (−3.84 eV), and lithium's electron gas is denser than at the Hartree-Fock minimum.

C E/N = +0.209 Ry (+2.85 eV), and lithium's electron gas is denser than at the Hartree-Fock minimum.

D E/N = −0.073 Ry (−0.99 eV), and lithium's electron gas is denser than at the Hartree-Fock minimum.

Answer

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11 CORE COURSE

STRUCTURED · DEMANDING · UAQM-04.09 · UAQM-06.01 · UAQM-06.02 · UAQM-06.04 · UAQM-06.05 · UAQM-06.08 **[14]**

A negative muon, a spin-half Dirac particle of charge q = −e and mass m about 207 times the electron's, is captured by a lead nucleus, ^{208}Pb (Z = 82), and cascades to the 1s orbit of muonic lead. That orbit is so small that the muon spends most of its time inside the nucleus, modelled as a uniformly charged sphere of radius R = 7.1 fm, where its potential energy is V(r) = −(Zαℏc/R)(3/2 − r²/2R²). In static potentials φ and A its Dirac Hamiltonian is H = cα · π + βmc² + qφ, with π = p − qA, p = −iℏ∇ and V = qφ. In the Dirac representation, writing a stationary state as e^(−i(mc² + W)t/ℏ) times the column (ξ, χ) of upper and lower two-component spinors, Hψ = (mc² + W)ψ becomes (W − V)ξ = cσ · π χ and (W − V + 2mc²)χ = cσ · π ξ. Pauli's anomalous-moment term adds to H, in a magnetic field B, the operator −a(qℏ/2m)βΣ · B, where Σ = diag(σ, σ). For vector operators a and b that commute with the Pauli matrices, (σ · a)(σ · b) = a · b + iσ · (a × b), and for any function f(r), pf − fp = −iℏ∇f. In a storage-ring measurement of the anomaly, muons of Lorentz factor γ = 29.3 circulate in a uniform field B = 1.4513 T, with no electric field acting on them, and the rate of their decay positrons oscillates at the frequency ω_a/2π = 229.08 kHz at which the spin turns relative to the momentum. Take mc² = 105.66 MeV, m = 1.8835 × 10⁻²⁸ kg, ℏc = 197.33 MeV fm, α = 1/137.036, e = 1.602 × 10⁻¹⁹ C and the muon's lifetime at rest τ = 2.197 μs.

(a) SHOW THAT **[2]**

Show that the Dirac equation iℏ∂ψ/∂t = Hψ keeps its form under the gauge transformation A → A + ∇Λ, φ → φ − ∂Λ/∂t, for any smooth function Λ(r, t), provided that ψ → e^(iqΛ/ℏ)ψ.

(b) ESTIMATE **[2]**

Inside the nucleus V is the potential energy of a three-dimensional harmonic oscillator. Estimate, stating your assumptions, the oscillator quantum ℏω, the muon's speed in the 1s state as a fraction of c and hence the size of χ relative to ξ, and judge whether an expansion in v/c is trustworthy for muonic lead.

(c) DERIVE **[3]**

For a slow muon in a uniform magnetic field B, with A = ½B × r and no electric field, eliminate χ to leading order in v/c and derive the Hamiltonian for ξ to first order in B. Hence find the g-factors with which the orbital angular momentum L = r × p and the spin S = (ħ/2)σ couple to B under minimal coupling alone, and the spin g-factor once Pauli's term is included.

(d) CALCULATE **[2]**

For a slow muon the Pauli Hamiltonian makes the spin precess at ω_s = g|q|B/2m while the momentum turns at ω_c = |q|B/m, so the spin turns relative to the momentum at ω_a = a|q|B/m; with no electric field the Thomas-BMT equation shows that this holds at any speed. Calculate a for the muon from the storage-ring measurement, and the number of cycles of the positron-rate oscillation in one dilated lifetime γτ.

(e) SHOW THAT **[3]**

The Foldy-Wouthuysen reduction to order $(v/c)^2$ contains the even term $-(1/8m^2c^4)[O, [O, V]]$, where $O = c\alpha \cdot p$ is the odd part of H when A = 0. Show that for a static V this term equals $(\hbar^2/8m^2c^2)\nabla^2 V + (\hbar/4m^2c^2)\Sigma \cdot (\nabla V \times p)$, and that inside the uniformly charged nucleus the Darwin part is the constant $3Z\alpha(\hbar c)^3/(8(mc^2)^2R^3)$. Evaluate it for muonic lead.

(f) EXPLAIN **[2]**

The 1s and 2s states of muonic lead are orthogonal four-component spinors. Using your estimate from (b), explain why their upper components ξ_{1s} and ξ_{2s} cannot be eigenfunctions of any Hermitian Hamiltonian at order $(v/c)^2$, so that no two-component Schrödinger equation for ξ can be exact to that order, and what the Foldy-Wouthuysen transformation does instead.

12 CORE COURSE

STRUCTURED · DEMANDING · UAQM-07.02 · UAQM-07.04 · UAQM-07.05 · UAQM-07.07 · UAQM-07.08 · UAQM-07.09 **[12]**

The Lyman-α X-rays of hydrogen-like uranium, U^{91+} ($Z = 92$), emitted when bare uranium nuclei circulating in a heavy-ion storage ring capture an electron that cascades to the ground state, test the Dirac-Coulomb spectrum where $Z\alpha$ is not small. The two lines are Lyα1, $2p_{3/2} \to 1s_{1/2}$, and Lyα2, $2p_{1/2} \to 1s_{1/2}$, and spectroscopy of the stored ions places Lyα1 at 102.17 keV. In a central potential $V(r)$ the radial functions $G(r)$ and $F(r)$ of the upper and lower components satisfy $dG/dr = -(\kappa/r)G + [(E + mc^2 - V)/\hbar c]F$ and $dF/dr = (\kappa/r)F - [(E - mc^2 - V)/\hbar c]G$, where E includes mc^2, $\kappa = (l - j)(2j + 1)$ and l is the orbital angular momentum of the upper component. For a point nucleus $V = -Z\alpha\hbar c/r$, and the exact energies are $E_{nj} = mc^2[1 + (Z\alpha)^2/(n - |\kappa| + \gamma)^2]^{(-1/2)}$, with $\gamma = \sqrt{(\kappa^2 - (Z\alpha)^2)}$. The normalised point-nucleus 1s density, both components together, is $|\psi_{1s}(r)|^2 = [(2Z/a_0)^{(2\gamma+1)}/(4\pi\Gamma(2\gamma + 1))]r^{(2\gamma-2)}e^{(-2Zr/a_0)}$, with $\Gamma(2\gamma + 1) = 1.313$ for $Z = 92$, while the Schrödinger 1s density is $(Z^3/\pi a_0^3)e^{(-2Zr/a_0)}$. The uranium nucleus has its surface at about $r = 7.5$ fm and is treated as infinitely heavy. Take $mc^2 = 510.999$ keV, $\alpha = 1/137.036$ and $a_0 = 5.2918 \times 10^4$ fm.

(a) STATE **[2]**

State which conserved operator takes the place of L^2 in labelling the electron's states, and state κ and the orbital angular momentum l' of the lower component for each of $1s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$, saying which of their components are s waves, able to reach the nucleus.

(b) DERIVE **[3]**

Derive from the radial equations how G and F rise from the origin. For the point nucleus, writing $G \approx ar^{\gamma}$ and $F \approx br^{\gamma}$, find γ and b/a, evaluate b/a for $1s_{1/2}$ and for $2p_{1/2}$ at $Z = 92$, and say which component dominates near the nucleus in each. Then find the powers of r with which G and F rise inside a nucleus of finite size, where $V(r)$ tends to a finite $V(0)$.

(c) CALCULATE **[2]**

Calculate, for a point nucleus, the energies of Lyα1 and Lyα2 in keV, and the 2p fine-structure splitting.

(d) DETERMINE **[2]**

Treating the $2p_{3/2}$ level as given correctly by the point-nucleus formula, determine from the measured Lyα1 energy the shift of the 1s level that the point-nucleus Dirac theory omits, its sign, and its size as a fraction of the 1s binding energy.

(e) COMPARE **[2]**

Compare the Dirac 1s density at the nuclear surface, r = 7.5 fm, with the Schrödinger 1s density there, and say what the comparison implies for the finite-size part of the shift found in (d).

(f) PREDICT **[1]**

In the point-nucleus Dirac theory $2s_{1/2}$ and $2p_{1/2}$ are exactly degenerate. Predict which of the two the finite size of the nucleus raises more, and so which lies higher.

13

CORE COURSE

STRUCTURED · DEMANDING · UAQM-09.01 · UAQM-09.03 · UAQM-09.04 **[12]**

A free complex scalar field φ(x, t), with φ and φ† treated as independent fields, describes spinless particles of mass m and charge q through the Lagrangian density $\mathcal{L} = (1/c^2)(\partial\varphi^\dagger/\partial t)(\partial\varphi/\partial t) - \nabla\varphi^\dagger \cdot \nabla\varphi - \kappa^2\varphi^\dagger\varphi$, where $\kappa = mc/\hbar$. In a box of volume V with periodic boundary conditions the quantised field is φ(x, t) = Σ_k N_k[a_k e^(i(k·x − ω_k t)) + b_k† e^(−i(k·x − ω_k t))], with ω_k = c√(k² + κ²), N_k = √(ħc²/(2ω_kV)), [a_k, a_k′†] = [b_k, b_k′†] = δ_kk′ and all other commutators zero. After normal ordering the Hamiltonian is H = Σ_k ħω_k(a_k†a_k + b_k†b_k). Take ħc = 197.3 MeV fm.

(a) DERIVE **[3]**

Apply the Euler-Lagrange equation for φ† to derive the field equation for φ, and use it to show that the density $\rho = (iq/\hbar c^2)(\varphi^\dagger\partial\varphi/\partial t - \varphi\partial\varphi^\dagger/\partial t)$ and the current $\mathbf{j} = -(iq/\hbar)(\varphi^\dagger\nabla\varphi - \varphi\nabla\varphi^\dagger)$ satisfy $\partial\rho/\partial t + \nabla \cdot \mathbf{j} = 0$.

(b) SHOW THAT **[3]**

By substituting the mode expansion, show that after normal ordering the charge $Q = \int\rho\, d^3x$ is qΣ_k(a_k†a_k − b_k†b_k), and hence that a_k† creates a quantum of charge +q and b_k† one of charge −q, each with energy ħω_k.

(c) EXPLAIN **[2]**

The field commutator is [φ(x), φ†(y)] = D(x − y) − D(y − x), where D(x − y) = ⟨0|φ(x)φ†(y)|0⟩ = Σ_k N_k² e^(i(k·(x − y) − ω_k(t_x − t_y))). Explain why the commutator vanishes when x − y is spacelike although D(x − y) does not, and what would go wrong if the field contained the a_k terms alone.

(d) ESTIMATE **[2]**

At equal times and separation r, D is proportional to $(1/r)K_1(\kappa r)$, which for $\kappa r \gg 1$ behaves as $r^{(-3/2)}e^{(-\kappa r)}$. For a π^+, with $mc^2 = 139.6$ MeV, estimate the length $1/\kappa$ over which the amplitude extends outside the light cone, and the factor by which D falls between r = 5 fm and r = 10 fm, stating your assumptions.

(e) COMPARE **[2]**

Compare the density ρ of this question, as an operator of the quantised field, with the same expression read as a probability density in the one-particle Klein-Gordon theory.

14 CORE COURSE

STRUCTURED · DEMANDING · UAQM-10.03 · UAQM-10.04 · UAQM-10.05 · UAQM-10.09 **[12]**

Two electrons occupy a molecule of two sites, 1 and 2, each with one orbital, described by the Hubbard Hamiltonian $H = -t\,\Sigma_\sigma\,(c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$, with hopping $t = 0.40$ eV and on-site repulsion $U = 3.2$ eV. The operators obey $\{c_{i\sigma}, c_{j\sigma'}^\dagger\} = \delta_{ij}\,\delta_{\sigma\sigma'}$ and $\{c_{i\sigma}, c_{j\sigma'}\} = 0$. In the sector with $S_z = 0$ a basis is $|D_1\rangle = c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle$ and $|D_2\rangle = c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle$, with both electrons on one site, and $|A\rangle = c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle$ and $|B\rangle = c_{2\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle$, with one electron on each.

(a) SHOW THAT **[3]**

Show that the hopping term T sends both $|A\rangle$ and $|B\rangle$ to $-t(|D_1\rangle + |D_2\rangle)$, and hence that in the basis $|S\rangle = (|A\rangle + |B\rangle)/\sqrt{2}$, $|D_+\rangle = (|D_1\rangle + |D_2\rangle)/\sqrt{2}$ the Hamiltonian is the matrix $[[0, -2t], [-2t, U]]$, while $(|A\rangle - |B\rangle)/\sqrt{2}$ and $(|D_1\rangle - |D_2\rangle)/\sqrt{2}$ are eigenstates with energies 0 and U.

(b) CALCULATE **[2]**

Calculate the ground-state energy and the singlet-triplet splitting, and compare the splitting with the superexchange coupling $4t^2/U$.

(c) DETERMINE **[2]**

Determine the probability of finding both electrons on the same site in the ground state.

(d) EVALUATE **[3]**

A student describes the ground state by the restricted Hartree-Fock determinant, with both electrons in the bonding orbital, created by $(c_{1\sigma}^\dagger + c_{2\sigma}^\dagger)/\sqrt{2}$. Evaluate this description: find its energy and its probability of double occupation, and say what it gets wrong as U/t grows.

(e) EXPLAIN **[2]**

Unrestricted Hartree-Fock lets the up-spin and down-spin electrons occupy different orbitals, and its best determinant has energy $-2t^2/U = -0.10$ eV. Explain why it recovers only about half of the exact binding, and what symmetry of H it breaks to do so.

15 CORE COURSE

STRUCTURED · DEMANDING · UAQM-13.02 · UAQM-13.03 · UAQM-13.04 · UAQM-13.05 · UAQM-13.07 **[12]**

In Bethe's non-relativistic calculation a hydrogen atom in a state $|n\rangle$ is coupled to the transverse vacuum field through $H' = (e/m)A \cdot p$ in the dipole approximation, photon energies being cut off at K. Summing the second-order energy over photon modes and polarisations gives $W_n = -(2\alpha/(3\pi m^2c^2)) \Sigma_m |\langle m|p|n\rangle|^2 \int_0^K k\, dk/(E_m - E_n + k)$, where $k = \hbar\omega$ is the photon energy and the sum runs over all bound and continuum states $|m\rangle$. For a free electron of momentum p the same calculation gives $W_{free} = -(2\alpha/(3\pi m^2c^2))p^2K$. Write $p_{mn} = \langle m|p|n\rangle$. Take $mc^2 = 0.510999$ MeV, $\alpha = 1/137.036$, 1 Ry = 13.6057 eV and $h = 4.1357 \times 10^{-15}$ eV s.

(a) SHOW THAT **[2]**

Show that, for K large compared with every $|E_m - E_n|$ that matters, W_n grows linearly with K, and that the coefficient of K is the free-electron result averaged over the state $|n\rangle$.

(b) EXPLAIN **[2]**

Explain why the linear term is removed by mass renormalisation, and why what remains depends on K only through a logarithm.

(c) SHOW THAT **[3]**

The sum rule $\Sigma_m |p_{mn}|^2(E_m - E_n) = (\hbar^2/2)\langle n|\nabla^2 V|n\rangle$ holds for $V = -e^2/(4\pi\varepsilon_0 r)$. Defining the mean excitation energy K_0 by $\ln K_0 = \Sigma_m |p_{mn}|^2(E_m - E_n) \ln|E_m - E_n| / \Sigma_m |p_{mn}|^2(E_m - E_n)$, show that $\Delta E_n = (4\alpha^5 mc^2/(3\pi n^3)) \ln(K/K_0)$ for s states, and that the term in ln K vanishes in every other state. You may use $|\psi_{ns}(0)|^2 = 1/(\pi n^3 a_0^3)$, $a_0 = \hbar/(\alpha mc)$ and $e^2/(4\pi\varepsilon_0) = \alpha\hbar c$.

(d) CALCULATE **[3]**

With the cutoff at $K = mc^2$ and the quoted mean excitation energies $K_0(2s) = 16.64$ Ry and $K_0(1s) = 19.77$ Ry, calculate the shifts of 2s and of 1s in frequency units, and compare them with the measured $2s_{1/2}$–$2p_{1/2}$ splitting of 1057.8 MHz and 1s shift of 8.17 GHz.

(e) EVALUATE **[2]**

The non-relativistic calculation fixes the cutoff K only to within a factor of about two. Evaluate whether the agreement found in (d) shows that Bethe's choice $K = mc^2$ is correct.

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16 CORE COURSE

DATA ANALYSIS · DEMANDING · UAQM-14.02 · UAQM-14.06 **[11]**

A single InGaAs quantum dot at 4 K is driven resonantly by a narrow-band laser, and its fluorescence is recorded as the laser frequency is scanned across the transition. For each laser power P at the dot, the full width at half maximum $\Delta\nu$ of the fluorescence line, as a frequency, is recorded. A separate time-resolved measurement gives the population relaxation time $T_1 = 0.80$ ns. For a two-level emitter with population relaxation time T_1 and coherence time T_2, driven at Rabi frequency Ω and detuning Δ (both angular frequencies), the steady state of the optical Bloch equations has the excited-state population $\rho_{ee} = \tfrac{1}{2}\Omega^2 T_1 T_2/(1 + \Delta^2 T_2^2 + \Omega^2 T_1 T_2)$, and Ω^2 is proportional to P. Without pure dephasing $T_2 = 2T_1$; in general $1/T_2 = 1/(2T_1) + 1/T_\varphi$, where T_φ is the pure dephasing time.

Fluorescence linewidth of one quantum dot at six laser powers

Run	Laser power P / nW	Linewidth $\Delta\nu$ / MHz
1	2.0	289
2	5.0	314
3	10.0	361
4	20.0	429
5	40.0	554
6	80.0	732

(a) SHOW THAT **[2]**

Show that $\Delta\nu = (1/\pi T_2)\sqrt{(1 + \Omega^2 T_1 T_2)}$, and hence that $\Delta\nu^2 = \Delta\nu_0^2(1 + P/P_{sat})$, where $\Delta\nu_0 = 1/(\pi T_2)$ and P_{sat} is the power at which $\Omega^2 T_1 T_2 = 1$.

(b) DESCRIBE **[2]**

Describe how the readings should be plotted to test this model and to extract $\Delta\nu_0$ and P_{sat}, and explain why the narrowest line measured is not itself $\Delta\nu_0$.

(c) DETERMINE **[3]**

Determine the gradient and intercept of your graph, and hence $\Delta\nu_0$ and P_sat.

(d) EVALUATE **[3]**

Evaluate whether this emitter's line is lifetime-limited, finding T_2 and, if it is needed, the pure dephasing time T_φ.

(e) PREDICT **[1]**

Predict the rate at which the dot scatters photons when it is driven on resonance at P = 40 nW.

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17 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-11.03 · UAQM-11.05 · UAQM-12.01 · UAQM-12.02 · UAQM-12.03 **[15]**

In a cubic box of volume V with periodic boundary conditions, the transverse vector potential in Coulomb gauge is, in the Heisenberg picture, A(r, t) = Σ_(k,λ) √(ħ/(2ε₀ω_kV)) [a_(kλ) ε_(kλ) e^(i(k·r − ω_k t)) + a_(kλ)† ε_(kλ)* e^(−i(k·r − ω_k t))], with ω_k = ck, two unit polarisation vectors ε_(kλ) perpendicular to k for each k, and [a_(kλ), a_(k′λ′)†] = δ_kk′ δ_λλ′. A one-electron atom or ion, with an infinitely heavy nucleus at the origin, has an excited state $|b\rangle$ and a lower state $|a\rangle$ separated by $\hbar\omega_0$. It couples to the field through $H' = (e/m)A(0)\cdot p$, the dipole approximation, where $-e$ and m are the electron's charge and mass, and $\langle a|p|b\rangle = -im\omega_0\langle a|r|b\rangle$. For a hydrogen-like ion of nuclear charge Ze, $|\langle 1s|r|2p, m\rangle|^2 = (2^{15}/3^{10})(a_0/Z)^2$ for each m, and the 2p–1s energy difference is $(3/4)Z^2 \times 13.606$ eV. Take $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C, $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, $c = 2.998 \times 10^8$ m s^{-1} and $a_0 = 5.292 \times 10^{-11}$ m.

(a) DERIVE **[2]**

Derive the transverse electric field operator from A, and show that in the vacuum each mode contributes $\hbar\omega_k/(2\varepsilon_0 V)$ to $\langle E^2\rangle$, so that its mean electric energy is half its zero-point energy.

(b) SHOW THAT **[2]**

Show that the matrix element for emission of a photon into a mode (k, λ) already holding n photons is $\langle a; n+1|H'|b; n\rangle = (e/m)\sqrt{\hbar/(2\varepsilon_0\omega_k V)}\,\sqrt{n+1}\,\varepsilon_{(k\lambda)}^*\cdot\langle a|p|b\rangle$, and explain why this makes an excited atom in the vacuum emit, though an atom in a classical field of zero amplitude never does.

(c) DERIVE **[4]**

Using Fermi's golden rule, and summing over all modes, directions and both polarisations, derive $A = \omega_0^3 e^2 |\langle a|r|b\rangle|^2/(3\pi\varepsilon_0\hbar c^3)$ for spontaneous emission from $|b\rangle$ to $|a\rangle$.

(d) CALCULATE **[3]**

Calculate A, the lifetime and the natural linewidth $\Gamma/2\pi$ of the 2p state of the hydrogen-like helium ion He^+ ($Z = 2$).

(e) EXPLAIN **[2]**

The atom now sits in thermal radiation at temperature T, whose modes near ω_0 hold $\bar{n}$ photons on average. Explain how the $\sqrt{n}$ and $\sqrt{(n+1)}$ factors of the field operators lead to Planck's distribution, rather than assuming it as Einstein did.

(f) JUSTIFY **[2]**

Justify the dipole approximation, and the use of the golden rule leading to exponential decay at the rate A, for the He^+ 2p state.

18

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UAQM-15.07 · UAQM-15.08 · UAQM-15.09 **[16]**

A uniform gas of N spinless bosons of mass m in a box of volume V, with periodic boundary conditions, interacts through a short-range potential represented at low energy by the contact coupling $g = 4\pi\hbar^2 a_s/m$, where a_s is the s-wave scattering length. In second quantisation $H = \Sigma_k \varepsilon_k a_k^\dagger a_k + (g/2V)\Sigma_{k,k',q} a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k$, with $\varepsilon_k = \hbar^2 k^2/2m$. Bogoliubov's approximation replaces a_0 and $a_0^\dagger$ by $\sqrt{N_0}$, where N_0 is the number of bosons in the condensate, and keeps the terms of H up to second order in the operators a_k with $k \neq 0$, the sums over such k being written Σ'. Write $n = N/V$. Take $\hbar = 1.055 \times 10^{-34}$ J s, $1\ u = 1.661 \times 10^{-27}$ kg and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) SHOW THAT **[3]**

Show that, to this order and with N_0 eliminated in favour of N, $H \approx gN^2/(2V) + \Sigma'[(\varepsilon_k + gn)a_k^\dagger a_k + (gn/2)(a_k^\dagger a_{-k}^\dagger + a_k a_{-k})]$.

(b) DERIVE **[4]**

Writing $a_k = u_k b_k - v_k b_{-k}^\dagger$ with u_k and v_k real, derive the condition on u_k and v_k that keeps $[b_k, b_k^\dagger] = 1$, the choice that removes the terms $b_k b_{-k}$ and $b_k^\dagger b_{-k}^\dagger$, and the excitation energy $E_k = \sqrt{(\varepsilon_k(\varepsilon_k + 2gn))}$.

(c) SKETCH **[2]**

Sketch E_k against k, with ε_k on the same axes, marking the behaviour at small and at large k and the scale $k \approx 1/\xi$, where $\xi = \hbar/\sqrt{(2mgn)}$, at which it changes.

(d) CALCULATE **[3]**

For a condensate of ^{87}Rb ($m = 87$ u, $a_s = 5.3$ nm) of density $n = 2.0 \times 10^{20}$ m^{-3}, calculate gn/k_B in nanokelvin, the speed of sound c, the healing length ξ and $n a_s^3$.

(e) DETERMINE **[2]**

Determine $\langle a_k^\dagger a_k \rangle$ for $k \neq 0$ in the ground state, the vacuum of every b_k, and hence, given that the sum over k gives a depletion $(N - N_0)/N = (8/(3\sqrt{\pi}))\sqrt{(n a_s^3)}$, the fraction of the atoms of (d) that lie outside the condensate at zero temperature.

(f) PREDICT **[2]**

A tightly focused repulsive laser beam is dragged through the condensate of (d) at 1.5 mm s^{-1}. Predict, using Landau's criterion, whether it heats the gas, and whether an obstacle moving at the same speed through an ideal Bose gas would.

MARK SCHEME

University Advanced Quantum Mechanics · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-01.06 · UAQM-01.07 **[1]**

A metallic grain a few nanometres across holds an odd number N of conduction electrons. Its shape is irregular, its impurities are random, and strong spin-orbit scattering mixes spin with orbital motion, so that no spatial symmetry and no component of the total spin is conserved. No magnetic field is applied, so every term in the grain's Hamiltonian H is even under time reversal, and for N spin-half particles the time-reversal operator satisfies Θ² = (−1)^N. Which statement about the energy levels of the grain is correct?

KEY **B** **Every level is at least twofold degenerate: for an eigenstate |ψ⟩, Θ|ψ⟩ has the same energy, and Θ² = −1 makes it orthogonal to |ψ⟩.**

B is correct. Because H is even under time reversal, ΘH = HΘ, so Θ|ψ⟩ is an eigenstate with the same energy as |ψ⟩. Antiunitarity, ⟨Θa|Θb⟩ = ⟨b|a⟩, applied with a = Θψ and b = ψ gives ⟨ψ|Θψ⟩ = ⟨Θ²ψ|Θψ⟩ = −⟨ψ|Θψ⟩ when Θ² = −1, so ⟨ψ|Θψ⟩ = 0 and the two states are distinct: every level is a Kramers doublet, whatever the shape, the disorder or the strength of the spin-orbit scattering. A is tempting because a system with no spatial symmetry is expected to have no degeneracy at all, which is right for an even number of electrons; it forgets that time reversal is itself a symmetry, and for odd N it forces the pairing. C pictures the doublets as spin-up and spin-down copies of one orbital, which they are only without spin-orbit coupling; Kramers' theorem pairs |ψ⟩ with Θ|ψ⟩, which need not be eigenstates of any spin component, so the degeneracy survives spin-orbit scattering of any strength, though a magnetic field would split each pair, with a g-factor that the scattering reduces. D treats the many-electron levels as if they were built from single-electron pairs; with N + 1 electrons Θ² = +1, Θ|ψ⟩ can be a multiple of |ψ⟩, nothing forces the levels to pair, and a grain with an even number of electrons generically has non-degenerate levels.

U1

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2 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UAQM-02.05 · UAQM-02.08 **[1]**

The elastic scattering of ^{16}O by ^{16}O is followed at a fixed centre-of-mass angle θ = 60° while the centre-of-mass energy is stepped down from 8.0 MeV to 3.0 MeV, below the Coulomb barrier of about 10 MeV; at this angle the nuclei never come closer than about 17 fm, so only the Coulomb interaction acts. ^{16}O nuclei are identical spin-0 bosons. The Coulomb amplitude the pair would have if the nuclei were distinguishable is, up to a phase that does not depend on θ, f(θ) = −[η/(2k sin²(θ/2))] exp[−iη ln sin²(θ/2)], where k is the relative wavenumber and the Sommerfeld parameter η = Z²α/β, with βc the relative speed, rises from 10.1 to 16.5 over the range. Let R be the measured differential cross-section divided by |f(60°)|², the Rutherford value for the projectile alone. How does R behave over the range?

KEY **C** **R oscillates between 4/9 and 16/9, passing through more than one full cycle as η ln 3 changes by about 7 rad.**

For identical spin-0 bosons the amplitude is f(θ) + f(π − θ). At θ = 60°, |f(120°)|/|f(60°)| = sin²30°/cos²30° = tan²30° = 1/3, and the relative phase of the two amplitudes is η ln[cos²(θ/2)/sin²(θ/2)] = η ln 3. So R = |1 + (1/3)e^(iη ln 3)|² = 1 + 1/9 + (2/3)cos(η ln 3). As η rises from 10.1 to 16.5 the phase η ln 3 runs from 11.1 rad to 18.1 rad, more than 2π, so R swings through its whole range, from (1 − 1/3)² = 4/9 to (1 + 1/3)² = 16/9: these are Mott oscillations (C). A keeps |f(θ)|² + |f(π − θ)|² and drops the cross term, as if the nuclei were distinguishable; the Coulomb phase does cancel in each squared amplitude, but not in the cross term, which is where exchange shows up. B would need the two amplitudes to have equal size, which happens only at 90°, and even there the relative phase η ln 1 vanishes, so R is fixed at 4 rather than oscillating; at 60° the exchanged amplitude is a third of the direct one and the cancellation can never be complete. D confuses symmetrisation with constructive interference: bosons add amplitudes, not intensities, and when the relative phase is near π the two amplitudes partly cancel, taking R below the incoherent sum 10/9.

U2

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-03.06 · UAQM-03.08 **[1]**

A negative kaon, K^-, is a spinless particle of charge $-e$ that can be captured into atomic orbits. Consider a kaon bound to a lead nucleus ($Z = 82$), treated as a point charge, with the strong interaction ignored. Its Klein-Gordon radial equation contains the centrifugal term $\hbar^2c^2l(l + 1)/r^2$ together with an attractive term $-(Z\alpha\hbar c)^2/r^2$ that comes from squaring the Coulomb energy, so that near the origin the radial function R(r) behaves as r^s, with $s(s + 1) = l(l + 1) - (Z\alpha)^2$. Take $\alpha = 1/137.04$. Which statement about the kaon's s, p and d orbitals ($l = 0$, 1 and 2) is correct?

KEY **A** **Only the s orbitals fail: for l = 0, s is complex and the solutions oscillate without limit near the origin, a fall to the centre, while the p and d orbitals have s ≈ 0.88 and s ≈ 1.93.**

Solving $s(s + 1) = l(l + 1) - (Z\alpha)^2$ gives $s = -½ + \sqrt{((l + ½)^2 - (Z\alpha)^2)}$, which is real only while $Z\alpha < l + ½$. With $Z\alpha = 82/137.04 = 0.598$, $(Z\alpha)^2 = 0.358$ exceeds ¼, so for the s waves $s = -½ \pm 0.329i$: R oscillates as r^(−1/2)cos(0.329 ln r) infinitely often near the origin, the point-nucleus problem has no lowest level, and the particle falls to the centre (A). For $l = 1$, $s = -½ + \sqrt{(2.25 - 0.358)} = 0.876$, and for $l = 2$, $s = -½ + \sqrt{(6.25 - 0.358)} = 1.927$, both real. B borrows the Dirac condition $Z\alpha < |\kappa| = 1$ and the small-$Z\alpha$ result $s \approx -(Z\alpha)^2$, which holds only for $Z\alpha \ll ½$; for the Klein-Gordon equation the threshold is $Z\alpha = l + ½$, which lead's s waves have already passed. C forgets that the centrifugal term has the same $1/r^2$ form: what matters is the net coefficient $(l + ½)^2 - (Z\alpha)^2$ in the indicial equation, which stays positive for $l \geq 1$ until $Z\alpha$ reaches 1.5. D picks the failing orbitals correctly but treats $(Z\alpha)^2$ as negligible; at $Z\alpha = 0.6$ it lowers the p-wave exponent by 12%, from 1 to 0.88.

U2

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-05.05 · UAQM-05.06 **[1]**

In Fermi's theory of beta decay and its generalisations, $n \to p\ e^-\ \bar{\nu}$ arises from contact terms in the Lagrangian density, each the product of a bilinear in the proton and neutron fields and a bilinear in the electron and neutrino fields, with $\bar{\psi} = \psi^\dagger\gamma^0$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and σ^μν = (i/2)[γ^μ, γ^ν]. Under parity each Dirac field becomes $\gamma^0\psi(t, -x)$ times an intrinsic phase, and the phases are chosen so that Fermi's vector term G_V(ψ̄_pγ^μψ_n)(ψ̄_eγ_μψ_ν) conserves parity. Which further term, with a real constant G, would make the interaction violate parity?

KEY **D** **G(ψ̄_pγ^μψ_n)(ψ̄_eγ_μγ⁵ψ_ν)**

With the phases fixed so that the vector term is even, every term carries the same overall phase factor, so each term's parity is set by its bilinears. Under parity a bilinear $\bar{\psi}_1\Gamma\psi_2$ becomes $\bar{\psi}_1\gamma^0\Gamma\gamma^0\psi_2$ evaluated at $-x$. For $\Gamma = 1$ it is unchanged, a scalar. For $\Gamma = \gamma^\mu$ the time component keeps its sign and the space components reverse, a polar vector. For $\Gamma = \gamma^\mu\gamma^5$, $\gamma^0\gamma^\mu\gamma^5\gamma^0 = -\gamma^0\gamma^\mu\gamma^0\gamma^5$, so the time component reverses and the space components do not, an axial vector. For $\Gamma = \sigma^{\mu\nu}$ the components $\sigma^{(0i)}$ reverse and $\sigma^{(ij)}$ do not, so a contraction of two tensors is even. D contracts a polar vector with an axial vector, which changes sign under parity, so adding it to the vector term leaves no choice of phases that makes both even: this cross term is how the V − A weak interaction violates parity (D). A is tempting to a candidate who suspects a coupling without γ^μ, but $\bar{\psi}\psi$ is a true scalar and a product of two scalars is even. B is tempting because σ^μν mixes components of opposite parity, but each component is contracted with one of the same kind, so the product is even. C is tempting because γ^5 marks a pseudo-quantity, but there are two axial vectors, and their two sign changes cancel.

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-08.02 · UAQM-08.06 **[1]**

In a uniform static electric field of strength $\mathcal{E}$, the potential energy of an electron changes steadily along the field, so that a positive-energy electron state at one point has the same total energy as a negative-energy state a distance d = 2mc²/(e$\mathcal{E}$) further along: Sauter's smooth version of the Klein step. In hole theory an electron tunnelling from the filled negative-energy states across this gap leaves a hole behind, so the field creates electron-positron pairs. Take mc² = 0.511 MeV, the reduced Compton wavelength ħ/mc = 3.86 × 10⁻¹³ m and e = 1.602 × 10⁻¹⁹ C. At about what field strength does pair creation stop being exponentially suppressed?

KEY **A** **About m²c³/(eħ) ≈ 1.3 × 10¹⁸ V m⁻¹, at which the gap d has shrunk to two reduced Compton wavelengths, so tunnelling across it is no longer suppressed.**

Tunnelling through a forbidden gap is exponentially suppressed unless the gap is no wider than the decay length of the wavefunction inside it, which for an electron between the two continua is of order ħ/mc. Setting d = 2mc²/(e$\mathcal{E}$) equal to 2ħ/mc gives $\mathcal{E}$ = mc²/(e × ħ/mc) = m²c³/(eħ) = 0.511 × 10⁶ V/3.86 × 10⁻¹³ m = 1.3 × 10¹⁸ V m⁻¹, the critical field; below it the pair-creation rate carries the factor exp(−πm²c³/(eħ$\mathcal{E}$)) (A). C applies Klein's sharp-step result to a smooth field. An abrupt step higher than 2mc² does transmit freely, but only because it is sharper than ħ/mc; spreading the same rise over a macroscopic distance makes the gap wide and the transmission vanishingly small, which is why laboratory fields do not break down the vacuum. B and D choose the wrong length. The Bohr radius has nothing to do with the vacuum, whose electrons are the negative-energy sea everywhere, and the field it gives is about 70 times too weak. The classical electron radius is α times the reduced Compton wavelength, so it gives a field about 270 times too strong.

U1

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6 CORE COURSE

MULTIPLE CHOICE · RECALL · UAQM-09.06 · UAQM-09.08 · UAQM-09.09 **[1]**

A free complex scalar field and a free Dirac field are each quantised in two ways: with commutators and with anticommutators of their mode operators. Two requirements are imposed: a Hamiltonian bounded below on a space of states of positive norm, and local observables that commute at spacelike separation. Which statement is correct?

KEY **B** **The Dirac field with commutators has an energy unbounded below, or states of negative norm if creators and annihilators are relabelled, while the scalar field with anticommutators fails microcausality; only the scalar with commutators and the Dirac field with anticommutators pass both tests.**

B is correct, and the two failures are different in kind. Quantised canonically with commutators, the Dirac field has mode operators with [b, b†] = [d, d†] = 1, where d multiplies the negative-frequency modes, and H = Σ E_p(b†b − d†d): every quantum that d† creates lowers the energy, without limit. Making d† the annihilation operator instead, to cure this, turns its commutator with its adjoint into −1 and gives states of negative norm. The canonical commutator of the Dirac field is nevertheless causal, so A puts the Dirac field's failure in the wrong place, and the positive energy it claims is exactly what commutators cannot deliver on states of positive norm. For the scalar field the failure is of the other kind: anticommutators replace the commutator D(x − y) − D(y − x) by the anticommutator D(x − y) + D(y − x), which does not vanish at spacelike separation, and local observables built from the field then fail to commute there, so measurements outside each other's light cones would interfere. C repeats the commonest misconception of the unit: the propagation amplitude D(x − y) does not vanish outside the light cone, falling only as e^(−mcr/ħ); only the commutator, in which the antiparticle term cancels it, vanishes. D offers a heuristic rather than the argument: the limit on occupation is a consequence of anticommutators, not the reason they are excluded, and the decisive failure is causality.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-10.01 · UAQM-10.03 **[1]**

Three fermion orbitals are ordered 1, 2, 3, and occupation-number states are defined by $|n_1\ n_2\ n_3\rangle = (c_1\dagger)^{n_1}(c_2\dagger)^{n_2}(c_3\dagger)^{n_3}|0\rangle$, with $\{c_i, c_j\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$. The one-body operator $T = t(c_3\dagger c_1 + c_1\dagger c_3)$ moves a fermion between orbitals 1 and 3. What is $T|1\ 1\ 0\rangle$?

KEY **D** **$-t|0\ 1\ 1\rangle$**

The term $c_1\dagger c_3$ gives zero because orbital 3 is empty. In the other term, c_1 acting on $c_1\dagger c_2\dagger|0\rangle$ gives $c_2\dagger|0\rangle$, and $c_3\dagger$ must then be moved past $c_2\dagger$ to reach its standard place on the right of the product: $c_3\dagger c_2\dagger|0\rangle = -c_2\dagger c_3\dagger|0\rangle = -|0\ 1\ 1\rangle$. So $T|1\ 1\ 0\rangle = -t|0\ 1\ 1\rangle$ (D). The sign is -1 raised to the number of occupied orbitals between the two that the fermion moves between, and it is physical: on a ring or a lattice it decides whether a hopping process lowers or raises the energy. A is the answer of a candidate who moves the electron but not the operator, which is what a product wavefunction without antisymmetry would give. B treats the occupied orbital as an obstacle; the hop is between orbitals, not along a line, and orbital 2 contributes only a sign. C moves the wrong fermion: $c_3\dagger c_1$ empties orbital 1, so the final state must have $n_1 = 0$.

U2

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8 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-11.07 **[1]**

A single mode of the quantised field is prepared first in the number state $|n = 4\rangle$ and then in the coherent state $|\alpha\rangle$ with $|\alpha|^2 = 4$. Its normalised second-order correlation at zero delay is $g^{(2)}(0) = \langle a\dagger a\dagger a a\rangle/\langle a\dagger a\rangle^2$. What are the two values of $g^{(2)}(0)$, number state first?

KEY **C** **0.75 and 1**

For a number state $a\dagger a\dagger a a|n\rangle = n(n-1)|n\rangle$, so $g^{(2)}(0) = n(n-1)/n^2 = 1 - 1/n = 0.75$. For a coherent state $a|\alpha\rangle = \alpha|\alpha\rangle$, so $\langle a\dagger a\dagger a a\rangle = |\alpha|^4 = \langle a\dagger a\rangle^2$ and $g^{(2)}(0) = 1$ whatever $|\alpha|$ is (C). The number state's value below 1 is impossible for any classical field, for which $g^{(2)}(0) = \langle I^2\rangle/\langle I\rangle^2$ is at least 1 by the Cauchy-Schwarz inequality: it records that once one photon has been detected only three remain. A supposes that $g^{(2)}(0)$ depends only on the mean photon number, which the two states share. B gives the coherent state the value 2 of thermal light, whose photon-number fluctuations are larger than Poissonian. D reasons that a number state has no fluctuations in photon number and so no correlation; its variance is indeed zero, but $g^{(2)}(0) = 1 + (\mathrm{Var}\ n - \langle n\rangle)/\langle n\rangle^2$, which is $1 - 1/n$, not 0.

U3

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9 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-12.08 **[1]**

A quantum dot emitting at a free-space wavelength $\lambda = 930$ nm sits at the field maximum of a GaAs micropillar cavity of refractive index $n = 3.5$, quality factor $Q = 6000$ and mode volume $V = 1.5(\lambda/n)^3$. Its transition is resonant with the cavity mode and its dipole is parallel to the cavity field. The Purcell factor, the ratio of its emission rate into the cavity mode to its spontaneous emission rate in bulk GaAs, is $F_P = (3/4\pi^2)(\lambda/n)^3(Q/V)$. What is F_P?

KEY **A** **304**

With $V = 1.5(\lambda/n)^3$ the factor $(\lambda/n)^3/V$ is $1/1.5$, so $F_P = 3 \times 6000/(4\pi^2 \times 1.5) = 304$ (A): the dot emits into the single cavity mode about 300 times faster than into the continuum of modes of bulk GaAs. B uses the free-space λ^3 against a mode volume measured in units of $(\lambda/n)^3$, which inflates the result by $n^3 = 42.9$; the wavelength that matters is the one in the medium, because the bulk rate it is compared with is the rate inside GaAs. C omits the factor $3/4\pi^2$, which comes from comparing one mode, of linewidth ω/Q, with the density of modes summed over all directions and both polarisations, as in the A coefficient. D overlooks that the dipole moment enters both rates as $|d|^2$ and cancels from their ratio: the Purcell factor belongs to the cavity, and the emitter has only to sit at the field maximum, aligned and on resonance.

U3

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UAQM-15.02 · UAQM-15.03 · UAQM-15.04 **[1]**

In Hartree-Fock theory the energy per electron of the uniform electron gas, in rydbergs, is $E/N = 2.21/r_s^2 - 0.916/r_s$, where r_s is the radius, in Bohr radii, of the sphere that contains one electron, and 1 Ry = 13.61 eV. The conduction electrons of lithium have $r_s = 3.25$. What is E/N at lithium's density, and how does that density compare with the density at which the Hartree-Fock E/N is lowest?

KEY **D** **E/N = −0.073 Ry (−0.99 eV), and lithium's electron gas is denser than at the Hartree-Fock minimum.**

At $r_s = 3.25$ the kinetic term is $2.21/10.56 = +0.209$ Ry and the exchange term is $-0.916/3.25 = -0.282$ Ry, so $E/N = -0.073$ Ry, or −0.99 eV. Setting $d(E/N)/dr_s = -4.42/r_s^3 + 0.916/r_s^2 = 0$ puts the Hartree-Fock minimum at $r_s = 4.42/0.916 = 4.83$. Lithium's r_s is smaller, and the electron density, $3/(4\pi r_s^3 a_0^3)$, rises as r_s falls, so lithium's electron gas is denser than the Hartree-Fock optimum (D): in this approximation, and on its own, it would push outward on the positive background that holds it. A has the right energy but reads a larger r_s as a higher density, forgetting that r_s is the radius of the volume each electron occupies. B keeps only the exchange term, forgetting the kinetic energy that opposes it. C keeps only the kinetic energy, which is the whole story only for electrons that do not interact. What actually binds lithium also involves the ions, which jellium smears into a uniform background, and correlation, quoted rather than derived in the course.

U3

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11 CORE COURSE

STRUCTURED · DEMANDING · UAQM-04.09 · UAQM-06.01 · UAQM-06.02 · UAQM-06.04 · UAQM-06.05 · UAQM-06.08 **[14]**

A negative muon, a spin-half Dirac particle of charge $q = -e$ and mass m about 207 times the electron's, is captured by a lead nucleus, ^{208}Pb ($Z = 82$), and cascades to the 1s orbit of muonic lead. That orbit is so small that the muon spends most of its time inside the nucleus, modelled as a uniformly charged sphere of radius R = 7.1 fm, where its potential energy is $V(r) = -(Z\alpha\hbar c/R)(3/2 - r^2/2R^2)$. In static potentials φ and A its Dirac Hamiltonian is $H = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 + q\varphi$, with $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$, $\mathbf{p} = -i\hbar\nabla$ and $V = q\varphi$. In the Dirac representation, writing a stationary state as $e^{-i(mc^2 + W)t/\hbar}$ times the column (ξ, χ) of upper and lower two-component spinors, $H\psi = (mc^2 + W)\psi$ becomes $(W - V)\xi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\chi$ and $(W - V + 2mc^2)\chi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\xi$. Pauli's anomalous-moment term adds to H, in a magnetic field B, the operator $-a(q\hbar/2m)\beta\Sigma\cdot\mathbf{B}$, where $\Sigma = \mathrm{diag}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$. For vector operators a and b that commute with the Pauli matrices, $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$, and for any function f(r), $pf - fp = -i\hbar\nabla f$. In a storage-ring measurement of the anomaly, muons of Lorentz factor $\gamma = 29.3$ circulate in a uniform field B = 1.4513 T, with no electric field acting on them, and the rate of their decay positrons oscillates at the frequency $\omega_a/2\pi = 229.08$ kHz at which the spin turns relative to the momentum. Take $mc^2 = 105.66$ MeV, $m = 1.8835 \times 10^{-28}$ kg, $\hbar c = 197.33$ MeV fm, $\alpha = 1/137.036$, $e = 1.602 \times 10^{-19}$ C and the muon's lifetime at rest $\tau = 2.197$ μs.

(a) SHOW THAT · U2 **[2]**

Show that the Dirac equation $i\hbar\partial\psi/\partial t = H\psi$ keeps its form under the gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$, $\varphi \to \varphi - \partial\Lambda/\partial t$, for any smooth function $\Lambda(\mathbf{r}, t)$, provided that $\psi \to e^{iq\Lambda/\hbar}\psi$.

1. $\mathbf{p}(e^{iq\Lambda/\hbar}\psi) = e^{iq\Lambda/\hbar}(\mathbf{p} + q\nabla\Lambda)\psi$, so $(\mathbf{p} - q\mathbf{A}')\psi' = e^{iq\Lambda/\hbar}(\mathbf{p} - q\mathbf{A})\psi$: the kinetic momentum acting on ψ' is the phase times $\boldsymbol{\pi}$ acting on ψ, and $c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}$ and βmc^2 pass through the phase unchanged
2. $i\hbar\partial\psi'/\partial t = e^{iq\Lambda/\hbar}(i\hbar\partial\psi/\partial t - q(\partial\Lambda/\partial t)\psi)$, and the extra term $-q(\partial\Lambda/\partial t)\psi'$ is exactly the change $q\varphi' - q\varphi = -q\partial\Lambda/\partial t$ in the potential energy, so ψ' obeys the same equation with the new potentials

Candidates who use a constant phase, or the wrong sign in the exponent, cannot cancel both the gradient and the time-derivative terms. The point of minimal coupling is that p and the energy enter only as $\mathbf{p} - q\mathbf{A}$ and $i\hbar\partial/\partial t - q\varphi$, so a local phase on ψ is absorbed by the potentials.

(b) ESTIMATE · U4 **[2]**

Inside the nucleus V is the potential energy of a three-dimensional harmonic oscillator. Estimate, stating your assumptions, the oscillator quantum $\hbar\omega$, the muon's speed in the 1s state as a fraction of c and hence the size of χ relative to ξ, and judge whether an expansion in v/c is trustworthy for muonic lead.

1. assuming the 1s muon lies mostly inside the nucleus, $V = V(0) + \tfrac{1}{2}kr^2$ with $k = Z\alpha\hbar c/R^3$, so $\hbar\omega = \hbar c\sqrt{(Z\alpha\hbar c/(mc^2R^3))} = 197.33\ \text{MeV fm} \times \sqrt{(0.5984 \times 197.33\ \text{MeV fm}/(105.66\ \text{MeV} \times 357.9\ \text{fm}^3))} \approx 11.0\ \text{MeV}$, and the oscillator length $\sqrt{(\hbar/m\omega)} \approx 5.8\ \text{fm}$, less than $R = 7.1\ \text{fm}$, supports the assumption
2. with $\langle T\rangle = \tfrac{3}{4}\hbar\omega \approx 8.3\ \text{MeV}$, $v^2/c^2 \approx 2\langle T\rangle/mc^2 \approx 0.16$ and $v/c \approx 0.4$; the lower equation gives $\chi \approx (\sigma \cdot p/2mc)\xi$, so $|\chi|/|\xi| \approx v/2c \approx 0.2$ and χ carries about 4% of the probability: the expansion converges, but its order-$(v/c)^2$ terms are corrections of order 15%, not the parts in 10^5 of hydrogen

Candidates who carry hydrogen's $v/c \approx \alpha$ across to muonic lead miss the point. The muon's speed is set by the nuclear charge it sits inside, not by the mass ratio, and at $v/c \approx 0.4$ the relativistic corrections are large though still perturbative.

(c) DERIVE · U2 **[3]**

For a slow muon in a uniform magnetic field B, with $A = \tfrac{1}{2}B \times r$ and no electric field, eliminate χ to leading order in v/c and derive the Hamiltonian for ξ to first order in B. Hence find the g-factors with which the orbital angular momentum $L = r \times p$ and the spin $S = (\hbar/2)\sigma$ couple to B under minimal coupling alone, and the spin g-factor once Pauli's term is included.

1. with $V = 0$ and W neglected beside $2mc^2$, the lower equation gives $\chi \approx (\sigma \cdot \pi/2mc)\xi$, so $W\xi = (\sigma \cdot \pi)^2\xi/2m$; $(\sigma \cdot \pi)^2 = \pi^2 + i\sigma \cdot (\pi \times \pi)$, and because the components of π do not commute, $\pi \times \pi = -q(p \times A + A \times p) = iq\hbar\nabla \times A = iq\hbar B$, so $(\sigma \cdot \pi)^2 = \pi^2 - q\hbar\sigma \cdot B$
2. with $A = \tfrac{1}{2}B \times r$, $\nabla \cdot A = 0$, so $p \cdot A = A \cdot p$ and $\pi^2 = p^2 - 2qA \cdot p + q^2A^2$, where $2A \cdot p = (B \times r) \cdot p = B \cdot L$; to first order in B, $H = p^2/2m - (q/2m)L \cdot B - (q/m)S \cdot B$: an orbital moment $(q/2m)L$, with $g_l = 1$, and a spin moment $(q/m)S = g(q/2m)S$, with $g = 2$
3. the upper block of $\beta\Sigma \cdot B$ is $\sigma \cdot B$, so Pauli's term adds $-a(q\hbar/2m)\sigma \cdot B = -a(q/m)S \cdot B$, making the spin term $-(1 + a)(q/m)S \cdot B$ and $g = 2(1 + a)$ while leaving the orbital term alone; a is a parameter the Dirac equation itself does not fix

Candidates who write $\pi \times \pi = 0$, as for ordinary vectors, lose the spin term entirely: the components of π do not commute in a magnetic field, and that is the whole origin of $g = 2$. Those who drop $p \cdot A$ halve the orbital term. Setting the two terms side by side shows the factor of two between spin and orbit that (d) exploits, and that the anomaly enters only through Pauli's term.

(d) CALCULATE · U3 **[2]**

For a slow muon the Pauli Hamiltonian makes the spin precess at $\omega_s = g|q|B/2m$ while the momentum turns at $\omega_c = |q|B/m$, so the spin turns relative to the momentum at $\omega_a = a|q|B/m$; with no electric field the Thomas-BMT equation shows that this holds at any speed. Calculate a for the muon from the storage-ring measurement, and the number of cycles of the positron-rate oscillation in one dilated lifetime $\gamma\tau$.

1. $a = \omega_a m/(eB) = 2\pi \times 229.08 \times 10^3\ \text{s}^{-1} \times 1.8835 \times 10^{-28}\ \text{kg}/(1.602 \times 10^{-19}\ \text{C} \times 1.4513\ \text{T}) = 1.166 \times 10^{-3}$
2. $\gamma\tau = 29.3 \times 2.197\ \mu\text{s} = 64.4\ \mu\text{s}$, so the positron rate completes $229.08\ \text{kHz} \times 64.4\ \mu\text{s} \approx 14.7$ cycles per dilated lifetime, enough to fit the frequency directly rather than measuring ω_s and ω_c separately

Candidates who divide by γm rather than m come out a factor 29 too large. The anomalous precession involves the rest mass, which is why a measurement on fast muons gives a directly, while time dilation lets many cycles be counted.

(e) SHOW THAT · U2 **[3]**

The Foldy-Wouthuysen reduction to order $(v/c)^2$ contains the even term $-(1/8m^2c^4)[O, [O, V]]$, where $O = c\alpha \cdot p$ is the odd part of H when A = 0. Show that for a static V this term equals $(\hbar^2/8m^2c^2)\nabla^2V + (\hbar/4m^2c^2)\Sigma \cdot (\nabla V \times p)$, and that inside the uniformly charged nucleus the Darwin part is the constant $3Z\alpha(\hbar c)^3/(8(mc^2)^2R^3)$. Evaluate it for muonic lead.

1. $[O, V] = c\alpha_j[p_j, V] = -i\hbar c\alpha \cdot \nabla V$, so $[O, [O, V]] = -i\hbar c^2\Sigma_{jk}[\alpha_j p_j, \alpha_k \partial_k V]$; using $p_j \partial_k V = (\partial_k V)p_j - i\hbar \partial_j \partial_k V$ this is $-i\hbar c^2\Sigma_{jk}[(\alpha_j\alpha_k - \alpha_k\alpha_j)(\partial_k V)p_j - i\hbar\alpha_j\alpha_k\partial_j\partial_k V]$
2. with $\alpha_j\alpha_k = \delta_{jk} + i\varepsilon_{jkl}\Sigma_l$ the symmetric part gives $-\hbar^2c^2\nabla^2V$ and the antisymmetric part $-2\hbar c^2\Sigma \cdot (\nabla V \times p)$, so $-(1/8m^2c^4)[O, [O, V]] = (\hbar^2/8m^2c^2)\nabla^2V + (\hbar/4m^2c^2)\Sigma \cdot (\nabla V \times p)$
3. inside the sphere $dV/dr = Z\alpha\hbar cr/R^3$, so $\nabla^2V = (1/r^2)d(r^2dV/dr)/dr = 3Z\alpha\hbar c/R^3$ and the Darwin term is $3Z\alpha(\hbar c)^3/(8(mc^2)^2R^3) = 3 \times 0.5984 \times (197.33\ \text{MeV fm})^3/(8 \times (105.66\ \text{MeV})^2 \times 357.9\ \text{fm}^3) \approx 0.43$ MeV, a uniform upward shift wherever the muon is inside the nucleus

Candidates who carry over the hydrogen result, a delta function at the origin, miss that ∇^2V is proportional to the charge density producing V. For a point nucleus the Darwin term is a contact term; a muon inside a uniform nucleus feels it everywhere as a constant.

(f) EXPLAIN · U1 **[2]**

The 1s and 2s states of muonic lead are orthogonal four-component spinors. Using your estimate from (b), explain why their upper components ξ_{1s} and ξ_{2s} cannot be eigenfunctions of any Hermitian Hamiltonian at order $(v/c)^2$, so that no two-component Schrödinger equation for ξ can be exact to that order, and what the Foldy-Wouthuysen transformation does instead.

1. orthonormality belongs to the whole spinor, $\int(\xi_{1s}\dagger\xi_{2s} + \chi_{1s}\dagger\chi_{2s})d^3r = 0$, and with $\chi \approx (\sigma \cdot p/2mc)\xi$ this gives $\int\xi_{1s}\dagger\xi_{2s}\, d^3r \approx -\langle\xi_{1s}|p^2|\xi_{2s}\rangle/(4m^2c^2)$, which is not zero, while $\int\xi_{1s}\dagger\xi_{1s}\, d^3r \approx 1 - \langle p^2\rangle/(4m^2c^2) \approx 0.96$; eigenfunctions of a Hermitian operator belonging to different eigenvalues are orthogonal, so no such operator has the ξ as its eigenfunctions, and expanding $1/(W - V + 2mc^2)$ one step further gives instead an operator that depends on the energy W
2. the Foldy-Wouthuysen transformation is unitary, so it keeps the spinors orthonormal while removing the odd operators order by order; the upper components of the transformed spinors, $\Phi \approx [1 + p^2/(8m^2c^2)]\xi$ to this order, are orthonormal two-component functions obeying one energy-independent Hermitian Hamiltonian, which contains the Darwin and spin-orbit terms of (e)

Candidates who treat the upper components as non-relativistic wavefunctions have forgotten that orthonormality belongs to the whole spinor; renormalising each ξ separately repairs the norms but not the overlaps. With 4% of the 1s probability in χ, as in muonic lead, the defect is as large as the order-$(v/c)^2$ terms being sought.

12

CORE COURSE

STRUCTURED · DEMANDING · UAQM-07.02 · UAQM-07.04 · UAQM-07.05 · UAQM-07.07 · UAQM-07.08 · UAQM-07.09 **[12]**

The Lyman-α X-rays of hydrogen-like uranium, U^{91+} (Z = 92), emitted when bare uranium nuclei circulating in a heavy-ion storage ring capture an electron that cascades to the ground state, test the Dirac-Coulomb spectrum where $Z\alpha$ is not small. The two lines are Lyα1, $2p_{3/2} \to 1s_{1/2}$, and Lyα2, $2p_{1/2} \to 1s_{1/2}$, and spectroscopy of the stored ions places Lyα1 at 102.17 keV. In a central potential V(r) the radial functions G(r) and F(r) of the upper and lower components satisfy $dG/dr = -(\kappa/r)G + [(E + mc^2 - V)/\hbar c]F$ and $dF/dr = (\kappa/r)F - [(E - mc^2 - V)/\hbar c]G$, where E includes mc^2, $\kappa = (l - j)(2j + 1)$ and l is the orbital angular momentum of the upper component. For a point nucleus $V = -Z\alpha\hbar c/r$, and the exact energies are $E_{nj} = mc^2[1 + (Z\alpha)^2/(n - |\kappa| + \gamma)^2]^{(-1/2)}$, with $\gamma = \sqrt{(\kappa^2 - (Z\alpha)^2)}$. The normalised point-nucleus 1s density, both components together, is $|\psi_{1s}(r)|^2 = [(2Z/a_0)^{(2\gamma+1)}/(4\pi\Gamma(2\gamma + 1))]r^{(2\gamma-2)}e^{(-2Zr/a_0)}$, with $\Gamma(2\gamma + 1) = 1.313$ for Z = 92, while the Schrödinger 1s density is $(Z^3/\pi a_0^3)e^{(-2Zr/a_0)}$. The uranium nucleus has its surface at about r = 7.5 fm and is treated as infinitely heavy. Take $mc^2 = 510.999$ keV, $\alpha = 1/137.036$ and $a_0 = 5.2918 \times 10^4$ fm.

(a) STATE · U1 **[2]**

State which conserved operator takes the place of L^2 in labelling the electron's states, and state κ and the orbital angular momentum l' of the lower component for each of $1s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$, saying which of their components are s waves, able to reach the nucleus.

1. L does not commute with H, because $c\alpha \cdot p$ couples an upper component of orbital angular momentum l to a lower one of $l' = 2j - l$; together with J^2, J_z and parity, $K = \beta(\Sigma \cdot L + \hbar)$ commutes with H, and its eigenvalue $-\kappa\hbar$ labels the state, $|\kappa| = j + \tfrac{1}{2}$ giving j and the sign of κ telling whether the upper component has $l = j - \tfrac{1}{2}$ ($\kappa < 0$) or $l = j + \tfrac{1}{2}$ ($\kappa > 0$)
2. $\kappa = -1$ for $1s_{1/2}$, $+1$ for $2p_{1/2}$ and -2 for $2p_{3/2}$, with $l' = 1$, 0 and 2: the s waves are the upper component of $1s_{1/2}$ and the lower component of $2p_{1/2}$, so both reach the nucleus, while $2p_{3/2}$, with $l = 1$ and $l' = 2$, has none

   Candidates who treat $2p_{1/2}$ as a pure p state miss that its lower component is an s wave, which is what makes a $p_{1/2}$ level, unlike $p_{3/2}$, sensitive to the nucleus of a heavy ion. Only the upper component has orbital angular momentum l; the state as a whole is labelled by κ.

(b) DERIVE · U2 **[3]**

Derive from the radial equations how G and F rise from the origin. For the point nucleus, writing $G \approx ar^\gamma$ and $F \approx br^\gamma$, find γ and b/a, evaluate b/a for $1s_{1/2}$ and for $2p_{1/2}$ at $Z = 92$, and say which component dominates near the nucleus in each. Then find the powers of r with which G and F rise inside a nucleus of finite size, where V(r) tends to a finite V(0).

1. for the point nucleus the Coulomb term is as singular as κ/r and dominates $E \pm mc^2$, and the $r^{(\gamma-1)}$ terms give $(\gamma + \kappa)a = Z\alpha b$ and $(\gamma - \kappa)b = -Z\alpha a$, which have a nonzero solution only if $\gamma^2 = \kappa^2 - (Z\alpha)^2$; the positive root $\gamma = \sqrt{(\kappa^2 - (Z\alpha)^2)}$ keeps the density normalisable, and $b/a = (\gamma + \kappa)/(Z\alpha)$
2. at $Z = 92$, $Z\alpha = 0.6714$ and $\gamma = \sqrt{(1 - 0.4507)} = 0.7411$ for $|\kappa| = 1$, so $b/a = (0.7411 + 1)/0.6714 = +2.59$ for $2p_{1/2}$, the s-wave lower component outweighing the upper one near the nucleus, and $b/a = (0.7411 - 1)/0.6714 = -0.386$ for $1s_{1/2}$, the upper component dominating; both components share the non-integer power, so G/r and F/r diverge weakly, as $r^{(\gamma-1)} = r^{(-0.26)}$
3. inside a finite nucleus V stays finite, so near the origin only the κ/r terms are singular: for $\kappa < 0$, $G \propto r^{(-\kappa)} = r^{(l+1)}$ and $F \propto r^{(l+2)} = r^{(l'+1)}$, and for $\kappa > 0$, $F \propto r^{\kappa} = r^{(l'+1)}$ and $G \propto r^{(\kappa+1)} = r^{(l+1)}$; the integer powers of the non-relativistic problem return, each component with its own orbital angular momentum, and the singularity of the point nucleus is removed

   Candidates who drop the Coulomb term against κ/r near a point nucleus recover the integer powers $r^{(l+1)}$ and $r^{(l'+1)}$, which hold only inside a finite nucleus. For the point nucleus the two terms are of the same order, both components share the non-integer power γ, and for $2p_{1/2}$ the small component is not small near the origin.

(c) CALCULATE · U3 **[2]**

Calculate, for a point nucleus, the energies of Lyα1 and Lyα2 in keV, and the 2p fine-structure splitting.

1. $E_{1s} = \gamma mc^2 = 0.74113 \times 510.999$ keV $= 378.72$ keV; for $2p_{1/2}$, $n - |\kappa| + \gamma = 1.7411$ and $E = 510.999 \text{ keV}/\sqrt{(1 + 0.4507/3.0315)} = 476.78$ keV; for $2p_{3/2}$, $\gamma = \sqrt{(4 - 0.4507)} = 1.8840$ and $E = 510.999 \text{ keV}/\sqrt{(1 + 0.4507/3.5493)} = 481.35$ keV
2. Lyα1 = 481.35 keV − 378.72 keV = 102.63 keV and Lyα2 = 476.78 keV − 378.72 keV = 98.06 keV, a fine-structure splitting of 4.57 keV, where the non-relativistic theory gives one line at $(3/8)(Z\alpha)^2mc^2 = 86.4$ keV

   Candidates who use the Bohr formula find a single line at 86.4 keV, 16% low, with no splitting. At $Z\alpha = 0.67$ the exact formula is needed, and the differences of large energies must be taken to five figures.

(d) DETERMINE · U3 **[2]**

Treating the $2p_{3/2}$ level as given correctly by the point-nucleus formula, determine from the measured Lyα1 energy the shift of the 1s level that the point-nucleus Dirac theory omits, its sign, and its size as a fraction of the 1s binding energy.

1. the measured line lies 102.63 keV − 102.17 keV = 0.46 keV below the point-nucleus value, so with $2p_{3/2}$ fixed the 1s level lies about 0.46 keV higher, less bound, than the point-nucleus Dirac prediction
2. the 1s binding energy is $mc^2 - E_{1s}$ = 510.999 keV − 378.72 keV = 132.28 keV, so the shift is 0.46/132.28 = 0.35% of it, the combined effect of the nucleus's finite size and of the radiative corrections of quantum electrodynamics, which together raise the s level

   Candidates who give the shift the wrong sign, or put it down to experimental error, miss what the line measures. The $2p_{3/2}$ level barely overlaps the nucleus, so Lyα1 isolates the ground-state shift and tests quantum electrodynamics where Zα is close to 1.

(e) COMPARE · U3 **[2]**

Compare the Dirac 1s density at the nuclear surface, r = 7.5 fm, with the Schrödinger 1s density there, and say what the comparison implies for the finite-size part of the shift found in (d).

1. the exponentials cancel and the ratio is [2/Γ(2γ + 1)](2Zr/a_0)^(2γ−2); with 2Zr/a_0 = 2 × 92 × 7.5 fm/(5.2918 × 10^4 fm) = 0.0261 and 2γ − 2 = −0.518 it is (2/1.313) × 0.0261^(−0.518) ≈ 10
2. the weak singularity r^(2γ−2) puts about ten times more of the Dirac electron's density inside the nucleus than the Schrödinger estimate, so the finite-size shift is about ten times what a non-relativistic calculation gives and makes up a substantial part of the 0.46 keV

   Candidates who take the density at the nucleus from the Schrödinger formula, as is fine for hydrogen, underestimate the overlap with the nucleus tenfold. The factor r^(2γ−2) is harmless to the normalisation but not to anything that samples the nuclear volume.

(f) PREDICT · U1 **[1]**

In the point-nucleus Dirac theory $2s_{1/2}$ and $2p_{1/2}$ are exactly degenerate. Predict which of the two the finite size of the nucleus raises more, and so which lies higher.

1. $2s_{1/2}$, whose s wave is the large upper component, overlaps the nucleus far more than $2p_{1/2}$, whose s wave is the lower component that is small over most of the orbit, so $2s_{1/2}$ is raised more and lies above $2p_{1/2}$, in the same direction as the radiative Lamb shift

   Candidates who reason from (b), where the lower component of $2p_{1/2}$ dominates near the origin, may predict the reverse. The ratio b/a fixes the shape near the origin, but the normalisation over the whole orbit, where the upper component dominates, makes the $2p_{1/2}$ density at the nucleus much smaller than that of $2s_{1/2}$.

13

CORE COURSE

STRUCTURED · DEMANDING · UAQM-09.01 · UAQM-09.03 · UAQM-09.04 **[12]**

A free complex scalar field φ(x, t), with φ and φ† treated as independent fields, describes spinless particles of mass m and charge q through the Lagrangian density $\mathcal{L} = (1/c^2)(\partial\varphi^\dagger/\partial t)(\partial\varphi/\partial t) - \nabla\varphi^\dagger \cdot \nabla\varphi - \kappa^2\varphi^\dagger\varphi$, where κ = mc/ħ. In a box of volume V with periodic boundary conditions the quantised field is φ(x, t) = Σ_k N_k[a_k e^(i(k·x − ω_k t)) + b_k† e^(−i(k·x − ω_k t))], with ω_k = c√(k² + κ²), N_k = √(ħc²/(2ω_kV)), [a_k, a_k′†] = [b_k, b_k′†] = δ_kk′ and all other commutators zero. After normal ordering the Hamiltonian is H = Σ_k ħω_k(a_k†a_k + b_k†b_k). Take ħc = 197.3 MeV fm.

(a) DERIVE · U2 **[3]**

Apply the Euler-Lagrange equation for φ† to derive the field equation for φ, and use it to show that the density ρ = (iq/ħc²)(φ†∂φ/∂t − φ∂φ†/∂t) and the current j = −(iq/ħ)(φ†∇φ − φ∇φ†) satisfy ∂ρ/∂t + ∇ · j = 0.

1. ∂ℒ/∂(∂_tφ†) = (1/c²)∂_tφ, ∂ℒ/∂(∇φ†) = −∇φ and ∂ℒ/∂φ† = −κ²φ, so ∂_t[∂ℒ/∂(∂_tφ†)] + ∇ · [∂ℒ/∂(∇φ†)] − ∂ℒ/∂φ† = 0 gives (1/c²)∂²φ/∂t² − ∇²φ + κ²φ = 0, the Klein-Gordon equation, with its conjugate for φ†
2. ∂ρ/∂t = (iq/ħc²)(φ†∂²_tφ − φ∂²_tφ†), the products of first time derivatives cancelling, and likewise ∇ · j = −(iq/ħ)(φ†∇²φ − φ∇²φ†)
3. substituting ∂²_tφ = c²(∇²φ − κ²φ) and its conjugate, the κ² terms cancel and ∂ρ/∂t = (iq/ħ)(φ†∇²φ − φ∇²φ†) = −∇ · j: the current is conserved, the Noether current of the symmetry φ → e^(−iθ)φ

   Candidates who treat φ and φ† as one real field lose the conserved charge altogether, since a real field has no phase to rotate. Varying with respect to φ instead gives the equation for φ†, which is equally valid once conjugated.

(b) SHOW THAT · U2 **[3]**

By substituting the mode expansion, show that after normal ordering the charge Q = ∫ρ d³x is qΣ_k(a_k†a_k − b_k†b_k), and hence that a_k† creates a quantum of charge +q and b_k† one of charge −q, each with energy ħω_k.

1. ∫e^(i(k − k′)·x) d³x = Vδ_kk′ pairs each operator with one of the same mode or of the opposite mode, and the terms in a_k b_(−k) e^(−2iω_k t) and in a_k†b_(−k)† e^(2iω_k t) enter φ†∂_tφ and φ∂_tφ† with the same coefficient and cancel, so Q is independent of time
2. each mode then contributes (iq/ħc²)(−iω_k)N_k²V(a_k†a_k + a_k a_k† − b_k b_k† − b_k†b_k) = (q/2)(a_k†a_k + a_k a_k† − b_k b_k† − b_k†b_k), since ω_kN_k²V/ħc² = ½; normal ordering gives Q = qΣ_k(a_k†a_k − b_k†b_k)
3. [Q, a_k†] = qa_k† and [Q, b_k†] = −qb_k†, so a_k†|0⟩ has charge +q and b_k†|0⟩ charge −q, while [H, a_k†] = ħω_k a_k† and [H, b_k†] = ħω_k b_k† give both the energy ħω_k: the antiparticle has the opposite charge and a positive energy

   Candidates who keep only the a-terms find a charge of one sign and miss the antiparticles; those who expect b-quanta to have negative energy have confused the sign of the charge with the sign of the energy, which H fixes as positive.

(c) EXPLAIN · U1 **[2]**

The field commutator is [φ(x), φ†(y)] = D(x − y) − D(y − x), where D(x − y) = ⟨0|φ(x)φ†(y)|0⟩ = Σ_k N_k² e^(i(k·(x − y) − ω_k(t_x − t_y))). Explain why the commutator vanishes when x − y is spacelike although D(x − y) does not, and what would go wrong if the field contained the a_k terms alone.

1. for spacelike x − y, D depends only on the invariant interval (at equal times it is even in x − y, by k → −k), and a proper Lorentz transformation can carry a spacelike x − y into −(x − y); so D(x − y) = D(y − x) and the commutator vanishes, although each term is nonzero
2. the second term comes from the b-operators: the amplitude for an antiparticle to travel from x to y cancels that for a particle to travel from y to x; with the a-terms alone, [φ(x), φ†(y)] = D(x − y), nonzero at spacelike separation, so a measurement at one point could affect another outside its light cone: a relativistic charged field needs its antiparticle

   The error this part catches is the belief that nothing propagates outside the light cone. The amplitude does; causality is a property of the commutator, and it is the antiparticle term that secures it.

(d) ESTIMATE · U4 **[2]**

At equal times and separation r, D is proportional to (1/r)$K_1(\kappa r)$, which for $\kappa r \gg 1$ behaves as r^(−3/2)e^(−κr). For a π^+, with mc^2 = 139.6 MeV, estimate the length 1/κ over which the amplitude extends outside the light cone, and the factor by which D falls between r = 5 fm and r = 10 fm, stating your assumptions.

1. $1/\kappa = \hbar c/mc^2$ = 197.3 MeV fm/139.6 MeV = 1.41 fm, the pion's reduced Compton wavelength
2. assuming the asymptotic form already holds at κr = 5/1.41 = 3.5, D(10 fm)/D(5 fm) ≈ (1/2)^(3/2) e^(−5/1.41) = 0.354 × 0.029 = 0.010 (the Bessel functions themselves give 0.0098): a hundredfold fall, but never zero

   Candidates who keep only the exponential find 0.029, three times too large, since the r^(−3/2) is not negligible over a factor of two in r. That κr = 3.5 is only moderately large is an assumption to state, not to hide.

(e) COMPARE · U1 **[2]**

Compare the density ρ of this question, as an operator of the quantised field, with the same expression read as a probability density in the one-particle Klein-Gordon theory.

1. in the one-particle theory the same bilinear (without the factor q) is negative on negative-frequency solutions and can be given either sign by the initial data, so it cannot be a probability density, and the negative-energy solutions have no consistent meaning
2. in the field theory it is the charge density: its integral has eigenvalues q(N_a − N_b), negative values counting antiparticles, which have positive energy and states of positive norm; the indefinite sign becomes the two signs of charge, not of probability

   The comparison must say what changes and why. Candidates who say the sign problem simply disappears miss that it is reinterpreted: the same indefinite quantity is now a charge, for which both signs are physical.

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CORE COURSE

STRUCTURED · DEMANDING · UAQM-10.03 · UAQM-10.04 · UAQM-10.05 · UAQM-10.09 **[12]**

Two electrons occupy a molecule of two sites, 1 and 2, each with one orbital, described by the Hubbard Hamiltonian $H = -t\,\Sigma_\sigma (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}) + U(n_{1\uparrow} n_{1\downarrow} + n_{2\uparrow} n_{2\downarrow})$, with hopping t = 0.40 eV and on-site repulsion U = 3.2 eV. The operators obey $\{c_{i\sigma}, c_{j\sigma'}^\dagger\} = \delta_{ij}\delta_{\sigma\sigma'}$ and $\{c_{i\sigma}, c_{j\sigma'}\} = 0$. In the sector with $S_z = 0$ a basis is $|D_1\rangle = c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle$ and $|D_2\rangle = c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle$, with both electrons on one site, and $|A\rangle = c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle$ and $|B\rangle = c_{2\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle$, with one electron on each.

(a) SHOW THAT · U2 **[3]**

Show that the hopping term T sends both $|A\rangle$ and $|B\rangle$ to $-t(|D_1\rangle + |D_2\rangle)$, and hence that in the basis $|S\rangle = (|A\rangle + |B\rangle)/\sqrt{2}$, $|D_+\rangle = (|D_1\rangle + |D_2\rangle)/\sqrt{2}$ the Hamiltonian is the matrix [[0, −2t], [−2t, U]], while $(|A\rangle - |B\rangle)/\sqrt{2}$ and $(|D_1\rangle - |D_2\rangle)/\sqrt{2}$ are eigenstates with energies 0 and U.

1. on $|A\rangle$ only $c_{2\uparrow}^\dagger c_{1\uparrow}$ and $c_{1\downarrow}^\dagger c_{2\downarrow}$ act: $c_{2\uparrow}^\dagger c_{1\uparrow} c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle = c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle = |D_2\rangle$, and $c_{1\downarrow}^\dagger c_{2\downarrow} c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle = -c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle = +|D_1\rangle$, two anticommutation signs cancelling; likewise $c_{1\uparrow}^\dagger c_{2\uparrow}^\dagger|B\rangle = |D_1\rangle$ and $c_{2\downarrow}^\dagger c_{1\downarrow}^\dagger|B\rangle = |D_2\rangle$, so $T|A\rangle = T|B\rangle = -t(|D_1\rangle + |D_2\rangle)$
2. the U term acts only on $|D_1\rangle$ and $|D_2\rangle$, each with eigenvalue U, and $|A\rangle$, $|B\rangle$ have energy 0; by Hermiticity $T|D_1\rangle = T|D_2\rangle = -t(|A\rangle + |B\rangle)$, so T annihilates $(|A\rangle - |B\rangle)/\sqrt{2}$ and $(|D_1\rangle - |D_2\rangle)/\sqrt{2}$, which are eigenstates with energies 0 and U
3. $\langle S|T|D_+\rangle = \tfrac{1}{2}(\langle A| + \langle B|)(-2t)(|A\rangle + |B\rangle) = -2t$, so on $\{|S\rangle, |D_+\rangle\}$ H = [[0, −2t], [−2t, U]]; $|S\rangle = (c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger)|0\rangle/\sqrt{2}$ is the spin singlet and $(|A\rangle - |B\rangle)/\sqrt{2}$ the triplet with m = 0

   Every sign here comes from moving one operator past another. A candidate who drops them can find $T|B\rangle = -t(|D_1\rangle - |D_2\rangle)$, couple the triplet to the doubly occupied states, and obtain a ground state of the wrong spin.

(b) CALCULATE · U3 **[2]**

Calculate the ground-state energy and the singlet-triplet splitting, and compare the splitting with the superexchange coupling $4t^2/U$.

1. $E_0 = [U - \sqrt{(U^2 + 16t^2)}]/2 = [3.2 - \sqrt{(10.24 + 2.56)}]/2$ eV $= (3.2 - 3.578)/2$ eV $= -0.189$ eV, a spin singlet
2. the triplet lies at 0, so the splitting is 0.189 eV, against $4t^2/U = 4(0.40)^2/3.2$ eV $= 0.200$ eV, 6% high because $t/U = 0.125$ is not very small; the coupling is antiferromagnetic

Candidates who quote $4t^2/U$ as the splitting have used the large-U limit without checking t/U. The exact 2 × 2 result is available, and the superexchange formula should be judged against it.

(c) DETERMINE · U3 **[2]**

Determine the probability of finding both electrons on the same site in the ground state.

1. the first row of $(H - E_0)$ acting on the amplitudes s of $|S\rangle$ and d of $|D_+\rangle$ gives $-2td = E_0 s$, so $d/s = -E_0/2t = 0.189/0.80 = 0.236$
2. $P = d^2/(s^2 + d^2) = 0.0557/1.0557 = 0.053$, about 5%, against 50% for two electrons that ignore each other

Normalising is the step most often missed: 0.236 is a ratio of amplitudes, and the probability is $d^2/(1 + d^2)$, not 0.236 or 0.236^2.

(d) EVALUATE · U1 **[3]**

A student describes the ground state by the restricted Hartree-Fock determinant, with both electrons in the bonding orbital, created by $(c_1\sigma\uparrow + c_2\sigma\uparrow)/\sqrt{2}$. Evaluate this description: find its energy and its probability of double occupation, and say what it gets wrong as U/t grows.

1. the determinant is $\tfrac{1}{2}(c_1\uparrow\uparrow + c_2\uparrow\uparrow)(c_1\downarrow\uparrow + c_2\downarrow\uparrow)|0\rangle = (|S\rangle + |D_+\rangle)/\sqrt{2}$, with energy $\tfrac{1}{2}(0 + U) - 2t = U/2 - 2t = +0.80$ eV, 0.99 eV above the exact -0.19 eV: the correlation energy, -0.99 eV, is about five times the magnitude of the exact energy itself
2. its probability of double occupation is 50% whatever U is, against the exact 5%: one determinant of delocalised orbitals cannot keep the two electrons apart
3. as U/t grows the restricted energy rises as U/2 while the exact energy tends to $-4t^2/U$, approaching zero from below: restricted Hartree-Fock fails worst exactly where the spins localise and couple antiferromagnetically, the dissociation problem of H_2

The expected answer, that the restricted determinant is a reasonable first approximation, is what this part tests. Its energy is positive here although the exact ground state is bound, and its error grows without limit with U.

(e) EXPLAIN · U1 **[2]**

Unrestricted Hartree-Fock lets the up-spin and down-spin electrons occupy different orbitals, and its best determinant has energy $-2t^2/U = -0.10$ eV. Explain why it recovers only about half of the exact binding, and what symmetry of H it breaks to do so.

1. for $U \gg t$ the unrestricted determinant approaches $|A\rangle = (|S\rangle + |T_0\rangle)/\sqrt{2}$, with $|T_0\rangle = (|A\rangle - |B\rangle)/\sqrt{2}$: an equal mixture of the singlet, near $-4t^2/U$, and the triplet, at 0, so its energy is about their average, $-2t^2/U$
2. to localise the spins it breaks the spin-rotation symmetry, since the determinant is not an eigenstate of S^2, and the symmetry exchanging the sites; the singlet $(|A\rangle + |B\rangle)/\sqrt{2}$, which the exact ground state $0.973|S\rangle + 0.230|D_+\rangle$ approaches as U/t grows, is a sum of two determinants, which no single determinant can represent

Candidates who take the lower unrestricted energy as proof of a better state should notice that it has no definite total spin. Broken symmetry buys energy at the price of a state that is not an eigenstate of S^2.

15 CORE COURSE

STRUCTURED · DEMANDING · UAQM-13.02 · UAQM-13.03 · UAQM-13.04 · UAQM-13.05 · UAQM-13.07 **[12]**

In Bethe's non-relativistic calculation a hydrogen atom in a state $|n\rangle$ is coupled to the transverse vacuum field through $H' = (e/m)A \cdot p$ in the dipole approximation, photon energies being cut off at K. Summing the second-order energy over photon modes and polarisations gives $W_n = -(2\alpha/(3\pi m^2c^2))\ \Sigma_m\ |(m|p|n)|^2\ \int_0^K k\ dk/(E_m - E_n + k)$, where $k = \hbar\omega$ is the photon energy and the sum runs over all bound and continuum states $|m\rangle$. For a free electron of momentum p the same calculation gives $W_{free} = -(2\alpha/(3\pi m^2c^2))p^2K$. Write $p_{mn} = \langle m|p|n\rangle$. Take $mc^2 = 0.510999$ MeV, $\alpha = 1/137.036$, 1 Ry = 13.6057 eV and $h = 4.1357 \times 10^{-15}$ eV s.

(a) SHOW THAT · U2 **[2]**

Show that, for K large compared with every $|E_m - E_n|$ that matters, W_n grows linearly with K, and that the coefficient of K is the free-electron result averaged over the state $|n\rangle$.

1. with $\Delta = E_m - E_n$, $\int_0^K k\ dk/(\Delta + k) = K - \Delta\ \ln|(\Delta + K)/\Delta| \approx K - \Delta\ \ln(K/|\Delta|)$ for $K \gg |\Delta|$, so $W_n = -(2\alpha/3\pi m^2c^2)[K\ \Sigma_m|p_{mn}|^2 - \Sigma_m|p_{mn}|^2\Delta\ \ln(K/|\Delta|)]$
2. by closure $\Sigma_m|p_{mn}|^2 = \langle n|p^2|n\rangle$, so the linear term is $-(2\alpha K/3\pi m^2c^2)\langle n|p^2|n\rangle = \langle n|W_{free}|n\rangle$: the bound electron's self-energy diverges exactly as a free electron's with the same momentum distribution

Candidates who try to evaluate the sum over states before separating the divergence get lost in the continuum. Closure does the sum in one line, and shows that the divergence knows nothing about the binding.

(b) EXPLAIN · U1 **[2]**

Explain why the linear term is removed by mass renormalisation, and why what remains depends on K only through a logarithm.

1. $W_{free} = -(2\alpha K/3\pi m^2c^2)p^2$ changes the coefficient of p^2: $p^2/2m_0 + W_{free} = p^2/2(m_0 + \delta m)$ with $\delta m = 4\alpha K/(3\pi c^2)$, an electromagnetic mass; the measured mass already contains it, so writing H with the observed mass subtracts W_{free} and removes the linear term from every state (the A^2 term, the same for every state in the dipole approximation, drops out of all intervals)
2. what remains, $\Delta E_n = (2\alpha/3\pi m^2c^2)\ \Sigma_m|p_{mn}|^2(E_m - E_n)\ \ln(K/|E_m - E_n|)$, exists only because the electron is bound, vanishing for a free one, and keeps K only in a logarithm, because photons much more energetic than the binding cannot tell a bound electron from a free one

The subtraction is not an arbitrary deletion of an infinity: the same divergent operator appears for the free electron, where it is observable only as part of the mass. A candidate who cannot say what is subtracted, and why it is the same for every state, has not explained it.

(c) SHOW THAT · U2 **[3]**

The sum rule $\Sigma_m|p_{mn}|^2(E_m - E_n) = (\hbar^2/2)\langle n|\nabla^2V|n\rangle$ holds for $V = -e^2/(4\pi\varepsilon_0 r)$. Defining the mean excitation energy K_0 by $\ln K_0 = \Sigma_m|p_{mn}|^2(E_m - E_n)\ \ln|E_m - E_n| / \Sigma_m|p_{mn}|^2(E_m - E_n)$, show that $\Delta E_n = (4\alpha^5mc^2/(3\pi n^3))\ \ln(K/K_0)$ for s states, and that the term in ln K vanishes in every other state. You may use $|\psi_{ns}(0)|^2 = 1/(\pi n^3a_0^3)$, $a_0 = \hbar/(\alpha mc)$ and $e^2/(4\pi\varepsilon_0) = \alpha\hbar c$.

1. $\nabla^2(1/r) = -4\pi\delta^3(r)$ gives $\nabla^2V = (e^2/\varepsilon_0)\delta^3(r) = 4\pi\alpha\hbar c\ \delta^3(r)$, so the sum rule gives $2\pi\alpha\hbar^3c|\psi_n(0)|^2$, which vanishes unless $l = 0$: the term in ln K acts on s states only
2. with the definition of K_0, $\Delta E_n = (2\alpha/3\pi m^2c^2) \times 2\pi\alpha\hbar^3c|\psi_n(0)|^2 \times \ln(K/K_0) = (4\alpha^2\hbar^3/3m^2c)|\psi_n(0)|^2\ \ln(K/K_0)$
3. with $|\psi_{ns}(0)|^2 = 1/(\pi n^3a_0^3) = \alpha^3m^3c^3/(\pi n^3\hbar^3)$, $\Delta E_n = (4\alpha^5mc^2/(3\pi n^3))\ \ln(K/K_0)$: of order α^5mc^2/n^3 times a large logarithm

The sum rule is the step that turns an intractable sum over states into a contact term. Candidates who miss the delta function in $\nabla^2(1/r)$ find zero, and those who forget that $\psi(0) = 0$ for $l \geq 1$ cannot explain why $2p_{1/2}$ stays put.

(d) CALCULATE · U3 **[3]**

With the cutoff at $K = mc^2$ and the quoted mean excitation energies $K_0(2s) = 16.64$ Ry and $K_0(1s) = 19.77$ Ry, calculate the shifts of 2s and of 1s in frequency units, and compare them with the measured $2s_{1/2}$–$2p_{1/2}$ splitting of 1057.8 MHz and 1s shift of 8.17 GHz.

1. for $n = 2$, $4\alpha^5 mc^2/(3\pi \times 8) = 5.610 \times 10^{-7}$ eV, which is 135.6 MHz
2. $\ln(mc^2/K_0) = \ln(510999\text{ eV}/226.4\text{ eV}) = 7.722$, so the 2s shift is 135.6 MHz × 7.722 = 1047 MHz, 1% below the measured 1057.8 MHz, $2p_{1/2}$ receiving no logarithmic shift
3. for 1s the prefactor is 8 × 135.6 MHz = 1085 MHz and $\ln(510999\text{ eV}/269.0\text{ eV}) = 7.549$, giving 8.19 GHz against the measured 8.17 GHz

   Leaving K_0 in rydbergs while mc^2 is in electronvolts gives a logarithm near 10.3 and a 2s shift a third too large. The 1s shift is less than eight times the 2s shift because its mean excitation energy is higher.

(e) EVALUATE · U4 **[2]**

The non-relativistic calculation fixes the cutoff K only to within a factor of about two. Evaluate whether the agreement found in (d) shows that Bethe's choice $K = mc^2$ is correct.

1. a factor of two in K changes the 2s shift by 135.6 MHz × ln 2 = 94 MHz, about 9%, and the 1s shift by 0.75 GHz; agreement at the 1% level is therefore much closer than the method can guarantee, and is partly fortunate
2. what the calculation does establish is robust against the cutoff: a shift of order $\alpha^5 mc^2/n^3$ times a logarithm, confined to s states, with a 1s to 2s ratio of 7.8 rather than 8; removing the cutoff needs the covariant QED calculation, which also adds vacuum polarisation and the anomalous moment, each worth tens of MHz

   A good evaluation sizes the uncertainty before admiring the agreement. Candidates who conclude that the non-relativistic theory is accurate to 1% have mistaken a fortunate cutoff for a precise one.

16

CORE COURSE

DATA ANALYSIS · DEMANDING · UAQM-14.02 · UAQM-14.06 **[11]**

A single InGaAs quantum dot at 4 K is driven resonantly by a narrow-band laser, and its fluorescence is recorded as the laser frequency is scanned across the transition. For each laser power P at the dot, the full width at half maximum $\Delta\nu$ of the fluorescence line, as a frequency, is recorded. A separate time-resolved measurement gives the population relaxation time $T_1 = 0.80$ ns. For a two-level emitter with population relaxation time T_1 and coherence time T_2, driven at Rabi frequency Ω and detuning Δ (both angular frequencies), the steady state of the optical Bloch equations has the excited-state population $\rho_{ee} = \tfrac{1}{2}\Omega^2 T_1 T_2/(1 + \Delta^2 T_2^2 + \Omega^2 T_1 T_2)$, and Ω^2 is proportional to P. Without pure dephasing $T_2 = 2T_1$; in general $1/T_2 = 1/(2T_1) + 1/T_\varphi$, where T_φ is the pure dephasing time.

(a) SHOW THAT · U2 **[2]**

Show that $\Delta\nu = (1/\pi T_2)\sqrt{(1 + \Omega^2 T_1 T_2)}$, and hence that $\Delta\nu^2 = \Delta\nu_0^2(1 + P/P_{sat})$, where $\Delta\nu_0 = 1/(\pi T_2)$ and P_{sat} is the power at which $\Omega^2 T_1 T_2 = 1$.

1. ρ_{ee} is a Lorentzian in Δ that falls to half its peak value when $\Delta^2 T_2^2 = 1 + \Omega^2 T_1 T_2$, so its full width in angular frequency is $(2/T_2)\sqrt{(1 + \Omega^2 T_1 T_2)}$
2. dividing by 2π, $\Delta\nu = (1/\pi T_2)\sqrt{(1 + \Omega^2 T_1 T_2)}$; with $\Omega^2 T_1 T_2 = P/P_{sat}$ this gives $\Delta\nu^2 = \Delta\nu_0^2(1 + P/P_{sat})$, linear in P

   Candidates who take the half-width for the full width, or forget the 2π between angular and ordinary frequency, carry a factor of 2 or 2π into T_2.

(b) DESCRIBE · U4 **[2]**

Describe how the readings should be plotted to test this model and to extract $\Delta\nu_0$ and P_sat, and explain why the narrowest line measured is not itself $\Delta\nu_0$.

1. plot $\Delta\nu^2$ against P: the model predicts a straight line whose intercept at P = 0 is $\Delta\nu_0^2$ and whose gradient is $\Delta\nu_0^2$/P_sat, so P_sat = intercept/gradient; a plot of $\Delta\nu$ against P curves and cannot be extrapolated reliably
2. even the lowest power, 2.0 nW, is about a sixth of P_sat, so the 289 MHz line is already power-broadened by about 8%; the zero-power width has to be found by extrapolation

   Taking the narrowest line as the unbroadened one is the error this part catches: it gives T_2 = 1.10 ns and a dephasing time that is too short.

(c) DETERMINE · U4 **[3]**

Determine the gradient and intercept of your graph, and hence $\Delta\nu_0$ and P_sat.

1. $\Delta\nu^2$ = 0.835, 0.986, 1.303, 1.840, 3.069 and 5.358 × 10^5 MHz^2, which lie on a straight line to within 2%
2. gradient = 5.8 × 10^3 MHz^2 nW^{-1} (accept 5.7 to 6.0 × 10^3 MHz^2 nW^{-1}) and intercept = 7.1 × 10^4 MHz^2 (accept 6.9 to 7.3 × 10^4 MHz^2)
3. $\Delta\nu_0$ = √(intercept) = 266 MHz (accept 263 to 270 MHz) and P_sat = intercept/gradient = 12 nW (accept 11.5 to 12.8 nW)

   A line forced through the origin, or fitted to $\Delta\nu$ rather than $\Delta\nu^2$, gives no meaningful intercept. The intercept is in MHz^2, so $\Delta\nu_0$ is its square root; candidates who forget the root report a width of 7 × 10^4 MHz.

(d) EVALUATE · U4 **[3]**

Evaluate whether this emitter's line is lifetime-limited, finding T_2 and, if it is needed, the pure dephasing time T_φ.

1. $T_2 = 1/(\pi\Delta\nu_0) = 1/(\pi \times 266 \times 10^6\ \text{Hz})$ = 1.20 ns (accept 1.18 to 1.21 ns)
2. a lifetime-limited line would have $T_2 = 2T_1$ = 1.60 ns and $\Delta\nu_0 = 1/(2\pi T_1)$ = 199 MHz; the measured 266 MHz is a third wider, so the line is not lifetime-limited
3. $1/T_\varphi = 1/T_2 - 1/(2T_1) = (0.836 - 0.625)\ ns^{-1} = 0.211\ ns^{-1}$, so T_φ = 4.7 ns (accept 4.5 to 5.0 ns): pure dephasing, from phonons or fluctuating charges near the dot, which a measurement of T_1 cannot see

   Candidates who compare T_2 with T_1 rather than with $2T_1$ conclude that coherence outlasts population and that something has gone wrong. The bound is $T_2 \le 2T_1$, and the shortfall from it measures the pure dephasing.

(e) PREDICT · U3 **[1]**

Predict the rate at which the dot scatters photons when it is driven on resonance at P = 40 nW.

1. with P/P_sat = 40/12 = 3.3, ρ_{ee} = ½ × 3.3/4.3 = 0.38, and the scattering rate ρ_{ee}/T_1 = 0.38/(0.80 ns) = 4.8 × 10^8 s^{-1}, about 77% of the saturated maximum $1/(2T_1)$ = 6.3 × 10^8 s^{-1}

   The rate is ρ_{ee}/T_1, not ρ_{ee}/T_2: only the decay of population emits photons, and however hard the dot is driven the rate cannot exceed $1/(2T_1)$.

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17 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UAQM-11.03 · UAQM-11.05 · UAQM-12.01 · UAQM-12.02 · UAQM-12.03 **[15]**

In a cubic box of volume V with periodic boundary conditions, the transverse vector potential in Coulomb gauge is, in the Heisenberg picture, A(r, t) = Σ_(k,λ) √(ħ/(2ε₀ω_kV)) [a_(kλ) ε_(kλ) e^(i(k·r − ω_k t)) + a_(kλ)† ε_(kλ)* e^(−i(k·r − ω_k t))], with ω_k = ck, two unit polarisation vectors ε_(kλ) perpendicular to k for each k, and [a_(kλ), a_(k′λ′)†] = δ_kk′ δ_λλ′. A one-electron atom or ion, with an infinitely heavy nucleus at the origin, has an excited state |b⟩ and a lower state |a⟩ separated by $\hbar\omega_0$. It couples to the field through H′ = (e/m)A(0) · p, the dipole approximation, where −e and m are the electron's charge and mass, and ⟨a|p|b⟩ = −imω₀⟨a|r|b⟩. For a hydrogen-like ion of nuclear charge Ze, |⟨1s|r|2p, m⟩|² = ($2^{15}/3^{10}$) (a_0/Z)² for each m, and the 2p–1s energy difference is (3/4)Z² × 13.606 eV. Take $\hbar$ = 1.055 × 10^{-34} J s, e = 1.602 × 10^{-19} C, ε_0 = 8.854 × 10^{-12} F m^{-1}, c = 2.998 × 10^8 m s^{-1} and a_0 = 5.292 × 10^{-11} m.

(a) DERIVE · U2 **[2]**

Derive the transverse electric field operator from A, and show that in the vacuum each mode contributes $\hbar\omega_k/(2\varepsilon_0 V)$ to $\langle E^2\rangle$, so that its mean electric energy is half its zero-point energy.

1. in Coulomb gauge the transverse field is $E = -\partial A/\partial t = \Sigma_{(k,\lambda)}\, i\sqrt{(\hbar\omega_k/(2\varepsilon_0 V))}\,[a_{(k\lambda)}\varepsilon_{(k\lambda)} e^{(i(k\cdot r - \omega_k t)) - a_{(k\lambda)}\dagger \varepsilon_{(k\lambda)}* e^{(-i(k\cdot r - \omega_k t))}]$
2. in $\langle 0|E^2|0\rangle$ only the product $a\, a\dagger$ of each mode survives, with $\langle 0|a\, a\dagger|0\rangle = 1$ and $|\varepsilon|^2 = 1$, so each mode contributes $\hbar\omega_k/(2\varepsilon_0 V)$; its electric energy $(\varepsilon_0/2)\langle E^2\rangle V = \hbar\omega_k/4$ is half the zero-point energy $\hbar\omega_k/2$, the magnetic field carrying the other half

Candidates who lose the relative minus sign between the a and a† terms write a field that is not Hermitian and find $\langle E^2\rangle$ negative, which should stop them at once. The factor $-i\omega_k$ from the time derivative is what turns the normalisation of A into $\sqrt{(\hbar\omega_k/2\varepsilon_0 V)}$.

(b) SHOW THAT · U2 **[2]**

Show that the matrix element for emission of a photon into a mode (k, λ) already holding n photons is $\langle a; n+1|H'|b; n\rangle = (e/m)\sqrt{(\hbar/(2\varepsilon_0\omega_k V))}\,\sqrt{(n+1)}\,\varepsilon_{(k\lambda)}* \cdot \langle a|p|b\rangle$, and explain why this makes an excited atom in the vacuum emit, though an atom in a classical field of zero amplitude never does.

1. only the a† term of A raises the photon number, and $a\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle$ gives the stated element, with atomic factor $\varepsilon_{(k\lambda)}* \cdot \langle a|p|b\rangle$
2. for $n = 0$ the factor is $\sqrt{1}$, not zero: the 1 in $n + 1$ has no classical counterpart, since a classical field of zero amplitude gives $H' = 0$ and leaves $|b\rangle$ stationary; stimulated emission, proportional to n, and spontaneous emission, the 1, come from the same matrix element

The error this part catches is to say that the atom decays because a classical random field shakes it. The rate is set by the $\sqrt{(n+1)}$ of the creation operator; a classical noise field with the same $\langle E^2\rangle$ would drive absorption as often as emission.

(c) DERIVE · U2 **[4]**

Using Fermi's golden rule, and summing over all modes, directions and both polarisations, derive $A = \omega_0^3 e^2|\langle a|r|b\rangle|^2/(3\pi\varepsilon_0\hbar c^3)$ for spontaneous emission from $|b\rangle$ to $|a\rangle$.

1. $\Gamma = (2\pi/\hbar)\Sigma_{(k,\lambda)}|M_{(k\lambda)}|^2\delta(\hbar\omega_k - \hbar\omega_0)$, with $|M_{(k\lambda)}|^2 = (e^2/m^2)(\hbar/(2\varepsilon_0\omega_k V))m^2\omega_0^2|\varepsilon_{(k\lambda)}* \cdot r_{ab}|^2$ for $n = 0$, using $p_{ab} = -im\omega_0 r_{ab}$
2. the mode sum becomes an integral, $\Sigma_k \to (V/(2\pi)^3)\int k^2\, dk\, d\Omega = (V/(8\pi^3 c^3))\int \omega^2\, d\omega\, d\Omega$, in which V cancels the $1/V$ of $|M|^2$, and $\delta(\hbar\omega - \hbar\omega_0) = \delta(\omega - \omega_0)/\hbar$ sets $\omega = \omega_0$
3. the two polarisations perpendicular to k give $\Sigma_\lambda|\varepsilon_\lambda* \cdot r_{ab}|^2 = |r_{ab}|^2 - |\hat{k} \cdot r_{ab}|^2$, and since $|\hat{k} \cdot r_{ab}|^2$ averages to $|r_{ab}|^2/3$ over directions, the angular integral is $4\pi|r_{ab}|^2 - (4\pi/3)|r_{ab}|^2 = (8\pi/3)|r_{ab}|^2$ (for a real r_{ab} this is $\int|r_{ab}|^2\sin^2\Theta\, d\Omega$, Θ the angle between k and r_{ab})
4. $\Gamma = (2\pi/\hbar)(e^2\hbar\omega_0/2\varepsilon_0)|r_{ab}|^2(\omega_0^2/(8\pi^3 c^3\hbar))(8\pi/3) = \omega_0^3 e^2|r_{ab}|^2/(3\pi\varepsilon_0\hbar c^3) = A$, obtained from the quantised field without any appeal to Planck's law

The polarisation sum is where marks are lost. Candidates who take $\Sigma_\lambda|\varepsilon_\lambda \cdot r|^2 = |r|^2$, forgetting that a photon cannot be polarised along its own direction, are out by a factor of 3/2.

(d) CALCULATE · U3 **[3]**

Calculate A, the lifetime and the natural linewidth $\Gamma/2\pi$ of the 2p state of the hydrogen-like helium ion He^+ ($Z = 2$).

1. $\hbar\omega_0 = (3/4)(4)(13.606\text{ eV}) = 40.82\text{ eV}$, so $\omega_0 = (40.82 \times 1.602 \times 10^{-19}\text{ J})/(1.055 \times 10^{-34}\text{ J s}) = 6.198 \times 10^{16}\text{ rad s}^{-1}$ (a wavelength of 30.4 nm), and $|r_{ab}|^2 = 0.5549 \times (2.646 \times 10^{-11}\text{ m})^2 = 3.885 \times 10^{-22}\text{ m}^2$
2. $A = (6.198 \times 10^{16}\text{ s}^{-1})^3(1.602 \times 10^{-19}\text{ C})^2(3.885 \times 10^{-22}\text{ m}^2)/[3\pi(8.854 \times 10^{-12}\text{ F m}^{-1})(1.055 \times 10^{-34}\text{ J s})(2.998 \times 10^8\text{ m s}^{-1})^3] = 1.00 \times 10^{10}\text{ s}^{-1}$
3. $\tau = 1/A = 100\text{ ps}$ and $\Gamma/2\pi = A/2\pi = 1.59\text{ GHz}$; A scales as Z^4, ω_0^3 growing as Z^6 and $|r_{ab}|^2$ falling as Z^{-2}, so it is sixteen times hydrogen's $6.26 \times 10^8\text{ s}^{-1}$

Candidates who scale ω_0 but not the matrix element obtain a Z^6 law and a rate four times too large. Hydrogen's 1.6 ns lifetime divided by sixteen is the check.

(e) EXPLAIN · U1 **[2]**

The atom now sits in thermal radiation at temperature T, whose modes near ω_0 hold $\bar{n}$ photons on average. Explain how the $\sqrt{n}$ and $\sqrt{(n+1)}$ factors of the field operators lead to Planck's distribution, rather than assuming it as Einstein did.

1. because $|\langle n+1|a^\dagger|n\rangle|^2 = n+1$ and $|\langle n-1|a|n\rangle|^2 = n$, the emission rate from $|b\rangle$ is $A(\bar{n}+1)$, stimulated plus spontaneous, and the absorption rate from $|a\rangle$ is $A\bar{n}$ for non-degenerate levels: the B coefficients and A come from the same matrix element
2. in equilibrium $N_b A(\bar{n}+1) = N_a A\bar{n}$ with $N_b/N_a = e^{(-\hbar\omega_0/k_B T)}$, the Boltzmann factor being the one thermal input, so $\bar{n} = 1/(e^{(\hbar\omega_0/k_B T)} - 1)$: Planck's distribution is an output of the quantised field, and Einstein's relations follow with it

   Candidates who reproduce Einstein's argument, using Planck's law to fix A, have run the logic backwards. Here A is already known from (c), and the thermal photon distribution is what the balance delivers.

(f) JUSTIFY · U4 **[2]**

Justify the dipole approximation, and the use of the golden rule leading to exponential decay at the rate A, for the He^+ 2p state.

1. the dipole approximation needs $ka \ll 1$ across the ion: $k = \omega_0/c = 2.07 \times 10^8\ m^{-1}$ and the orbit size $a_0/Z = 2.65 \times 10^{-11}$ m give $ka = 5.5 \times 10^{-3}$, so $e^{(ik\cdot r)} \approx 1$ to better than 1%
2. the golden rule and the Markov step need a weak coupling and a mode density smooth across the line: $\Gamma/\omega_0 = (1.00 \times 10^{10}\ s^{-1})/(6.20 \times 10^{16}\ s^{-1}) = 1.6 \times 10^{-7}$, so the free-space modes are flat over many linewidths and the decay is exponential, with departures only at times of order $1/\omega_0$ and at very long times, both negligible

   A justification must put numbers to its small parameters. 'The ion is small' earns nothing until ka is evaluated, and 'the coupling is weak' nothing until Γ is set beside ω_0.

18

CORE COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UAQM-15.07 · UAQM-15.08 · UAQM-15.09 **[16]**

A uniform gas of N spinless bosons of mass m in a box of volume V, with periodic boundary conditions, interacts through a short-range potential represented at low energy by the contact coupling $g = 4\pi\hbar^2 a_s/m$, where a_s is the s-wave scattering length. In second quantisation $H = \Sigma_k \varepsilon_k a_k^\dagger a_k + (g/2V)\Sigma_{(k,k',q)} a_{(k+q)}^\dagger a_{(k'-q)}^\dagger a_{(k')} a_k$, with $\varepsilon_k = \hbar^2 k^2/2m$. Bogoliubov's approximation replaces a_0 and $a_0^\dagger$ by $\sqrt{N_0}$, where N_0 is the number of bosons in the condensate, and keeps the terms of H up to second order in the operators a_k with $k \neq 0$, the sums over such k being written Σ'. Write $n = N/V$. Take $\hbar = 1.055 \times 10^{-34}$ J s, 1 u $= 1.661 \times 10^{-27}$ kg and $k_B = 1.381 \times 10^{-23}$ J K^{-1}.

(a) SHOW THAT · U2 **[3]**

Show that, to this order and with N_0 eliminated in favour of N, $H \approx gN^2/(2V) + \Sigma'[(\varepsilon_k + gn)a_k^\dagger a_k + (gn/2)(a_k^\dagger a_{(-k)}^\dagger + a_k a_{(-k)})]$.

1. the term with all four momenta zero gives $(g/2V)N_0^2$; terms with one nonzero momentum vanish by momentum conservation, and those with three or four are dropped as of higher order
2. with two nonzero momenta, the four ways of placing one condensate operator among the creators and one among the annihilators each give $N_0 a_k^\dagger a_k$, and $a_q^\dagger a_{(-q)}^\dagger a_0 a_0$ and $a_0^\dagger a_0^\dagger a_q a_{(-q)}$ give $N_0(a_q^\dagger a_{(-q)}^\dagger + a_q a_{(-q)})$, so the interaction is $(g/2V)[N_0^2 + 4N_0\Sigma' a_k^\dagger a_k + N_0\Sigma'(a_k^\dagger a_{(-k)}^\dagger + a_k a_{(-k)})]$
3. with $N_0 = N - \Sigma' a_k^\dagger a_k$, $N_0^2 \approx N^2 - 2N\Sigma' a_k^\dagger a_k$ cancels half of the $4N_0$ term, leaving $gN/V = gn$ for each particle outside the condensate; elsewhere N_0 may be replaced by N to this order, which gives the stated H

   Candidates who replace N_0 by N before counting obtain 2gn instead of gn on the diagonal, and with it a spectrum with a gap at k = 0. The depletion hidden in N_0^2 is what makes the phonon gapless.

(b) DERIVE · U2 **[4]**

Writing $a_k = u_k b_k - v_k b_{(-k)}^\dagger$ with u_k and v_k real, derive the condition on u_k and v_k that keeps $[b_k, b_k^\dagger] = 1$, the choice that removes the terms $b_k b_{(-k)}$ and $b_k^\dagger b_{(-k)}^\dagger$, and the excitation energy $E_k = \sqrt{(\varepsilon_k(\varepsilon_k + 2gn))}$.

1. $[a_k, a_k^\dagger] = u_k^2[b_k, b_k^\dagger] - v_k^2[b_{(-k)}, b_{(-k)}^\dagger] = u_k^2 - v_k^2$ must equal 1, so $u_k = \cosh\theta_k$ and $v_k = \sinh\theta_k$: a hyperbolic rotation rather than an ordinary one
2. for the pair $k, -k$, with $X_k = \varepsilon_k + gn$ and $Y = gn$, $H_k = X_k(a_k^\dagger a_k + a_{(-k)}^\dagger a_{(-k)}) + Y(a_k^\dagger a_{(-k)}^\dagger + a_k a_{(-k)})$; after substitution the coefficient of $b_k b_{(-k)} + b_k^\dagger b_{(-k)}^\dagger$ is $-2X_k u_k v_k + Y(u_k^2 + v_k^2) = -X_k \sinh 2\theta_k + Y \cosh 2\theta_k$
3. setting it to zero gives $\tanh 2\theta_k = gn/(\varepsilon_k + gn)$, which has a solution for every $k \neq 0$ because gn is less than $\varepsilon_k + gn$
4. the coefficient of $b_k^\dagger b_k + b_{(-k)}^\dagger b_{(-k)}$ is $X_k \cosh 2\theta_k - Y \sinh 2\theta_k = \sqrt{(X_k^2 - Y^2)}$, so $H =$ constant $+ \Sigma' E_k b_k^\dagger b_k$ with $E_k = \sqrt{((\varepsilon_k + gn)^2 - (gn)^2)} = \sqrt{(\varepsilon_k(\varepsilon_k + 2gn))}$

Candidates who take $u^2 + v^2 = 1$, by analogy with a rotation, destroy the boson commutator. For bosons the transformation is hyperbolic, and that is what lets u_k and v_k both grow without bound as k approaches zero.

(c) SKETCH · U1 **[2]**

Sketch E_k against k, with ε_k on the same axes, marking the behaviour at small and at large k and the scale $k \approx 1/\xi$, where $\xi = \hbar/\sqrt{(2mgn)}$, at which it changes.

1. both curves start at the origin, E_k rising linearly with slope $\hbar c$, where $c = \sqrt{(gn/m)}$, a phonon branch, while ε_k starts as a parabola of zero slope; both curves are labelled and E_k lies above ε_k everywhere
2. the change from linear to quadratic is marked near $k \approx 1/\xi$, where $\varepsilon_k \approx gn$, and at large k E_k approaches from below the parabola $\varepsilon_k + gn$, a free particle displaced upward by the mean field gn

A sketch with a gap at $k = 0$ shows the error described in (a). The two features to show are the linear start, which makes the gas a superfluid, and the offset gn at large k.

(d) CALCULATE · U3 **[3]**

For a condensate of ^{87}Rb ($m = 87$ u, $a_s = 5.3$ nm) of density $n = 2.0 \times 10^{20}$ m^{-3}, calculate gn/k_B in nanokelvin, the speed of sound c, the healing length ξ and $n a_s^3$.

1. $m = 87 \times 1.661 \times 10^{-27}$ kg $= 1.445 \times 10^{-25}$ kg and $g = 4\pi\hbar^2 a_s/m = 4\pi(1.055 \times 10^{-34}$ J s$)^2(5.3 \times 10^{-9}$ m$)/(1.445 \times 10^{-25}$ kg$) = 5.13 \times 10^{-51}$ J m^3, so $gn = 1.03 \times 10^{-30}$ J and $gn/k_B = 74$ nK
2. $c = \sqrt{(gn/m)} = \sqrt{(1.03 \times 10^{-30}}$ J$/1.445 \times 10^{-25}$ kg$) = 2.66 \times 10^{-3}$ m s^{-1} and $\xi = \hbar/\sqrt{(2mgn)} = 1.9 \times 10^{-7}$ m, about 0.19 μm
3. $n a_s^3 = 2.0 \times 10^{20}$ m$^{-3} \times (5.3 \times 10^{-9}$ m$)^3 = 3.0 \times 10^{-5} \ll 1$, so the gas is dilute and Bogoliubov's quadratic theory, an expansion in $\sqrt{(n a_s^3)}$, is well justified

Using h for $\hbar$ in g is out by $4\pi^2$, and leaving the mass in atomic mass units gives nonsense. The diluteness check belongs to the answer: without it the numbers have no claim to be right.

(e) DETERMINE · U3 **[2]**

Determine $\langle a_k^\dagger a_k\rangle$ for $k \neq 0$ in the ground state, the vacuum of every b_k, and hence, given that the sum over k gives a depletion $(N - N_0)/N = (8/(3\sqrt{\pi}))\sqrt{(n a_s^3)}$, the fraction of the atoms of (d) that lie outside the condensate at zero temperature.

1. $a_k = u_k b_k - v_k b_{(-k)}^\dagger$ gives $\langle a_k^\dagger a_k\rangle = v_k^2\langle b_{(-k)} b_{(-k)}^\dagger\rangle = v_k^2 = \tfrac{1}{2}[(\varepsilon_k + gn)/E_k - 1]$, nonzero for every k: the pair term of the interaction holds atoms out of the condensate in pairs of opposite momentum even at $T = 0$
2. $(N - N_0)/N = 1.505 \times \sqrt{(3.0 \times 10^{-5})} = 8.2 \times 10^{-3}$: less than 1% of the atoms are outside the condensate, consistent with keeping H to quadratic order

Candidates who take the ground state to be the vacuum of the a_k conclude that every atom is in the condensate. The quasiparticle vacuum is not the particle vacuum: it contains correlated pairs.

(f) PREDICT · U1 **[2]**

A tightly focused repulsive laser beam is dragged through the condensate of (d) at 1.5 mm s^{-1}. Predict, using Landau's criterion, whether it heats the gas, and whether an obstacle moving at the same speed through an ideal Bose gas would.

1. an obstacle moving at v can create an excitation of momentum p only if $E(p) - p \cdot v \leq 0$, which needs $v \geq E(p)/p$ for some p; for the Bogoliubov spectrum $E/p = \sqrt{(c^2 + (p/2m)^2)}$ is least as $p \to 0$, so the critical speed is $c = 2.66$ mm s^{-1}, and at 1.5 mm s^{-1} no excitation can be made: the flow is frictionless and the gas is not heated (vortex nucleation, beyond this theory, is neglected)
2. for the ideal gas $E = p^2/2m$, so $E/p = p/2m$ can be made as small as desired and the critical speed is zero: the obstacle creates excitations at any speed and heats the gas, so superfluidity comes from the interaction, which turns the low-energy spectrum into phonons

   Candidates who argue that a condensate flows without friction simply because its atoms share one state miss the point of Landau's criterion. The ideal condensate shares that property and is not a superfluid.

University Quantum Field Theory I · Midterm Examination

The first eight units of Quantum Field Theory I on one paper: the Feynman propagator derived by contour, the Gell-Mann–Low formula and Wick's theorem, symmetry factors and Feynman rules read off a Lagrangian, Källén–Lehmann and LSZ, decay rates and cross-sections carried to seconds and barns, one-loop φ^4 in dimensional regularisation with its beta function and Landau pole, and the signs and traces of Yukawa theory with Dirac fermions. The data question fits a resonance against the s-wave unitarity limit, and the extended derivation quantises the photon covariantly, derives its propagator in every covariant gauge and checks the Ward identity on Compton scattering by a charged scalar, down to Thomson's cross-section.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	**73**	**14**	**8**

SECTION	**UQFTI · Midterm · University Quantum Field Theory I — Midterm examination**
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	**Recall 2 · Routine 7 · Demanding 3 · Discriminating 2** *Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.*
COURSE UNITS	UQFTI-01 · UQFTI-02 · UQFTI-03 · UQFTI-04 · UQFTI-05 · UQFTI-06 · UQFTI-07 · UQFTI-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

> *Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.*
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THE PAPER

University Quantum Field Theory I · Midterm Examination

Answer all questions. 73 marks in 90 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-02.01 · UQFTI-06.01 **[1]**

With $\hbar = c = 1$ the action is dimensionless, so in d spacetime dimensions the Lagrangian density has mass dimension d. The kinetic terms then give a real scalar field φ the mass dimension $(d-2)/2$ and a Dirac field ψ the mass dimension $(d-1)/2$. For which interaction is the coupling constant dimensionless, so that the theory is renormalisable by power counting, in $d = 6$?

A $(\lambda/4!)\varphi^4$

B $(g/6!)\varphi^6$

C $(g/3!)\varphi^3$

D $g\varphi\bar{\psi}\psi$

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-02.05 · UQFTI-02.06 **[1]**

The Gell-Mann–Low formula expresses a correlation function in the interacting vacuum $|\Omega\rangle$ through free fields: $\langle\Omega|T\varphi_H(x)\varphi_H(y)|\Omega\rangle = \langle 0|T[\varphi_I(x)\varphi_I(y)\exp(-i\int H_I\,dt)]|0\rangle / \langle 0|T[\exp(-i\int H_I\,dt)]|0\rangle$, where $|0\rangle$ is the free vacuum, φ_I and H_I are in the interaction picture, the time integrals run from $-T$ to T, and the limit is taken as $T \to \infty(1-i\varepsilon)$. What does taking the limit along the slightly tilted direction $T(1-i\varepsilon)$ accomplish?

A It damps every eigenstate of the full Hamiltonian above the ground state by e^(−(E_n − E₀)εT) relative to $|\Omega\rangle$, so the long evolution projects $|0\rangle$ onto $|\Omega\rangle$, provided $\langle\Omega|0\rangle \neq 0$ and there is a gap above E_0.

B It switches the interaction on and off slowly, so that $|0\rangle$ evolves into $|\Omega\rangle$ with $\langle\Omega|0\rangle = 1$, and the denominator is then needed only to cancel the ultraviolet divergences of the numerator.

C It makes the Dyson series converge by damping the high-frequency modes of H_I, while the projection of $|0\rangle$ onto $|\Omega\rangle$ is supplied separately by normal ordering H_I.

D It makes the contractions retarded rather than Feynman propagators, so that the formula respects causality, while the time ordering itself projects $|0\rangle$ onto $|\Omega\rangle$.

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-03.02 · UQFTI-03.03 **[1]**

In φ^4 theory with $\mathcal{L}_{int} = -(\lambda/4!)\varphi^4$, two connected vacuum diagrams appear at order λ^2. In diagram (i) two vertices are joined by four propagators. In diagram (ii) two vertices are joined by two propagators, and each vertex also carries one propagator that begins and ends on it. Each diagram contributes its spacetime integral times $(-i\lambda)^2/S$. What are the symmetry factors S of (i) and (ii)?

A 24 and 8

B 24 and 16

C 48 and 8

D 48 and 16

Answer

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-04.02 · UQFTI-04.03 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. A real scalar field φ, canonically normalised so that $[\varphi(x, t), \pi(y, t)] = i\delta^3(x - y)$, is interacting and invariant under $\varphi \to -\varphi$. Its lightest particle has physical mass m, and the field can create states of any odd number of these particles. Its exact two-point function has the Källén–Lehmann form $\int_0^\infty (d\mu^2/2\pi)\, \rho(\mu^2)\, i/(p^2 - \mu^2 + i\varepsilon)$, with $\rho(\mu^2) = 2\pi Z\delta(\mu^2 - m^2)$ plus a continuum. Which statement is correct?

A The continuum begins at $\mu^2 = 4m^2$, and $Z < 1$ because φ also creates two-particle states.

B The continuum begins at $\mu^2 = 9m^2$, $\rho(\mu^2) \geq 0$, and the canonical commutator fixes the total weight $\int \rho\, d\mu^2/2\pi = 1$, so $0 \leq Z \leq 1$.

C The continuum begins at $\mu^2 = 9m^2$, and Z may exceed 1 if the interaction is attractive, because ρ is then negative above threshold.

D The continuum begins at $\mu^2 = 3m^2$, and $Z = 1$ exactly, because the canonical commutator fixes the residue of the pole.

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-04.06 · UQFTI-04.07 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model, $\mathcal{L}_{int} = -g\psi\dagger\psi\varphi$, with ψ a complex scalar of mass m_ψ and φ a real scalar of mass $m_\varphi < 2m_\psi$. To order g^2 the amplitude for $\psi(p_1)\bar{\psi}(p_2) \to \psi(p_3)\bar{\psi}(p_4)$ is $M = -g^2[1/(s - m_\varphi^2) + 1/(t - m_\varphi^2)]$, with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$, the first term from annihilation through φ and the second from φ exchange. Defining s, t and u in the same way for $\psi(p_1)\psi(p_2) \to \psi(p_3)\psi(p_4)$, what is the order-$g^2$ amplitude for that process?

A $-g^2[1/(s - m_\varphi^2) + 1/(t - m_\varphi^2)]$

B $-g^2[1/(t - m_\varphi^2) - 1/(u - m_\varphi^2)]$

C $-g^2[1/(s - m_\varphi^2) + 1/(t - m_\varphi^2) + 1/(u - m_\varphi^2)]$

D $-g^2[1/(t - m_\varphi^2) + 1/(u - m_\varphi^2)]$

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.01 · UQFTI-05.04 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+,-,-,-)$. In φ^4 theory with $\mathcal{L}_{\text{int}} = -(\lambda/4!)\varphi^4$ and $\lambda = 0.60$, two particles collide at a centre-of-mass energy $\sqrt{s} = 4.0$ GeV, far enough above threshold that their mass can be neglected. At tree level the amplitude for $\varphi\varphi \to \varphi\varphi$ is $\mathcal{M} = -\lambda$, and for a $2 \to 2$ process with equal masses in and out, $d\sigma/d\Omega = |\mathcal{M}|^2/(64\pi^2 s)$ in the centre-of-mass frame. Take $(\hbar c)^2 = 0.3894$ GeV^2 mb. What is the total cross-section for $\varphi\varphi \to \varphi\varphi$?

A 87 nb

B 174 nb

C 14 nb

D 0.15 nb

Answer

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-07.02 · UQFTI-07.05 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+,-,-,-)$. Two identical Dirac fermions of mass m scatter, $\psi(p_1)\psi(p_2) \to \psi(p_3)\psi(p_4)$, through the Yukawa interaction $\mathcal{L}_{\text{int}} = -g\varphi\bar{\psi}\psi$, where φ is a real scalar of mass m_φ. At tree level two diagrams contribute, one with momentum transfer squared $t = (p_1 - p_3)^2$ and one with $u = (p_1 - p_4)^2$. In the non-relativistic limit $\bar{u}(p_3)u(p_1) \to 2m$ times a Kronecker delta between the spin labels. Which statement is correct?

A The two diagrams add with the same sign, as they would for bosons, because the particle exchanged between the fermion lines is the boson φ.

B The two diagrams have a relative minus sign, which makes scalar exchange repulsive between two fermions although it attracts a fermion and an antifermion.

C The two diagrams have a relative minus sign, because their final states differ by an exchange of anticommuting operators; the t-channel term alone is the Born amplitude of an attractive Yukawa potential.

D Only the t-channel diagram contributes, because the u-channel diagram counts the same final state again and is removed by the 1/2 for identical particles.

Answer

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8 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-08.02 · UQFTI-08.03 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In Feynman gauge the four photon operators of one momentum k, taken along z, obey $[a^{\lambda}(k), a^{\lambda'\dagger}(k')] = -g^{\lambda\lambda'}(2\pi)^3 2|k|\delta^3(k - k')$ for $\lambda, \lambda' = 0, 1, 2, 3$, where $\lambda = 0$ labels the timelike polarisation $\varepsilon^0 = (1, 0, 0, 0)$, $\lambda = 1, 2$ the transverse ones and $\lambda = 3$ the longitudinal one, $\varepsilon^3 = (0, 0, 0, 1)$. The Gupta–Bleuler condition requires the positive-frequency part of $\partial_\mu A^\mu$ to annihilate every physical state, which for this momentum reads $(a^0 - a^3)|\psi\rangle = 0$. Which statement is correct?

A The condition removes timelike and longitudinal photons from every physical state, which may contain only photons created by $a^{1\dagger}$ and $a^{2\dagger}$.

B Physical states may contain $(a^{0\dagger} - a^{3\dagger})$ acting on a transverse state; such a state has zero norm and adds nothing to the energy or to any gauge-invariant expectation value.

C A longitudinal photon $a^{3\dagger}|0\rangle$ has negative norm like the timelike one, and the condition admits only states in which the two negative norms cancel.

D The full condition $\partial_\mu A^\mu|\psi\rangle = 0$, creation part included, would serve equally well; only its positive-frequency part is kept, to simplify the algebra.

Answer

9 CORE COURSE

STRUCTURED · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07 · UQFTI-01.09 **[8]**

A free real scalar field of mass m is written, in natural units $\hbar = c = 1$ with the metric (+, −, −, −), as $\varphi(x) = \int d^3p/((2\pi)^3 2E_p)\,[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}]$, where $E_p = \sqrt{|p|^2 + m^2}$, $p^0 = E_p$ in the exponents and $[a(p), a^\dagger(q)] = (2\pi)^3 2E_p\,\delta^3(p - q)$. Its vacuum two-point function is $D(x - y) = \langle 0|\varphi(x)\varphi(y)|0\rangle = \int d^3p/((2\pi)^3 2E_p)\, e^{-ip\cdot(x-y)}$. Take $\hbar c = 197.327$ MeV fm.

(a) DERIVE **[3]**

Starting from the definition $D_F(x - y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$, derive the representation $D_F(x - y) = \int d^4p/(2\pi)^4\, i\, e^{-ip\cdot(x-y)}/(p^2 - m^2 + i\varepsilon)$, evaluating the p^0 integral with the contour closed in whichever half-plane each time ordering requires.

(b) SKETCH **[2]**

The retarded propagator is $D_R(x - y) = \theta(x^0 - y^0)\langle 0|[\varphi(x), \varphi(y)]|0\rangle$, whose momentum-space form is $i/((p^0 + i\varepsilon)^2 - |p|^2 - m^2)$. Sketch the complex p^0-plane at fixed p, marking the poles of D_F and of D_R and the contour used when $x^0 < y^0$, and state where in spacetime each propagator is non-zero.

(c) CALCULATE **[2]**

Two static point sources, $j(x) = g[\delta^3(x - x_1) + \delta^3(x - x_2)]$, coupled through $\mathcal{L}_{int} = j\varphi$ and held a distance $r = |x_1 - x_2|$ apart for a long time T, give a vacuum persistence amplitude $\langle 0|0\rangle \propto e^{-iE(r)T}$, with interaction energy $E(r) = -(g^2/(4\pi r))e^{-mr}$. For a scalar of mass m = 550 MeV, $g^2/(4\pi) = 0.80$ and r = 0.70 fm, calculate E(r) in MeV and state the range of the force in femtometres.

(d) EXPLAIN **[1]**

$D_F(x - y)$ is non-zero when $x - y$ is spacelike. Explain why this does not allow a signal to travel faster than light.

10 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-02.01 · UQFTI-02.09 · UQFTI-03.02 · UQFTI-03.07 · UQFTI-04.05 · UQFTI-04.07 · UQFTI-04.09 · UQFTI-05.03 **[10]**

Two real scalar fields, φ of mass m_φ and χ of mass m_χ, live in four spacetime dimensions and interact through $\mathcal{L}_{int} = -(g/2)\varphi\chi^2 - (\kappa/2)\varphi\partial_\mu\chi\partial\hat{}\mu\chi$, in natural units $\hbar = c = 1$ with the metric (+, −, −, −). Treat both couplings to the lowest order at which each process occurs. For the two-body decay of a particle of mass m_φ into particles of momentum p in its rest frame, Lorentz-invariant phase space gives $\int d\Pi_2 = p/(4\pi m_\varphi)$ when the two final particles are distinguishable, and $\Gamma = (1/(2m_\varphi))\int |M|^2 d\Pi_2$. Take $\hbar = 6.582 \times 10^{-25}$ GeV s.

(a) DETERMINE **[2]**

Determine the Feynman rule for the φχχ vertex, taking every momentum as incoming and p_2 and p_3 as the momenta of the two χ lines, and give the mass dimensions of g and κ.

(b) SHOW THAT **[2]**

Set κ = 0. By counting the Wick contractions in the order-g^2 term of the Gell-Mann–Low numerator, show that the one-loop φ self-energy, a loop of two χ propagators, carries a symmetry factor of 2, and that the one-loop χ self-energy, a loop of one φ and one χ propagator, carries a symmetry factor of 1. Check each by counting the diagram's automorphisms.

(c) DETERMINE **[2]**

Still with κ = 0, determine the tree-level amplitude M for $\chi(p_1)\chi(p_2) \to \chi(p_3)\chi(p_4)$ in terms of the Mandelstam variables s, t and u, check its symmetry under t ↔ u and the value of s + t + u, and say why √Z = 1 may be used on each external leg at this order.

(d) PREDICT **[2]**

The masses are now chosen so that m_φ > 2m_χ. Predict what becomes of the s-channel term of the amplitude in (c) as s approaches m_φ^2, and of the φ particle's place in the LSZ formula, giving the reason, and say whether the t- and u-channel poles are affected.

(e) CALCULATE **[2]**

Restore κ. With m_φ = 0.90 GeV, m_χ = 0.14 GeV, g = 0.50 GeV and κ = 0.80 GeV^{-1}, calculate the decay rate Γ(φ → χχ) in MeV and the lifetime of φ in seconds.

11 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-07.02 · UQFTI-07.03 · UQFTI-07.04 · UQFTI-07.06 · UQFTI-07.07 · UQFTI-07.08 **[10]**

A real spin-0 boson φ of mass m_φ couples to a Dirac fermion ψ of mass m either as a scalar, $\mathcal{L}_S = -g\phi\bar{\psi}\psi$, or as a pseudoscalar, $\mathcal{L}_P = -ig\phi\bar{\psi}\gamma^5\psi$, in natural units $\hbar = c = 1$ with the metric (+, −, −, −). Use the spin sums $\Sigma u(k)\bar{u}(k) = \not{k} + m$ and $\Sigma v(k)\bar{v}(k) = \not{k} - m$; the identities tr 1 = 4 and $\mathrm{tr}(\not{a}\not{b}) = 4a\cdot b$; the vanishing of the trace of an odd number of γ matrices; $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, $(\gamma^5)^2 = 1$ and $\gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$. For the decay $\phi \to \psi(k_1)\bar{\psi}(k_2)$ write $\beta = \sqrt{(1 - 4m^2/m_\phi^2)}$, the speed of either fermion in the φ rest frame; then $\int d\Pi_2 = \beta/(8\pi)$, and $\Gamma = (1/(2m_\phi)) \times$ (spin-summed $|M|^2$) $\times \int d\Pi_2$. Take $\hbar = 6.582 \times 10^{-25}$ GeV s.

(a) DERIVE **[2]**

For the scalar coupling the amplitude is $M_S = -g\,\bar{u}(k_1)v(k_2)$. Using Casimir's trick with the spin sums, derive the spin-summed $|M_S|^2$ in terms of g, m_ϕ and β.

(b) SHOW THAT **[2]**

For the pseudoscalar coupling the amplitude is $M_P = -g\,\bar{u}(k_1)i\gamma^5 v(k_2)$. Show that the spin-summed $|M_P|^2$ is $2g^2m_\phi^2$, with no factor of β^2.

(c) CALCULATE **[2]**

For $m_\phi = 1.50$ GeV, m = 0.60 GeV and g = 0.30, calculate $\Gamma(\phi \to \psi\bar{\psi})$ and the φ lifetime in seconds for each coupling.

(d) EXPLAIN **[2]**

Explain the powers β^3 and β in the two rates by the parity of φ and the orbital angular momentum of the fermion pair near threshold.

(e) JUSTIFY **[2]**

In the one-loop φ self-energy built from a ψ loop, Wick's theorem supplies a factor −1. Justify this factor from the anticommutation of the fields, and confirm it from the optical theorem, which requires $\mathrm{Im}\,M(\phi \to \phi) = m_\phi\Gamma(\phi \to \psi\bar{\psi})$ once $m_\phi > 2m$.

12 CORE COURSE

STRUCTURED · DISCRIMINATING · UQFTI-06.02 · UQFTI-06.03 · UQFTI-06.04 · UQFTI-06.06 · UQFTI-06.07 · UQFTI-06.08 · UQFTI-06.09 **[11]**

Real scalar φ^4 theory, $\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - (\lambda/4!)\varphi^4$, in natural units $\hbar = c = 1$ with the metric $(+,-,-,-)$, is regularised dimensionally in $d = 4 - 2\varepsilon$ dimensions, with λ replaced by $\lambda\mu^{2\varepsilon}$ so that λ stays dimensionless, μ being an arbitrary scale. Use the Feynman-parameter identity $1/(AB) = \int_0^1 dx\,[xA + (1-x)B]^{-2}$, the Euclidean master integral $\int d^d\ell_E/(2\pi)^d\,(\ell_E^2 + \Delta)^{-2} = \Gamma(2 - d/2)(4\pi)^{-d/2}\Delta^{d/2-2}$, and $\Gamma(\varepsilon) = 1/\varepsilon - \gamma_E + O(\varepsilon)$. At one loop the field-strength counterterm vanishes, so the bare coupling is $\lambda_0 = \mu^{2\varepsilon}(\lambda + \delta_\lambda)$, where the counterterm vertex is $-i\delta_\lambda$.

(a) SHOW THAT **[3]**

The s-channel one-loop diagram for $\varphi\varphi \to \varphi\varphi$, with incoming momenta p_1 and p_2, $p = p_1 + p_2$ and $s = p^2$, carries a symmetry factor of 2. Show that, below the threshold $s = 4m^2$, its contribution to iM, with an overall factor $\mu^{2\varepsilon}$ removed, is $(i\lambda^2/(32\pi^2))\int_0^1 dx\,[1/\varepsilon - \gamma_E + \ln 4\pi - \ln(\Delta_s/\mu^2)] + O(\varepsilon)$, where $\Delta_s = m^2 - x(1-x)s$.

(b) SHOW THAT **[2]**

Including the t- and u-channel diagrams and the counterterm, the renormalised coupling λ is defined by the condition $M = -\lambda$ at $s = 4m^2$, $t = u = 0$. Show that $M(s,t,u) = -\lambda - (\lambda^2/(32\pi^2))\Sigma\int_0^1 dx\,\ln[\Delta(q^2)/\Delta(q_0^2)]$, the sum running over the three channels $q^2 = s, t, u$ with reference values $q_0^2 = 4m^2, 0, 0$ and $\Delta(q^2) = m^2 - x(1-x)q^2$, and that neither ε nor μ survives in it.

(c) DERIVE **[2]**

In minimal subtraction δ_λ contains only the poles of the three channels. Derive the one-loop beta function $\beta(\lambda) = \mu\,d\lambda/d\mu$ from the μ-independence of the bare coupling λ_0.

(d) CALCULATE **[2]**

In a model whose coupling is $\lambda(\mu_0) = 1.5$ at $\mu_0 = 1.0$ TeV, calculate the scale at which the one-loop running coupling has doubled and the scale of its Landau pole, both in GeV.

(e) ESTIMATE **[2]**

Estimate the highest scale at which this one-loop running can be trusted, stating the criterion you use, and say what the Landau pole of (d) then tells you about the theory.

13 CORE COURSE

DATA ANALYSIS · DEMANDING · UQFTI-01.01 · UQFTI-05.02 · UQFTI-05.03 · UQFTI-05.07 · UQFTI-05.09 **[11]**

Two distinct spinless particles, a of mass 0.400 GeV and b of mass 0.600 GeV, scatter elastically, a + b → a + b, and near a centre-of-mass energy of 1.85 GeV the cross-section is dominated by an s-channel resonance X of mass m_X and total width Γ. Parts (a) to (d) treat X as a spin-0 particle coupled to ab through $\mathcal{L}_{\text{int}} = -g X a b$, which gives $\Gamma(X \to ab) = g^2 p/(8\pi m_X^2)$, where p is the momentum of a or b in the centre-of-mass frame at $\sqrt{s} = m_X$; near the peak the amplitude is $M = -g^2/(s - m_X^2 + i m_X \Gamma)$. For two particles in and two out with these masses, $\sigma = |M|^2/(16\pi s)$. To the accuracy of these data the line shape is $\sigma(\sqrt{s}) = \sigma_{\text{peak}}(\Gamma^2/4)/[(\sqrt{s} - m_X)^2 + \Gamma^2/4]$. The optical theorem reads Im M(forward) = $2\sqrt{s}\, p\, \sigma_{\text{tot}}$. Work in natural units with the metric (+, −, −, −), and take $(\hbar c)^2 = 0.3894$ GeV² mb and $\hbar = 6.582 \times 10^{-25}$ GeV s. The table gives the measured elastic cross-section at seven energies.

Elastic cross-section for a + b → a + b near the resonance

√s (GeV)	σ (mb)	Uncertainty in σ (mb)
1.826	0.198	0.005
1.834	0.363	0.009
1.842	0.805	0.020
1.850	1.300	0.033
1.858	0.809	0.020
1.866	0.364	0.009
1.874	0.190	0.005

(a) SHOW THAT **[2]**

Show that $\sigma_{\text{peak}} = (4\pi/p^2)B^2$, where $B = \Gamma(X \to ab)/\Gamma$ is the branching fraction to ab, so that $4\pi/p^2$ is the largest cross-section an s-wave resonance can give in this channel.

(b) DETERMINE **[3]**

By plotting a suitable linear function of the data, determine m_X, Γ and σ_peak.

(c) CALCULATE **[2]**

Calculate p at $\sqrt{s} = m_X$ and the s-wave unitarity limit $4\pi/p^2$ in millibarns, and hence B and the lifetime of X in seconds.

(d) DETERMINE **[2]**

Use the optical theorem to determine the total and the inelastic cross-sections at the peak, and determine $\Gamma(X \to ab)$ and g.

(e) EVALUATE **[2]**

A separate experiment counts the decays of X and finds $B = 0.43 \pm 0.05$. For a resonance of spin J the peak elastic cross-section is $(2J + 1)(4\pi/p^2)B^2$. Evaluate whether the data support $J = 0$ against $J = 1$, and say what further measurement on the same events would settle the spin directly.

14

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQFTI-08.01 · UQFTI-08.02 · UQFTI-08.04 · UQFTI-08.05 · UQFTI-08.06 · UQFTI-08.08 · UQFTI-08.09 **[15]**

The free electromagnetic field, in natural units $\hbar = c = 1$ and Heaviside–Lorentz units with $e^2 = 4\pi\alpha$ and the metric $(+,-,-,-)$, has the Maxwell Lagrangian $\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A complex scalar φ of mass m and charge e couples to it through $\mathcal{L}_\varphi = (D_\mu\varphi)^*(D^\mu\varphi) - m^2\varphi^*\varphi$ with $D_\mu = \partial_\mu + ieA_\mu$, whose Feynman rules are a vertex $-ie(p + p')^\mu$, with p and p' the incoming and outgoing scalar momenta, and a two-photon seagull vertex $2ie^2g^{\mu\nu}$. Write Π^μ for the momentum conjugate to A_μ. Take $\alpha = 1/137.036$, $(\hbar c)^2 = 0.3894\ \text{GeV}^2\,\text{mb}$, and $m = 139.570\ \text{MeV}$, the mass of the charged pion, in (f).

(a) SHOW THAT **[2]**

Show that the Maxwell Lagrangian gives A_0 no conjugate momentum, so that canonical quantisation of all four components fails, and that adding the gauge-fixing term $-(1/(2\xi))(\partial_\mu A^\mu)^2$ supplies one.

(b) DERIVE **[3]**

From the Euler–Lagrange equations, derive the field equation of the gauge-fixed Lagrangian and, from its quadratic action in momentum space, the photon propagator in a general covariant gauge.

(c) SHOW THAT **[3]**

Two conserved currents, $k_\mu j_1{}^\mu(k) = k_\mu j_2{}^\mu(k) = 0$, exchange a photon with amplitude proportional to $j_1{}^\mu D_{\mu\nu}(k) j_2{}^\nu$. Show that ξ drops out and, taking k along z with $|k| = \kappa$, that the exchange splits into an instantaneous Coulomb term $i\, j_1{}^0 j_2{}^0/\kappa^2$ and a transverse term $i\, j_1{}^T \cdot j_2{}^T/(k^2 + i\varepsilon)$, where j^T is the part of the three-vector current perpendicular to k.

(d) COMPARE **[2]**

A massive vector field of mass M_V, the Proca field, has the propagator $-i[g_{\mu\nu} - k_\mu k_\nu/M_V^2]/(k^2 - M_V^2 + i\varepsilon)$. Compare its polarisations and propagator with the photon's, and explain from the little group of a massless particle why the photon must couple to a conserved current while the Proca field need not.

(e) SHOW THAT **[3]**

For Compton scattering on the charged scalar, $\gamma(k_1) + \varphi(p) \to \gamma(k_2) + \varphi(p')$, write $iM^{(\mu\nu)}$, with μ belonging to the incoming photon and ν to the outgoing one, as the sum of the s-channel, u-channel and seagull diagrams. Show that $k_{1\mu} M^{(\mu\nu)} = 0$ for the sum, but not for any one diagram or for the two exchange diagrams alone.

(f) CALCULATE **[2]**

In the rest frame of the initial scalar choose both photon polarisations with $\varepsilon^0 = 0$, so that $p\cdot\varepsilon = k\cdot\varepsilon = 0$ for each photon. For photon energies $\omega \ll m$, where $d\sigma/d\Omega = |M|^2/(64\pi^2 m^2)$, calculate the total cross-section for a point-like scalar with the charged pion's mass, in microbarns, averaging over initial and summing over final polarisations.

MARK SCHEME

University Quantum Field Theory I · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-02.01 · UQFTI-06.01 **[1]**

With $\hbar = c = 1$ the action is dimensionless, so in d spacetime dimensions the Lagrangian density has mass dimension d. The kinetic terms then give a real scalar field φ the mass dimension $(d - 2)/2$ and a Dirac field ψ the mass dimension $(d - 1)/2$. For which interaction is the coupling constant dimensionless, so that the theory is renormalisable by power counting, in $d = 6$?

KEY **C** **$(g/3!)\varphi^3$**

A is the renormalisable self-coupling of four dimensions: $[\lambda] = d - 4(d - 2)/2 = 4 - d$ vanishes there, but in six dimensions λ has mass dimension −2, the superficial degree of divergence grows with every vertex added, and φ^4 is non-renormalisable. B has $[g] = d - 6(d - 2)/2 = 6 - 2d$, which vanishes in three dimensions, where φ^6 is the renormalisable self-coupling; in six it is −6. D is the Yukawa coupling, with $[g] = d - (d - 2)/2 - (d - 1) = (4 - d)/2$: dimensionless in four dimensions, which is why Yukawa theory is renormalisable there, but of dimension −1 in six. C is correct: $[g] = d - 3(d - 2)/2 = (6 - d)/2$, which is +1 in four dimensions, where φ^3 is super-renormalisable, and 0 in six, where it is the renormalisable cubic theory.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-02.05 · UQFTI-02.06 **[1]**

The Gell-Mann–Low formula expresses a correlation function in the interacting vacuum $|\Omega\rangle$ through free fields: $\langle\Omega|T\varphi_H(x)\varphi_H(y)|\Omega\rangle = \langle 0|T[\varphi_I(x)\varphi_I(y)\exp(-i\int H_I\,dt)]|0\rangle / \langle 0|T[\exp(-i\int H_I\,dt)]|0\rangle$, where $|0\rangle$ is the free vacuum, φ_I and H_I are in the interaction picture, the time integrals run from −T to T, and the limit is taken as $T \to \infty(1 - i\varepsilon)$. What does taking the limit along the slightly tilted direction $T(1 - i\varepsilon)$ accomplish?

KEY **A** **It damps every eigenstate of the full Hamiltonian above the ground state by e^(−(E_n − E₀)εT) relative to |Ω⟩, so the long evolution projects |0⟩ onto |Ω⟩, provided ⟨Ω|0⟩ ≠ 0 and there is a gap above E₀.**

B confuses the tilt with adiabatic switching, and even adiabatic switching does not give unit overlap: the evolved state is $|\Omega\rangle$ multiplied by a phase and by $\langle\Omega|0\rangle$, and in infinite volume that overlap goes to zero, which is what Haag's theorem says about this derivation. The denominator, the sum of vacuum bubbles, removes exactly the phase and the overlap; it is present in a finite theory too and has nothing to do with ultraviolet divergences. C attaches the damping to the wrong operator: the factor e^(−iHT(1 − iε)) acts on the energy eigenstates of the full Hamiltonian, not on modes of H_I, and normal ordering H_I removes contractions within one vertex but cannot turn the free vacuum into the interacting one. D reverses the job of the iε: time-ordered contractions of interaction-picture fields are Feynman propagators, whose iε is inherited from this same tilt, and the retarded propagator belongs to the classical response to a source. A is correct: writing $|0\rangle = \Sigma_n|n\rangle\langle n|0\rangle$, the evolution weights each term by $e^{-iE_nT(1 - i\varepsilon)}$, so every state above E_0 dies away relative to $|\Omega\rangle$; the non-zero overlap and the gap are the two assumptions, and the phase and overlap left over cancel between numerator and denominator.

U1

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3 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-03.02 · UQFTI-03.03 **[1]**

In φ^4 theory with $\mathcal{L}_{int} = -(\lambda/4!)\varphi^4$, two connected vacuum diagrams appear at order λ^2. In diagram (i) two vertices are joined by four propagators. In diagram (ii) two vertices are joined by two propagators, and each vertex also carries one propagator that begins and ends on it. Each diagram contributes its spacetime integral times $(-i\lambda)^2/S$. What are the symmetry factors S of (i) and (ii)?

KEY **D** **48 and 16**

A counts, for (i), only the permutations of the four lines, 4! = 24, and for (ii) only the swap of the two connecting lines and the reversal of each self-loop, 2 × 2 × 2 = 8; both forget that the two vertices can also be exchanged: when a diagram is unchanged by that swap, the 1/2! of the second-order Dyson term is not cancelled by relabelling the vertices. B and C each make that omission in one diagram but not the other. D is correct. By contractions, the second-order term carries 1/(2! × 4! × 4!) = 1/1152; diagram (i) arises in 4! = 24 ways, giving 24/1152 = 1/48, and diagram (ii) in 6 × 6 × 2 = 72 ways, choosing two of the four fields at each vertex for its self-loop and then pairing the remaining two with two, giving 72/1152 = 1/16. Counting automorphisms agrees: 4! × 2 = 48 for (i) and 2 × 2 × 2 × 2 = 16 for (ii).

U2

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4 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-04.02 · UQFTI-04.03 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. A real scalar field φ, canonically normalised so that $[\varphi(x, t), \pi(y, t)] = i\delta^3(x - y)$, is interacting and invariant under $\varphi \to -\varphi$. Its lightest particle has physical mass m, and the field can create states of any odd number of these particles. Its exact two-point function has the Källén–Lehmann form $\int_0^\infty (d\mu^2/2\pi)\, \rho(\mu^2)\, i/(p^2 - \mu^2 + i\varepsilon)$, with $\rho(\mu^2) = 2\pi Z\delta(\mu^2 - m^2)$ plus a continuum. Which statement is correct?

KEY **B** **The continuum begins at $\mu^2 = 9m^2$, $\rho(\mu^2) \geq 0$, and the canonical commutator fixes the total weight $\int \rho\, d\mu^2/2\pi = 1$, so $0 \leq Z \leq 1$.**

A ignores the symmetry: under $\varphi \to -\varphi$ a two-particle state is even and φ is odd, so $\langle\Omega|\varphi(0)|$two particles$\rangle = 0$, and the lightest multiparticle state φ can create has three particles. C contradicts the derivation itself: $\rho(\mu^2) = \Sigma_\lambda\, 2\pi\delta(\mu^2 - m_\lambda^2)|\langle\Omega|\varphi(0)|\lambda_0\rangle|^2$ is a sum of squared moduli, non-negative whatever the sign of the interaction, so Z cannot exceed 1. D makes two errors. The threshold is the square of the summed masses, $(3m)^2 = 9m^2$, not the sum of the squares; and the canonical commutator fixes the sum $Z + \int \rho_{cont}\, d\mu^2/2\pi = 1$, not Z alone, so $Z = 1$ only when the continuum is empty, for a free field. B is correct: the threshold is $(3m)^2$, positivity holds term by term, and the sum rule with a non-negative continuum gives $0 \leq Z \leq 1$.

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-04.06 · UQFTI-04.07 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model, $\mathcal{L}_{int} = -g\psi^\dagger\psi\varphi$, with ψ a complex scalar of mass m_ψ and φ a real scalar of mass $m_\varphi < 2m_\psi$. To order g^2 the amplitude for $\psi(p_1)\bar{\psi}(p_2) \to \psi(p_3)\bar{\psi}(p_4)$ is $M = -g^2[1/(s - m_\varphi^2) + 1/(t - m_\varphi^2)]$, with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$, the first term from annihilation through φ and the second from φ exchange. Defining s, t and u in the same way for $\psi(p_1)\psi(p_2) \to \psi(p_3)\psi(p_4)$, what is the order-$g^2$ amplitude for that process?

KEY **D** **$-g^2[1/(t - m_\varphi^2) + 1/(u - m_\varphi^2)]$**

A copies the $\psi\bar{\psi}$ amplitude without crossing it. Crossing the incoming antiparticle into an outgoing particle and the outgoing antiparticle into an incoming one exchanges the roles of s and u, so the annihilation term becomes a u-channel exchange. C keeps an s-channel pole as well, but two ψ particles carry ψ number 2, which the vertex $\psi^\dagger\psi\varphi$ conserves, so they cannot annihilate into a single φ. B treats the identical particles as fermions; ψ is a scalar, so the diagrams with the final particles exchanged add, and the amplitude must be symmetric, not antisymmetric, under $t \leftrightarrow u$. D is correct: φ is exchanged in the t and u channels with the same sign, the amplitude is symmetric under $t \leftrightarrow u$ as Bose symmetry requires, and $s + t + u = 4m_\psi^2$ in both processes.

U2

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.01 · UQFTI-05.04 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In φ^4 theory with $\mathcal{L}_{int} = -(\lambda/4!)\varphi^4$ and $\lambda = 0.60$, two particles collide at a centre-of-mass energy $\sqrt{s} = 4.0$ GeV, far enough above threshold that their mass can be neglected. At tree level the amplitude for $\varphi\varphi \to \varphi\varphi$ is $M = -\lambda$, and for a $2 \to 2$ process with equal masses in and out, $d\sigma/d\Omega = |M|^2/(64\pi^2 s)$ in the centre-of-mass frame. Take $(\hbar c)^2 = 0.3894$ GeV^2 mb. What is the total cross-section for $\varphi\varphi \to \varphi\varphi$?

KEY **A** **87 nb**

B integrates $d\sigma/d\Omega$ over the full sphere without the factor 1/2 for two identical particles in the final state, so every final configuration is counted twice, once at θ and once at $\pi - \theta$. C is $d\sigma/d\Omega$ itself, 13.9 nb per steradian, never integrated over angles. D takes the vertex to be $-i\lambda/4!$, reading it straight off the Lagrangian; the 4! ways of attaching four external lines to the four identical fields at the vertex cancel the 1/4!, which is why the coupling is written with it, and leaving it in divides σ by 576. A is correct: $\sigma = \tfrac{1}{2} \times 4\pi \times \lambda^2/(64\pi^2 s) = \lambda^2/(32\pi s) = 0.36/(32\pi \times 16\ \text{GeV}^2) = 2.24 \times 10^{-4}\ \text{GeV}^{-2}$, and multiplying by $(\hbar c)^2 = 0.3894\ \text{GeV}^2\ \text{mb}$ gives 8.72×10^{-5} mb = 87 nb.

U3

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-07.02 · UQFTI-07.05 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. Two identical Dirac fermions of mass m scatter, $\psi(p_1)\psi(p_2) \to \psi(p_3)\psi(p_4)$, through the Yukawa interaction $\mathcal{L}_{int} = -g\varphi\bar{\psi}\psi$, where φ is a real scalar of mass m_φ. At tree level two diagrams contribute, one with momentum transfer squared $t = (p_1 - p_3)^2$ and one with $u = (p_1 - p_4)^2$. In the non-relativistic limit $\bar{u}(p_3)u(p_1) \to 2m$ times a Kronecker delta between the spin labels. Which statement is correct?

KEY **C** **The two diagrams have a relative minus sign, because their final states differ by an exchange of anticommuting operators; the t-channel term alone is the Born amplitude of an attractive Yukawa potential.**

A takes the sign from the wrong line: the relative sign is fixed by the external fermions, not by the exchanged boson, since the two Wick contractions pair the final-state operators with the vertex fields in orders that differ by one transposition of anticommuting operators. B mistakes an exchange sign for the sign of a force. The force is read from the t-channel term alone, which in the non-relativistic limit is the Born amplitude of $V(r) = -(g^2/4\pi r)e^{-m_\varphi r}$; the u-channel term is the exchange term that antisymmetrises the two-fermion state, and scalar exchange attracts every pair, fermion–fermion and fermion–antifermion alike. D confuses the amplitude with the phase-space integral: the u-channel contraction is a distinct term of the amplitude and must be included, while the 1/2 for identical particles belongs to the integral over final states in the total cross-section. C is correct: M is antisymmetric under exchange of the two final fermions, and each term on its own describes an attractive Yukawa force.

U1

8

CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-08.02 · UQFTI-08.03 **[1]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In Feynman gauge the four photon operators of one momentum k, taken along z, obey $[a^\lambda(k), a^{\lambda'\dagger}(k')] = -g^{(\lambda\lambda')}(2\pi)^3 2|k|\delta^3(k - k')$ for $\lambda, \lambda' = 0, 1, 2, 3$, where $\lambda = 0$ labels the timelike polarisation $\varepsilon^0 = (1, 0, 0, 0)$, $\lambda = 1, 2$ the transverse ones and $\lambda = 3$ the longitudinal one, $\varepsilon^3 = (0, 0, 0, 1)$. The Gupta–Bleuler condition requires the positive-frequency part of $\partial_\mu A^\mu$ to annihilate every physical state, which for this momentum reads $(a^0 - a^3)|\psi\rangle = 0$. Which statement is correct?

KEY **B** **Physical states may contain $(a^{0\dagger} - a^{3\dagger})$ acting on a transverse state; such a state has zero norm and adds nothing to the energy or to any gauge-invariant expectation value.**

A overstates the condition: it does not remove timelike and longitudinal photons but admits them in the combination created by $a^{0\dagger} - a^{3\dagger}$, which commutes with $a^0 - a^3$ because $[a^0, a^{0\dagger}] = -1$ and $[a^3, a^{3\dagger}] = +1$ cancel. C has the sign of the longitudinal norm wrong: $-g^{(33)} = +1$, so $a^{3\dagger}|0\rangle$ has positive norm, and the zero norm of the admitted combination comes from a positive norm cancelling the timelike photon's negative one. D asks the creation part, proportional to $a^{0\dagger} - a^{3\dagger}$, to annihilate the state as well; no non-zero state of the Fock space, the vacuum included, satisfies that, which is why Gupta and Bleuler kept only the positive-frequency half, enough to make $\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0$. B is correct: $(a^{0\dagger} - a^{3\dagger})|T\rangle$ with $|T\rangle$ transverse satisfies the condition, has norm $\langle T|(a^0 - a^3)(a^{0\dagger} - a^{3\dagger})|T\rangle = 0$, is orthogonal to every physical state, and drops out of $\langle H\rangle$ because $a^{3\dagger}a^3 - a^{0\dagger}a^0$ has zero expectation value whenever $(a^0 - a^3)|\psi\rangle = 0$.

U1

9

CORE COURSE

STRUCTURED · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07 · UQFTI-01.09 **[8]**

A free real scalar field of mass m is written, in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$, as $\varphi(x) = \int d^3p/((2\pi)^3 2E_p)\,[a(p)e^{(-ip\cdot x)} + a^\dagger(p)e^{(ip\cdot x)}]$, where $E_p = \sqrt{(|p|^2 + m^2)}$, $p^0 = E_p$ in the exponents and $[a(p), a^\dagger(q)] = (2\pi)^3 2E_p\,\delta^3(p - q)$. Its vacuum two-point function is $D(x - y) = \langle 0|\varphi(x)\varphi(y)|0\rangle = \int d^3p/((2\pi)^3 2E_p)\, e^{(-ip\cdot(x - y))}$. Take $\hbar c = 197.327$ MeV fm.

(a)

DERIVE · U2 **[3]**

Starting from the definition $D_F(x - y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$, derive the representation $D_F(x - y) = \int d^4p/(2\pi)^4\, i\, e^{(-ip\cdot(x - y))}/(p^2 - m^2 + i\varepsilon)$, evaluating the p^0 integral with the contour closed in whichever half-plane each time ordering requires.

1. $p^2 - m^2 + i\varepsilon = (p^0)^2 - E_p^2 + i\varepsilon$ vanishes at $p^0 = \pm(E_p - i\varepsilon')$, with $\varepsilon' = \varepsilon/(2E_p)$: the positive-energy pole lies just below the real axis and the negative-energy pole just above it
2. for $x^0 > y^0$ the factor $e^{(-ip^0(x^0 - y^0))}$ decays in the lower half-plane, so the contour is closed below, clockwise, enclosing $p^0 = E_p$: $\int dp^0/(2\pi)\, i\, e^{(-ip^0(x^0 - y^0))}/[(p^0 - E_p)(p^0 + E_p)] = (-2\pi i/2\pi) \times i\, e^{(-iE_p(x^0 - y^0))}/(2E_p) = e^{(-iE_p(x^0 - y^0))}/(2E_p)$, and the remaining d^3p integral is $D(x - y)$
3. for $x^0 < y^0$ the contour is closed above, enclosing $p^0 = -E_p$, which gives $e^{(iE_p(x^0 - y^0))}/(2E_p)$; relabelling $p \to -p$ turns the d^3p integral into $D(y - x)$, so the integral equals $\theta(x^0 - y^0)D(x - y) + \theta(y^0 - x^0)D(y - x) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$

   The usual slip is to close the contour in the wrong half-plane for a given time order, which yields $D(y - x)$ for $x^0 > y^0$, an anti-time-ordered product. The sign of $x^0 - y^0$ decides where the exponential decays, and nothing else decides the half-plane.

(b) SKETCH · U1 **[2]**

The retarded propagator is $D_R(x-y) = \theta(x^0-y^0)\langle 0|[\varphi(x),\varphi(y)]|0\rangle$, whose momentum-space form is $i/((p^0+i\varepsilon)^2-|\mathbf{p}|^2-m^2)$. Sketch the complex p^0-plane at fixed $\mathbf{p}$, marking the poles of D_F and of D_R and the contour used when $x^0<y^0$, and state where in spacetime each propagator is non-zero.

1. the sketch shows the real p^0 axis with the poles of D_F at $+E_p-i\varepsilon'$, below, and $-E_p+i\varepsilon'$, above; the poles of D_R both below the axis, at $\pm E_p-i\varepsilon'$; and the contour for $x^0<y^0$ closed by a large semicircle in the upper half-plane, enclosing the negative-energy pole of D_F and neither pole of D_R
2. D_R is therefore zero for $x^0<y^0$ and, being a commutator, also zero at spacelike separation, so it is supported on and inside the forward light cone; D_F is non-zero for both time orders and at spacelike separation, where it falls off as e^{-mr}, r being the invariant spacelike distance

(c) CALCULATE · U3 **[2]**

Two static point sources, $j(x)=g[\delta^3(\mathbf{x}-\mathbf{x}_1)+\delta^3(\mathbf{x}-\mathbf{x}_2)]$, coupled through $\mathcal{L}_{\text{int}}=j\varphi$ and held a distance $r=|\mathbf{x}_1-\mathbf{x}_2|$ apart for a long time T, give a vacuum persistence amplitude $\langle 0|0\rangle \propto e^{-iE(r)T}$, with interaction energy $E(r)=-(g^2/(4\pi r))e^{-mr}$. For a scalar of mass $m=550$ MeV, $g^2/(4\pi)=0.80$ and $r=0.70$ fm, calculate $E(r)$ in MeV and state the range of the force in femtometres.

1. in natural units $r=0.70\ \text{fm}/(197.327\ \text{MeV fm})=3.547\times10^{-3}\ \text{MeV}^{-1}$, so $1/r=281.9$ MeV, and $mr=550\times0.70/197.327=1.951$, with $e^{-1.951}=0.1421$
2. $E=-0.80\times281.9\ \text{MeV}\times0.1421=-32.1$ MeV, negative because scalar exchange between like sources attracts; the range is $1/m=197.327\ \text{MeV fm}/550\ \text{MeV}=0.359$ fm

(d) EXPLAIN · U1 **[1]**

$D_F(x-y)$ is non-zero when $x-y$ is spacelike. Explain why this does not allow a signal to travel faster than light.

1. whether a measurement at y can affect one at x is decided by the commutator $[\varphi(x),\varphi(y)]=D(x-y)-D(y-x)$, not by an amplitude; for spacelike $x-y$ a proper Lorentz transformation carries $x-y$ into $-(x-y)$ and leaves D invariant, so the commutator vanishes, the amplitude for propagation from y to x cancelling that from x to y, while D_F, an amplitude, need not vanish

   Answers that call D_F 'exponentially small' outside the light cone miss the point: a small non-zero commutator would still be a violation. Causality lives in the commutator, and the retarded propagator built from it is the one that describes the response to a source.

10

CORE COURSE

STRUCTURED · DEMANDING · UQFTI-02.01 · UQFTI-02.09 · UQFTI-03.02 · UQFTI-03.07 · UQFTI-04.05 · UQFTI-04.07 · UQFTI-04.09 · UQFTI-05.03 **[10]**

Two real scalar fields, φ of mass m_φ and χ of mass m_χ, live in four spacetime dimensions and interact through $\mathcal{L}_{\text{int}}=-(g/2)\varphi\chi^2-(\kappa/2)\varphi\partial_\mu\chi\partial^\mu\chi$, in natural units $\hbar=c=1$ with the metric $(+,-,-,-)$. Treat both couplings to the lowest order at which each process occurs. For the two-body decay of a particle of mass m_φ into particles of momentum p in its rest frame, Lorentz-invariant phase space gives $\int d\Pi_2 = p/(4\pi m_\varphi)$ when the two final particles are distinguishable, and $\Gamma=(1/(2m_\varphi))\int|\mathcal{M}|^2 d\Pi_2$. Take $\hbar=6.582\times10^{-25}$ GeV s.

(a) DETERMINE · U2 **[2]**

Determine the Feynman rule for the $\varphi\chi\chi$ vertex, taking every momentum as incoming and p_2 and p_3 as the momenta of the two χ lines, and give the mass dimensions of g and κ.

1. the term $-(g/2)\varphi\chi^2$ gives $i \times (-g/2) \times 2 = -ig$, the 2 counting the ways the two χ fields meet the two χ lines; $[g] = 4 - 3 = 1$, a super-renormalisable coupling
2. each ∂_μ acting on an incoming χ line of momentum p gives $-ip_\mu$, so $-(\kappa/2)\varphi\partial_\mu\chi\partial\hat{}\mu\chi$ gives $i \times (-\kappa/2) \times 2 \times (-ip_2)\cdot(-ip_3) = i\kappa\, p_2\cdot p_3$; the full vertex is $-ig + i\kappa\, p_2\cdot p_3$, with $[\kappa] = 4 - 1 - 2 - 2 = -1$, a non-renormalisable coupling

   Reading the vertex straight off the coefficient gives $-ig/2$ and loses the factor 2 from the identical χ fields, in the derivative term as well. Candidates who attach a momentum to only one χ line miss that the derivative vertex depends on both, and on the product of the two.

(b) SHOW THAT · U2 **[2]**

Set $\kappa = 0$. By counting the Wick contractions in the order-g^2 term of the Gell-Mann–Low numerator, show that the one-loop φ self-energy, a loop of two χ propagators, carries a symmetry factor of 2, and that the one-loop χ self-energy, a loop of one φ and one χ propagator, carries a symmetry factor of 1. Check each by counting the diagram's automorphisms.

1. the order-g^2 term is $(1/2!)(-ig/2)^2\int\int\varphi(z)\chi(z)^2\varphi(w)\chi(w)^2$, a prefactor $(-ig)^2/8$; in $\langle 0|T\varphi(x)\varphi(y)\ldots|0\rangle$, $\varphi(x)$ joins $\varphi(z)$ or $\varphi(w)$ (2 ways), $\varphi(y)$ the other, and the two $\chi(z)$ fields pair with the two $\chi(w)$ fields in 2 ways, so 4 contractions give $(-ig)^2 \times 4/8 = (-ig)^2/2$: $S = 2$, matching the one automorphism that swaps the two χ lines
2. in $\langle 0|T\chi(x)\chi(y)\ldots|0\rangle$, $\chi(x)$ joins any of the four χ fields (4 ways), $\chi(y)$ either χ field at the other vertex (2 ways), and the remaining χ and φ fields pair uniquely, so 8 contractions give $(-ig)^2 \times 8/8$: $S = 1$, as the loop's two lines, one φ and one χ, cannot be exchanged

   Candidates who carry over from φ^4 the habit of a 1/2 for every bubble put it on the χ self-energy too. A loop of two lines of different fields has no symmetry, and the direct count shows the 1/2! and both 1/2s cancelled completely.

(c) DETERMINE · U2 **[2]**

Still with $\kappa = 0$, determine the tree-level amplitude M for $\chi(p_1)\chi(p_2) \to \chi(p_3)\chi(p_4)$ in terms of the Mandelstam variables s, t and u, check its symmetry under $t \leftrightarrow u$ and the value of $s + t + u$, and say why $\sqrt{Z} = 1$ may be used on each external leg at this order.

1. φ is exchanged in all three channels, each giving $(-ig)^2 \times i/(q^2 - m_\varphi^2)$, so $M = -g^2[1/(s - m_\varphi^2) + 1/(t - m_\varphi^2) + 1/(u - m_\varphi^2)]$, dimensionless as a $2 \to 2$ amplitude in four dimensions must be, since $[g^2] = 2$
2. M is symmetric under $t \leftrightarrow u$, as identical bosons in the final state require, and $s + t + u = 4m_\chi^2$; because $Z = 1 + O(g^2)$ and the mass shift is $O(g^2)$, $\sqrt{Z}$ on each leg and the physical mass change M only at order g^4

   The commonest error is to drop the s-channel by analogy with $\psi\psi \to \psi\psi$ in the scalar Yukawa model, where ψ number forbids it. Here χ carries only the charge of the symmetry $\chi \to -\chi$, which two χ particles do not carry, so they can annihilate into one φ.

(d) PREDICT · U1 **[2]**

The masses are now chosen so that $m_\varphi > 2m_\chi$. Predict what becomes of the s-channel term of the amplitude in (c) as s approaches m_φ^2, and of the φ particle's place in the LSZ formula, giving the reason, and say whether the t- and u-channel poles are affected.

1. the pole at $s = m_\varphi^2$ now lies inside the physical region $s \geq 4m_\chi^2$, so the tree amplitude diverges there; the φ self-energy acquires an imaginary part from the open χχ channel, and resumming it turns $1/(s - m_\varphi^2)$ into $1/(s - m_\varphi^2 + i m_\varphi \Gamma)$, a Breit–Wigner peak of finite height
2. φ is no longer stable, so it is not an asymptotic state: its pole leaves the real axis, LSZ's isolated one-particle pole is lost and φ appears only as a resonance in χχ scattering; the t- and u-channel poles at $t = m_\varphi^2$ and $u = m_\varphi^2$ stay outside the physical region, where $t \leq 0$ and $u \leq 0$, and still describe a force of range $1/m_\varphi$

Answers that keep φ as an external state of LSZ with a real mass have missed the assumption the reduction formula states: stable particles. A pole inside the physical region is the signature of a particle that can be produced on shell, and therefore decays.

(e) CALCULATE · U3 **[2]**

Restore κ. With $m_\varphi = 0.90$ GeV, $m_\chi = 0.14$ GeV, $g = 0.50$ GeV and $\kappa = 0.80$ GeV^{-1}, calculate the decay rate $\Gamma(\varphi \to \chi\chi)$ in MeV and the lifetime of φ in seconds.

1. on shell $2p_2\cdot p_3 = m_\varphi^2 - 2m_\chi^2$, so $M = -g + \kappa(m_\varphi^2 - 2m_\chi^2)/2 = -0.500$ GeV $+$ 0.308 GeV $= -0.192$ GeV, and the χ momentum is $p = \sqrt{(m_\varphi^2/4 - m_\chi^2)} = 0.428$ GeV
2. with 1/2 for two identical χ, $\Gamma = (1/(2m_\varphi)) \times |M|^2 \times \tfrac{1}{2} \times p/(4\pi m_\varphi) = |M|^2 p/(16\pi m_\varphi^2)$ $= 3.86 \times 10^{-4}$ GeV $= 0.386$ MeV, and $\tau = \hbar/\Gamma = 1.71 \times 10^{-21}$ s

Two slips are set for here. Omitting the 1/2 for identical final particles doubles Γ to 0.772 MeV. Giving the derivative vertex the wrong sign makes the two couplings reinforce, Γ = 6.86 MeV, eighteen times too large: the sign fixed in (a) decides whether they cancel or add.

11

CORE COURSE

STRUCTURED · DEMANDING · UQFTI-07.02 · UQFTI-07.03 · UQFTI-07.04 · UQFTI-07.06 · UQFTI-07.07 · UQFTI-07.08 **[10]**

A real spin-0 boson φ of mass m_φ couples to a Dirac fermion ψ of mass m either as a scalar, $\mathcal{L}_S = -g\varphi\bar\psi\psi$, or as a pseudoscalar, $\mathcal{L}_P = -ig\varphi\bar\psi\gamma^5\psi$, in natural units $\hbar = c = 1$ with the metric $(+,-,-,-)$. Use the spin sums $\Sigma u(k)\bar u(k) = \not k + m$ and $\Sigma v(k)\bar v(k) = \not k - m$; the identities $\mathrm{tr}\,1 = 4$ and $\mathrm{tr}(\not a \not b) = 4a\cdot b$; the vanishing of the trace of an odd number of γ matrices; $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, $(\gamma^5)^2 = 1$ and $\gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$. For the decay $\varphi \to \psi(k_1)\bar\psi(k_2)$ write $\beta = \sqrt{(1 - 4m^2/m_\varphi^2)}$, the speed of either fermion in the φ rest frame; then $\int d\Pi_2 = \beta/(8\pi)$, and $\Gamma = (1/(2m_\varphi)) \times$ (spin-summed $|M|^2$) $\times \int d\Pi_2$. Take $\hbar = 6.582 \times 10^{-25}$ GeV s.

(a) DERIVE · U2 **[2]**

For the scalar coupling the amplitude is $M_S = -g\,\bar u(k_1)v(k_2)$. Using Casimir's trick with the spin sums, derive the spin-summed $|M_S|^2$ in terms of g, m_φ and β.

1. $(\bar u\, v)^* = \bar v\, u$, so $\Sigma|M_S|^2 = g^2\,\Sigma\,\bar u(k_1)v(k_2)\bar v(k_2)u(k_1) = g^2\,\mathrm{tr}[(\not k_1 + m)(\not k_2 - m)] = g^2(4k_1\cdot k_2 - 4m^2)$, the terms linear in m containing an odd number of γ matrices
2. on shell $2k_1\cdot k_2 = m_\varphi^2 - 2m^2$, so $\Sigma|M_S|^2 = 2g^2(m_\varphi^2 - 4m^2) = 2g^2 m_\varphi^2\beta^2$

Candidates who use $\Sigma v\bar v = \not k + m$ lose the sign of the mass term and obtain $2g^2m_\varphi^2$, which is the pseudoscalar result, for the scalar; the β^2 is the whole physics of the part.

(b) SHOW THAT · U2 **[2]**

For the pseudoscalar coupling the amplitude is $M_P = -g\,\bar u(k_1)i\gamma^5 v(k_2)$. Show that the spin-summed $|M_P|^2$ is $2g^2m_\varphi^2$, with no factor of β^2.

1. $(\bar u\, i\gamma^5\, v)^* = \bar v\,\gamma^0(i\gamma^5)^\dagger\gamma^0\,u = \bar v(-i)(-\gamma^5)u = \bar v\, i\gamma^5\, u$, so $\Sigma|M_P|^2 = g^2\,\mathrm{tr}[(\not k_1 + m)i\gamma^5(\not k_2 - m)i\gamma^5] = -g^2\,\mathrm{tr}[(\not k_1 + m)\gamma^5(\not k_2 - m)\gamma^5]$
2. anticommuting γ^5 through, $\gamma^5(\not k_2 - m)\gamma^5 = -\not k_2 - m$, so $\Sigma|M_P|^2 = g^2\,\mathrm{tr}[(\not k_1 + m)(\not k_2 + m)] = g^2(4k_1\cdot k_2 + 4m^2) = 2g^2m_\varphi^2$: the γ^5 reverses the sign of the mass term

The error to catch is forgetting that $\gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$ when forming the conjugate amplitude. It flips the overall sign and gives $-2g^2m_\varphi^2$, which a candidate should recognise at once as impossible for a sum of squared moduli.

(c) CALCULATE · U3 **[2]**

For $m_\varphi = 1.50$ GeV, $m = 0.60$ GeV and $g = 0.30$, calculate $\Gamma(\varphi \to \psi\bar{\psi})$ and the φ lifetime in seconds for each coupling.

1. $\beta = \sqrt{(1 - 4 \times 0.36/2.25)} = 0.600$, and $\Gamma_S = (1/(2m_\varphi)) \times 2g^2 m_\varphi^2 \beta^2 \times \beta/(8\pi) = g^2 m_\varphi \beta^3/(8\pi) = 1.16 \times 10^{-3}$ GeV $= 1.16$ MeV, so $\tau_S = \hbar/\Gamma_S = 5.67 \times 10^{-22}$ s
2. $\Gamma_P = g^2 m_\varphi \beta/(8\pi) = 3.22$ MeV and $\tau_P = 2.04 \times 10^{-22}$ s; the ratio $\Gamma_P/\Gamma_S = 1/\beta^2 = 2.78$

   Candidates who include a factor 1/2 for identical particles halve both rates; ψ and $\bar{\psi}$ are distinct particles, so no such factor applies.

(d) EXPLAIN · U1 **[2]**

Explain the powers β^3 and β in the two rates by the parity of φ and the orbital angular momentum of the fermion pair near threshold.

1. a fermion and its antifermion have opposite intrinsic parities, so a pair with orbital angular momentum L has parity $(-1)^{(L+1)}$; total J = 0 requires the pair's spin S to equal L, so S and L are both 0 or both 1
2. the scalar coupling makes φ a 0^+ state, which needs L odd, the ${}^3P_0$ pair with L = S = 1, and a P-wave rate carries $p^{(2L+1)} \propto \beta^3$; the pseudoscalar is 0^-, which allows the ${}^1S_0$ pair with L = 0 and a rate $\propto \beta$, so near threshold the pseudoscalar decays faster by $1/\beta^2$

   A frequent answer attributes β^3 to 'suppression by the fermion mass' without saying why only the scalar is suppressed. The difference is angular momentum: parity forbids the scalar from producing the pair in an S-wave.

(e) JUSTIFY · U1 **[2]**

In the one-loop φ self-energy built from a ψ loop, Wick's theorem supplies a factor -1. Justify this factor from the anticommutation of the fields, and confirm it from the optical theorem, which requires $\mathrm{Im}\, M(\varphi \to \varphi) = m_\varphi \Gamma(\varphi \to \psi\bar{\psi})$ once $m_\varphi > 2m$.

1. the two vertices supply $\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)$; writing the contractions in the order of a trace, $\psi(x)\bar{\psi}(y)\psi(y)\bar{\psi}(x) \to \mathrm{tr}[S_F(x - y)S_F(y - x)]$, means moving $\bar{\psi}(x)$ from the front to the back past three anticommuting fields, which gives $(-1)^3 = -1$
2. on the cut the fermion carries k_1 and the antifermion line carries $-k_2$, so the numerator trace is $\mathrm{tr}[(\not{k}_1 + m)(-\not{k}_2 + m)] = -\mathrm{tr}[(\not{k}_1 + m)(\not{k}_2 - m)] = -\Sigma|M_S|^2/g^2$; only the loop's -1 makes $2\,\mathrm{Im}\, M(\varphi \to \varphi) = \int d\Pi_2 \Sigma|M_S|^2 = 2m_\varphi \Gamma$ positive, as unitarity demands

   Candidates often treat the -1 as a convention that could be absorbed into g^2. It cannot: g^2 is positive in both places, and unitarity fixes the sign of the imaginary part. Commuting fields would give a negative width, a propagator growing in time: the perturbative face of the spin–statistics theorem.

12

CORE COURSE

STRUCTURED · DISCRIMINATING · UQFTI-06.02 · UQFTI-06.03 · UQFTI-06.04 · UQFTI-06.06 · UQFTI-06.07 · UQFTI-06.08 · UQFTI-06.09 **[11]**

Real scalar φ^4 theory, $\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^{\wedge}\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - (\lambda/4!)\varphi^4$, in natural units $\hbar = c = 1$ with the metric (+, −, −, −), is regularised dimensionally in $d = 4 - 2\varepsilon$ dimensions, with λ replaced by $\lambda\mu^{(2\varepsilon)}$ so that λ stays dimensionless, μ being an arbitrary scale. Use the Feynman-parameter identity $1/(AB) = \int_0^1 dx\,[xA + (1 - x)B]^{(-2)}$, the Euclidean master integral $\int d^{\wedge}d\ell_E/(2\pi)^{\wedge}d\ (\ell_E^2 + \Delta)^{(-2)} = \Gamma(2 - d/2)(4\pi)^{(-d/2)}\Delta^{(d/2 - 2)}$, and $\Gamma(\varepsilon) = 1/\varepsilon - \gamma_E + O(\varepsilon)$. At one loop the field-strength counterterm vanishes, so the bare coupling is $\lambda_0 = \mu^{(2\varepsilon)}(\lambda + \delta_\lambda)$, where the counterterm vertex is $-i\delta_\lambda$.

(a) SHOW THAT · U2 **[3]**

The s-channel one-loop diagram for φφ → φφ, with incoming momenta p_1 and p_2, $p = p_1 + p_2$ and $s = p^2$, carries a symmetry factor of 2. Show that, below the threshold $s = 4m^2$, its contribution to iM, with an overall factor μ^(2ε) removed, is $(i\lambda^2/(32\pi^2))\int_0^1 dx\,[1/\varepsilon - \gamma_E + \ln 4\pi - \ln(\Delta_s/\mu^2)] + O(\varepsilon)$, where $\Delta_s = m^2 - x(1-x)s$.

1. two vertices, each −iλμ^(2ε), two propagators and the factor ½ give $iM_s = \tfrac{1}{2}(-i\lambda\mu^{2\varepsilon})^2\int d^d\ell/(2\pi)^d\,[i/(\ell^2 - m^2)][i/((\ell+p)^2 - m^2)] = \tfrac{1}{2}\lambda^2\mu^{4\varepsilon}\int d^d\ell/(2\pi)^d\,1/[(\ell^2 - m^2)((\ell+p)^2 - m^2)]$
2. Feynman parameters turn the denominator into $[(\ell + xp)^2 - \Delta_s]^2$ with $\Delta_s = m^2 - x(1-x)s$; shifting $\ell \to \ell - xp$ and rotating $\ell^0 = i\ell_E^0$, so that $\ell^2 = -\ell_E^2$ and $d^d\ell = i\,d^d\ell_E$, gives the loop integral as $i\int_0^1 dx\,\Gamma(\varepsilon)(4\pi)^{-2+\varepsilon}\Delta_s^{-\varepsilon}$
3. expanding, $(4\pi)^{-2+\varepsilon} = (1 + \varepsilon\ln 4\pi)/(16\pi^2)$ and $\mu^{2\varepsilon}\Delta_s^{-\varepsilon} = 1 - \varepsilon\ln(\Delta_s/\mu^2)$, so $iM_s = \mu^{2\varepsilon}(i\lambda^2/(32\pi^2))\int_0^1 dx\,[1/\varepsilon - \gamma_E + \ln 4\pi - \ln(\Delta_s/\mu^2)] + O(\varepsilon)$: one $\mu^{2\varepsilon}$ belongs with the tree-level coupling and the other makes the logarithm dimensionless

   The factor i from the Wick rotation is the one most often lost, and with it the relative phase between the loop and the tree amplitude −iλ, so the result can no longer be combined with the counterterm. Candidates who expand Γ(ε) but not Δ^(−ε) are left with the logarithm of a dimensionful quantity, which the μ^(2ε) is there to prevent.

(b) SHOW THAT · U2 **[2]**

Including the t- and u-channel diagrams and the counterterm, the renormalised coupling λ is defined by the condition $M = -\lambda$ at $s = 4m^2$, $t = u = 0$. Show that $M(s,t,u) = -\lambda - (\lambda^2/(32\pi^2))\Sigma\int_0^1 dx\,\ln[\Delta(q^2)/\Delta(q_0^2)]$, the sum running over the three channels $q^2 = s, t, u$ with reference values $q_0^2 = 4m^2$, 0, 0 and $\Delta(q^2) = m^2 - x(1-x)q^2$, and that neither ε nor μ survives in it.

1. at one loop $M = -\lambda + (\lambda^2/(32\pi^2))\Sigma\int dx[1/\varepsilon - \gamma_E + \ln 4\pi - \ln(\Delta(q^2)/\mu^2)] - \delta_\lambda$, and the condition at the reference point fixes $\delta_\lambda = (\lambda^2/(32\pi^2))\Sigma\int dx[1/\varepsilon - \gamma_E + \ln 4\pi - \ln(\Delta(q_0^2)/\mu^2)]$, which carries the whole pole
2. subtracting, 1/ε, γ_E, $\ln 4\pi$ and μ^2 cancel channel by channel, leaving $M = -\lambda - (\lambda^2/(32\pi^2))\Sigma\int dx\,\ln[\Delta(q^2)/\Delta(q_0^2)]$: finite, independent of the regulator and of μ, and a relation between the amplitude at (s, t, u) and the coupling measured at the reference point

   Candidates who fix δ_λ by minimal subtraction and then present the amplitude as a prediction have left $\ln\mu^2$ in it. It becomes a prediction only when λ is tied to a measurement at a stated point, or when the μ-dependence of λ is itself accounted for, which is what (c) does.

(c) DERIVE · U2 **[2]**

In minimal subtraction δ_λ contains only the poles of the three channels. Derive the one-loop beta function $\beta(\lambda) = \mu\,d\lambda/d\mu$ from the μ-independence of the bare coupling λ_0.

1. each channel contributes a pole $\lambda^2/(32\pi^2\varepsilon)$, so $\delta_\lambda = 3\lambda^2/(32\pi^2\varepsilon)$, and $\mu\,d\lambda_0/d\mu = 0$ gives $0 = 2\varepsilon(\lambda + 3\lambda^2/(32\pi^2\varepsilon)) + \beta(1 + 6\lambda/(32\pi^2\varepsilon))$
2. solving to order λ^2, $\beta = -2\varepsilon\lambda - 6\lambda^2/(32\pi^2) + 12\lambda^2/(32\pi^2) = -2\varepsilon\lambda + 3\lambda^2/(16\pi^2)$, so as $\varepsilon \to 0$, $\beta(\lambda) = 3\lambda^2/(16\pi^2)$, positive: the coupling grows with the scale

   The $-2\varepsilon\lambda$ term is not optional. Candidates who drop it before solving lose the cross term that turns $-6\lambda^2/(32\pi^2)$ into $+6\lambda^2/(32\pi^2)$, and obtain a beta function of the wrong sign: an asymptotically free φ^4 theory.

(d) CALCULATE · U3 **[2]**

In a model whose coupling is $\lambda(\mu_0) = 1.5$ at $\mu_0 = 1.0$ TeV, calculate the scale at which the one-loop running coupling has doubled and the scale of its Landau pole, both in GeV.

1. integrating $d\lambda/\lambda^2 = (3/(16\pi^2))d \ln \mu$ gives $1/\lambda(\mu) = 1/\lambda(\mu_0) - (3/(16\pi^2))\ln(\mu/\mu_0)$; $\lambda = 3.0$ when $\ln(\mu/\mu_0) = (16\pi^2/3)(1/1.5 - 1/3.0) = 17.55$, that is at $\mu = 1000$ GeV $\times$ e^17.55 = 4.2×10^{10} GeV
2. $1/\lambda$ vanishes when $\ln(\mu_L/\mu_0) = 16\pi^2/(3 \times 1.5) = 35.09$, so the Landau pole is at $\mu_L = 1000$ GeV $\times$ e^35.09 = 1.7×10^{18} GeV

   Candidates who integrate $d\lambda/d \ln \mu = 3\lambda^2/(16\pi^2)$ as if one of the two factors of λ on the right were fixed at λ_0 obtain a power law, $\lambda \propto \mu^{(3\lambda_0/(16\pi^2))}$, which grows for ever and misses the pole. Using $\log_{10}$ for the logarithm moves both scales by many orders of magnitude.

(e) ESTIMATE · U4 **[2]**

Estimate the highest scale at which this one-loop running can be trusted, stating the criterion you use, and say what the Landau pole of (d) then tells you about the theory.

1. a stated criterion, for example $\lambda(\mu) \leq 4\pi$, where the one-loop term $3\lambda/(16\pi^2)$ has grown to about 0.24: $\ln(\mu/\mu_0) = (16\pi^2/3)(1/1.5 - 1/(4\pi)) = 30.9$, so $\mu \approx 3 \times 10^{16}$ GeV; whatever the criterion $\lambda \leq \lambda_{max}$, the trusted range stops short of the pole by $\ln(\mu_L/\mu) = 16\pi^2/(3\lambda_{max})$, which is $4\pi/3 = 4.19$ for $\lambda_{max} = 4\pi$, a factor e^(4.19) ≈ 66 in scale and small beside the 15 orders of magnitude between μ_0 and μ_L
2. the pole lies beyond the scale at which the formula that predicts it holds, so one-loop running does not establish a singularity; it signals strong coupling, and with the non-perturbative evidence for triviality it means φ^4 makes sense only as an effective theory with a cutoff below the pole

   The judgement tested is that the Landau pole marks where the approximation fails rather than a computed property of the theory. A candidate who quotes μ_L as a physical scale, or trusts the one-loop formula up to $\lambda \to \infty$, has not asked where the expansion parameter stops being small.

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CORE COURSE

DATA ANALYSIS · DEMANDING · UQFTI-01.01 · UQFTI-05.02 · UQFTI-05.03 · UQFTI-05.07 · UQFTI-05.09 **[11]**

Two distinct spinless particles, a of mass 0.400 GeV and b of mass 0.600 GeV, scatter elastically, a + b → a + b, and near a centre-of-mass energy of 1.85 GeV the cross-section is dominated by an s-channel resonance X of mass m_X and total width Γ. Parts (a) to (d) treat X as a spin-0 particle coupled to ab through $\mathcal{L}_{int} = -gXab$, which gives $\Gamma(X \to ab) = g^2 p/(8\pi m_X^2)$, where p is the momentum of a or b in the centre-of-mass frame at $\sqrt{s} = m_X$; near the peak the amplitude is $M = -g^2/(s - m_X^2 + i m_X\Gamma)$. For two particles in and two out with these masses, $\sigma = |M|^2/(16\pi s)$. To the accuracy of these data the line shape is $\sigma(\sqrt{s}) = \sigma_{peak}(\Gamma^2/4)/[(\sqrt{s} - m_X)^2 + \Gamma^2/4]$. The optical theorem reads Im M(forward) = $2\sqrt{s}\, p\, \sigma_{tot}$. Work in natural units with the metric (+, −, −, −), and take $(\hbar c)^2 = 0.3894$ GeV² mb and $\hbar = 6.582 \times 10^{-25}$ GeV s. The table gives the measured elastic cross-section at seven energies.

(a) SHOW THAT · U2 **[2]**

Show that $\sigma_{peak} = (4\pi/p^2)B^2$, where $B = \Gamma(X \to ab)/\Gamma$ is the branching fraction to ab, so that $4\pi/p^2$ is the largest cross-section an s-wave resonance can give in this channel.

1. at $s = m_X^2$, $|M|^2 = g^4/(m_X^2\Gamma^2)$, so $\sigma_{peak} = g^4/(16\pi m_X^4\Gamma^2)$
2. with $g^2 = 8\pi m_X^2\Gamma(X \to ab)/p$, $g^4 = 64\pi^2 m_X^4\Gamma(X \to ab)^2/p^2$, so $\sigma_{peak} = (4\pi/p^2)[\Gamma(X \to ab)/\Gamma]^2 = (4\pi/p^2)B^2$, which reaches $4\pi/p^2$ only for a purely elastic resonance, B = 1

   Candidates who do not set $s = m_X^2$ at the peak keep a stray m_X^2/s, harmless here but a sign that the 1/s of the flux has not been followed. The point to see is that the bound arises because the same coupling sets both the width and the amplitude.

(b) DETERMINE · U4 **[3]**

By plotting a suitable linear function of the data, determine m_X, Γ and σ_{peak}.

1. the readings are symmetric about 1.850 GeV within their uncertainties, so $m_X = 1.850$ GeV, and the line shape gives $1/\sigma = (1/\sigma_{peak})[1 + 4(\sqrt{s} - m_X)^2/\Gamma^2]$, linear in $x = (\sqrt{s} - m_X)^2$
2. $1/\sigma$ at $x = 0$, 64, 256 and 576 MeV^2, averaging each symmetric pair, is 0.769, 1.239, 2.751 and 5.157 mb^{-1}; a straight-line fit gives an intercept of 0.768 mb^{-1} and a slope of 7.64×10^{-3} mb^{-1} MeV^{-2}
3. $\sigma_{peak} = 1/(0.768\ \text{mb}^{-1}) = 1.30$ mb, and $\Gamma = 2\sqrt{(\text{intercept/slope})} = 2\sqrt{(0.768/0.00764)}$ MeV $= 20.1$ MeV, accepting 19.5 to 20.5 MeV

   Reading the half-maximum points by straight-line interpolation between readings 8 MeV apart biases Γ upwards, to about 21.7 MeV here, because the line shape is convex there, beyond its inflection points at $|\sqrt{s} - m_X| = \Gamma/(2\sqrt{3}) \approx 6$ MeV, so each chord lies above the curve; the linearisation uses every reading. Candidates who plot $1/\sigma$ against $\sqrt{s} - m_X$ instead of its square obtain two branches that no single line fits.

(c) CALCULATE · U3 **[2]**

Calculate p at $\sqrt{s} = m_X$ and the s-wave unitarity limit $4\pi/p^2$ in millibarns, and hence B and the lifetime of X in seconds.

1. $p = \sqrt{[(s - (m_a + m_b)^2)(s - (m_a - m_b)^2)]}/(2\sqrt{s}) = \sqrt{[(3.4225 - 1.0000)(3.4225 - 0.0400)]}/3.700$ GeV $= 0.774$ GeV, so $4\pi/p^2 = 21.0$ GeV$^{-2} = 21.0 \times 0.3894$ mb $= 8.18$ mb
2. $B = \sqrt{(\sigma_{peak}/(4\pi/p^2))} = \sqrt{(1.30/8.18)} = 0.399$, and $\tau = \hbar/\Gamma = 6.582 \times 10^{-25}$ GeV s$/0.0201$ GeV $= 3.3 \times 10^{-23}$ s

   Converting with $\hbar c$ instead of $(\hbar c)^2$, or comparing the peak in millibarns with $4\pi/p^2$ left in GeV^{-2}, gives $B = 0.56$ or $B = 0.25$; the second is close enough to the spin-1 value in (e) to mislead the whole analysis.

(d) DETERMINE · U2 **[2]**

Use the optical theorem to determine the total and the inelastic cross-sections at the peak, and determine $\Gamma(X \to ab)$ and g.

1. at the peak $M = ig^2/(m_X\Gamma)$, so $\text{Im}\ M = g^2/(m_X\Gamma) = 2m_X p\sigma_{tot}$ gives $\sigma_{tot} = g^2/(2m_X^2 p\Gamma) = (4\pi/p^2)B = 8.18\ \text{mb} \times 0.399 = 3.26$ mb, and $\sigma_{inel} = \sigma_{tot} - \sigma_{peak} = 1.96$ mb
2. $\Gamma(X \to ab) = B\Gamma = 0.399 \times 20.1$ MeV $= 8.0$ MeV, and $g^2 = 8\pi m_X^2 \Gamma(X \to ab)/p = 8\pi \times 3.4225$ GeV$^2 \times 0.0080$ GeV$/0.774$ GeV $= 0.89$ GeV^2, so $g = 0.94$ GeV

   The total cross-section at the peak is linear in B and the elastic one quadratic, so the inelastic share is $(4\pi/p^2)B(1 - B)$. Candidates who take σ_{tot} to be the measured elastic peak have assumed $B = 1$, which (c) has just excluded.

(e) EVALUATE · U4 **[2]**

A separate experiment counts the decays of X and finds $B = 0.43 \pm 0.05$. For a resonance of spin J the peak elastic cross-section is $(2J + 1)(4\pi/p^2)B^2$. Evaluate whether the data support $J = 0$ against $J = 1$, and say what further measurement on the same events would settle the spin directly.

1. the measured peak requires $B = \sqrt{(1.30/8.18)} = 0.40$ for $J = 0$ and $B = \sqrt{(1.30/(3 \times 8.18))} = 0.23$ for $J = 1$; against 0.43 ± 0.05 these lie 0.6σ and 4.0σ away, so $J = 0$ is supported and $J = 1$ excluded at about 4σ
2. the angular distribution in the centre-of-mass frame settles it directly: a spin-0 resonance decays isotropically, so $d\sigma/d\Omega$ is flat at the peak, while spin 1 gives $d\sigma/d\Omega \propto [P_1(\cos\theta)]^2 = \cos^2\theta$

   Candidates who reject $J = 1$ because its peak 'would exceed the unitarity limit' have the inequality backwards: a higher spin raises the limit, so the same peak needs a smaller B. The test needs the independent branching fraction, because the peak alone fixes only $(2J + 1)B^2$.

14 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQFTI-08.01 · UQFTI-08.02 · UQFTI-08.04 · UQFTI-08.05 · UQFTI-08.06 · UQFTI-08.08 · UQFTI-08.09 **[15]**

The free electromagnetic field, in natural units $\hbar = c = 1$ and Heaviside–Lorentz units with $e^2 = 4\pi\alpha$ and the metric $(+,-,-,-)$, has the Maxwell Lagrangian $\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A complex scalar φ of mass m and charge e couples to it through $\mathcal{L}_\varphi = (D_\mu\varphi)^*(D^\mu\varphi) - m^2\varphi^*\varphi$ with $D_\mu = \partial_\mu + ieA_\mu$, whose Feynman rules are a vertex $-ie(p + p')^\mu$, with p and p' the incoming and outgoing scalar momenta, and a two-photon seagull vertex $2ie^2g^{\mu\nu}$. Write Π^μ for the momentum conjugate to A_μ. Take $\alpha = 1/137.036$, $(\hbar c)^2 = 0.3894\ \text{GeV}^2\,\text{mb}$, and $m = 139.570\ \text{MeV}$, the mass of the charged pion, in (f).

(a) SHOW THAT · U2 **[2]**

Show that the Maxwell Lagrangian gives A_0 no conjugate momentum, so that canonical quantisation of all four components fails, and that adding the gauge-fixing term $-(1/(2\xi))(\partial_\mu A^\mu)^2$ supplies one.

1. $\Pi^\mu = \partial\mathcal{L}/\partial(\partial_0 A_\mu) = -F^{0\mu}$, and $F^{00} = 0$ by antisymmetry, so $\Pi^0 = 0$ identically and $[A_0(x), \Pi^0(y)] = i\delta^3(x - y)$ cannot be imposed: A_0 enters as a Lagrange multiplier enforcing Gauss's law, not as a dynamical field
2. with the gauge-fixing term $\Pi^\mu = -F^{0\mu} - (1/\xi)g^{\mu 0}\partial_\nu A^\nu$, so $\Pi^0 = -(1/\xi)\partial_\nu A^\nu$, which is non-zero and makes A_0 dynamical

Candidates who argue only that a gauge field has redundant components have not located the failure. It is the vanishing of one canonical momentum, which no choice of commutator can repair without changing the Lagrangian.

(b) DERIVE · U2 **[3]**

From the Euler–Lagrange equations, derive the field equation of the gauge-fixed Lagrangian and, from its quadratic action in momentum space, the photon propagator in a general covariant gauge.

1. $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = -F^{\mu\nu} - (1/\xi)g^{\mu\nu}\partial\cdot A$, so $\partial_\mu[\partial\mathcal{L}/\partial(\partial_\mu A_\nu)] = 0$ gives $\partial_\mu F^{\mu\nu} + (1/\xi)\partial^\nu(\partial\cdot A) = 0$, that is $\partial^2 A^\nu - (1 - 1/\xi)\partial^\nu(\partial\cdot A) = 0$, so that at $\xi = 1$ each component obeys $\partial^2 A^\nu = 0$
2. integrating by parts, the action is $\tfrac{1}{2}\int A_\mu[g^{\mu\nu}\partial^2 - (1 - 1/\xi)\partial^\mu\partial^\nu]A_\nu\, d^4x$, whose momentum-space kernel is $-k^2g^{\mu\nu} + (1 - 1/\xi)k^\mu k^\nu$, and the propagator must satisfy $[-k^2g^{\mu\nu} + (1 - 1/\xi)k^\mu k^\nu]D_{\nu\rho}(k) = i\delta^\mu{}_\rho$
3. the ansatz $D_{\nu\rho} = (-i/k^2)[g_{\nu\rho} + c\, k_\nu k_\rho/k^2]$ leaves a $k^\mu k_\rho$ term that cancels only if $(1 - 1/\xi)(1 + c) = c$, that is $c = \xi - 1$, so $D_{\mu\nu}(k) = -i[g_{\mu\nu} - (1 - \xi)k_\mu k_\nu/k^2]/(k^2 + i\varepsilon)$, reducing to $-ig_{\mu\nu}/(k^2 + i\varepsilon)$ in Feynman gauge

Without the gauge-fixing term the kernel $-k^2g^{\mu\nu} + k^\mu k^\nu$ annihilates k_ν and has no inverse. Candidates who try to invert it, or quote the Feynman-gauge propagator without deriving the general-ξ form, miss the step on which (c) depends.

(c) SHOW THAT · U2 **[3]**

Two conserved currents, $k_\mu j_1^\mu(k) = k_\mu j_2^\mu(k) = 0$, exchange a photon with amplitude proportional to $j_1^\mu D_{\mu\nu}(k) j_2^\nu$. Show that ξ drops out and, taking k along z with $|k| = \kappa$, that the exchange splits into an instantaneous Coulomb term $i\, j_1^0 j_2^0/\kappa^2$ and a transverse term $i\, j_1^{\mathrm{T}}\cdot j_2^{\mathrm{T}}/(k^2 + i\varepsilon)$, where j^{T} is the part of the three-vector current perpendicular to k.

1. the ξ-dependent term carries $k_\mu j_1^\mu$ and $k_\nu j_2^\nu$, both zero, so $j_1^\mu D_{\mu\nu} j_2^\nu = -i\, j_1\cdot j_2/(k^2 + i\varepsilon)$ in every covariant gauge
2. with $k^\mu = (k^0, 0, 0, \kappa)$, conservation reads $k^0 j^0 = \kappa j^3$, so $j^3 = (k^0/\kappa)j^0$ and $j_1\cdot j_2 = j_1^0 j_2^0[1 - (k^0)^2/\kappa^2] - j_1^{\mathrm{T}}\cdot j_2^{\mathrm{T}} = -(k^2/\kappa^2) j_1^0 j_2^0 - j_1^{\mathrm{T}}\cdot j_2^{\mathrm{T}}$
3. hence $-i\, j_1\cdot j_2/k^2 = i\, j_1^0 j_2^0/\kappa^2 + i\, j_1^{\mathrm{T}}\cdot j_2^{\mathrm{T}}/(k^2 + i\varepsilon)$: the first term does not depend on k^0, so it acts instantaneously and, as $1/\kappa^2$, Fourier-transforms to the Coulomb interaction $1/(4\pi r)$; only the second has the photon pole, carried by the two transverse polarisations, which is Coulomb gauge recovered

Candidates often show the ξ-independence and stop, or eliminate j^0 rather than j^3 and obtain a term that is not instantaneous. The split works because current conservation trades the timelike and longitudinal photons, which have no pole of their own once combined, for an interaction local in time.

(d) COMPARE · U1 **[2]**

A massive vector field of mass M_V, the Proca field, has the propagator $-i[g_{\mu\nu} - k_\mu k_\nu/M_V^2]/(k^2 - M_V^2 + i\varepsilon)$. Compare its polarisations and propagator with the photon's, and explain from the little group of a massless particle why the photon must couple to a conserved current while the Proca field need not.

1. a massive vector carries the three polarisations of spin 1 under its little group SO(3), with $\Sigma\varepsilon_\mu\varepsilon_\nu^* = -g_{\mu\nu} + k_\mu k_\nu/M_V^2$, and its propagator has no gauge parameter; its $k_\mu k_\nu/M_V^2$ term grows with energy unless it meets conserved currents, when it drops out as the photon's ξ term does
2. a massless particle's little group is ISO(2), whose translations leave the helicity ±1 unchanged but shift $\varepsilon^\mu \to \varepsilon^\mu + ck^\mu$, so ε^μ is a four-vector only up to a multiple of k^μ and amplitudes are Lorentz invariant only if $k_\mu M^\mu = 0$ for every external photon, which requires coupling to a conserved current; no such shift exists when $M_V \neq 0$

   Answers that say the photon has two polarisations because it is massless have restated the fact. The little group explains it, and shows that $k_\mu M^\mu = 0$ is a requirement on every amplitude, not a choice of gauge.

(e) SHOW THAT · U2 **[3]**

For Compton scattering on the charged scalar, $\gamma(k_1) + \varphi(p) \to \gamma(k_2) + \varphi(p')$, write $iM^{\mu\nu}$, with μ belonging to the incoming photon and ν to the outgoing one, as the sum of the s-channel, u-channel and seagull diagrams. Show that $k_{1\mu}M^{\mu\nu} = 0$ for the sum, but not for any one diagram or for the two exchange diagrams alone.

1. $iM^{\mu\nu} = (-ie)^2[(2p + k_1)^\mu(2p' + k_2)^\nu\, i/(s - m^2) + (2p' - k_1)^\mu(2p - k_2)^\nu\, i/(u - m^2)] + 2ie^2g^{\mu\nu} = -ie^2[(2p + k_1)^\mu(2p' + k_2)^\nu/(s - m^2) + (2p' - k_1)^\mu(2p - k_2)^\nu/(u - m^2) - 2g^{\mu\nu})]$, with $s = (p + k_1)^2$ and $u = (p - k_2)^2$
2. with $k_1^2 = 0$, $k_1\cdot(2p + k_1) = 2p\cdot k_1 = s - m^2$ and $k_1\cdot(2p' - k_1) = 2p'\cdot k_1 = m^2 - u$, so contracting with $k_{1\mu}$ cancels each propagator: the s-channel gives $-ie^2(2p' + k_2)^\nu$, the u-channel $+ie^2(2p - k_2)^\nu$ and the seagull $+2ie^2k_1^\nu$, each non-zero
3. the two exchange diagrams together give $-2ie^2(p' - p + k_2)^\nu = -2ie^2k_1^\nu$, using $p + k_1 = p' + k_2$, which is not zero; the seagull's $+2ie^2k_1^\nu$ cancels it exactly, so $k_{1\mu}M^{\mu\nu} = 0$ holds only for the sum of all three

   The seagull is the term candidates omit, having learned Compton scattering on a Dirac electron, where there is none; the Ward identity then fails by exactly $2ie^2k_1^\nu$. The identity also needs the external lines on shell, $k_1^2 = 0$ and $p^2 = p'^2 = m^2$, which every step above uses.

(f) CALCULATE · U3 **[2]**

In the rest frame of the initial scalar choose both photon polarisations with $\varepsilon^0 = 0$, so that $p\cdot\varepsilon = k\cdot\varepsilon = 0$ for each photon. For photon energies $\omega \ll m$, where $d\sigma/d\Omega = |M|^2/(64\pi^2m^2)$, calculate the total cross-section for a point-like scalar with the charged pion's mass, in microbarns, averaging over initial and summing over final polarisations.

1. with $p = (m, 0)$ and these polarisations, $(2p + k_1)\cdot\varepsilon_1 = 0$ and $(2p - k_2)\cdot\varepsilon_2^* = 0$, so only the seagull survives: $|M|^2 = 4e^4|\varepsilon_1\cdot\varepsilon_2^*|^2$, $d\sigma/d\Omega = (\alpha/m)^2|\varepsilon_1\cdot\varepsilon_2^*|^2$, and the polarisation average gives $(\alpha/m)^2(1 + \cos^2\theta)/2$
2. integrating over angles, $\sigma = (8\pi/3)(\alpha/m)^2 = (8\pi/3)[1/(137.036 \times 0.13957\ \text{GeV})]^2 = 0.0229\ \text{GeV}^{-2}$, and $0.0229 \times 0.3894\ \text{mb} = 8.92\ \mu\text{b}$: Thomson's cross-section with the pion's mass in place of the electron's

   In this frame and gauge both exchange diagrams vanish identically, so the classical Thomson result is carried entirely by the seagull. Summing over the initial polarisations instead of averaging gives twice the answer, 17.8 μb.

University Quantum Field Theory I · Final Examination

The whole course, weighted towards its second half: QED at tree level by traces and crossing, C, P and T on quantised fields, the self-energy and the Ward–Takahashi identity, vacuum polarisation and the running of α, $g-2$ with the infrared cancellation, and the Lamb shift at order $\alpha(Z\alpha)^4$, with the first half's propagators, rates and one-loop φ^4 sampled.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	**117**	**19**	**10**

SECTION — **UQFTI · Final · University Quantum Field Theory I — Final examination**

IN THE FORMAT OF — A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND — **Recall 2 · Routine 7 · Demanding 7 · Discriminating 3**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS — UQFTI-01 · UQFTI-02 · UQFTI-03 · UQFTI-04 · UQFTI-05 · UQFTI-06 · UQFTI-07 · UQFTI-08 · UQFTI-09 · UQFTI-10 · UQFTI-11 · UQFTI-12 · UQFTI-13 · UQFTI-14 · UQFTI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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THE PAPER

University Quantum Field Theory I · Final Examination

Answer all questions. 117 marks in 150 minutes.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07 **[1]**

For a free real scalar field of mass m, in natural units $\hbar = c = 1$ with the metric (+, −, −, −), the Feynman propagator is D_F(x − y) = ∫ d⁴p/(2π)⁴ i e^(−ip·(x − y))/(p² − m² + iε), with E_p = √(|p|² + m²) for the three-momentum p and ε → 0⁺. The p^0 integral is done first, by closing the contour in whichever half of the complex p^0 plane makes the exponential decay. Which statement is correct?

A The iε places both poles just below the real p^0 axis, so D_F vanishes whenever $x^0 < y^0$, as the response of a causal system must.

B The poles sit at $p^0 = \pm(E_p - i\varepsilon')$; for $x^0 > y^0$ the contour closes in the lower half-plane around $p^0 = +E_p$ and gives $\langle 0|\varphi(x)\varphi(y)|0\rangle$, and for $x^0 < y^0$ it closes in the upper half-plane around $p^0 = -E_p$ and gives $\langle 0|\varphi(y)\varphi(x)|0\rangle$, so the placement of the iε is the time ordering.

C Because it is time-ordered, D_F(x − y) vanishes when x − y is spacelike, and that is how the theory keeps measurements outside each other's light cones from interfering.

D For $x^0 > y^0$ the contour closes in the upper half-plane around $p^0 = -E_p$, so for that time order D_F describes an antiparticle carried from y to x.

Answer ☐

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-03.02 · UQFTI-03.06 **[1]**

Two theories are expanded to second order in their coupling, in natural units. In theory (i) a real scalar φ has $L_{int} = -(g/3!)\varphi^3$. In theory (ii), the scalar Yukawa model, a complex scalar ψ and a real scalar φ have $L_{int} = -g\psi^\dagger\psi\varphi$. In each, the order-$g^2$ correction to the φ two-point function $\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle$ contains a one-loop diagram in which x_1 joins one vertex, x_2 joins the other, and the two vertices are linked by two internal lines; in (ii) these form a closed ψ loop. If each diagram is written with its vertex factors and propagators and an overall 1/S, what are the symmetry factors S in (i) and (ii)?

A S = 2 in both

B S = 6 in (i) and S = 1 in (ii)

C S = 1 in both

D S = 2 in (i) and S = 1 in (ii)

Answer ☐

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-04.02 · UQFTI-04.03 **[1]**

In φ⁴ theory, L = ½∂_μφ∂^μφ − ½m₀²φ² − (λ/4!)φ⁴ with 0 < λ ≪ 1, the single particle has physical mass m. The Källén–Lehmann representation writes the exact two-point function as ⟨Ω|Tφ(x)φ(y)|Ω⟩ = ∫₀^∞ (dM²/2π) ρ(M²) D_F(x − y; M²), with ρ(M²) = 2πZδ(M² − m²) plus a continuum. Where does the continuum begin, and what bounds Z?

A At M² = 4m², the threshold for two particles, with 0 ≤ Z ≤ 1 because ρ is non-negative and the canonical commutator fixes ∫(dM²/2π)ρ(M²) = 1.

B At M² = 4m², with Z ≥ 1, because interactions add multiparticle states to the one-particle state and so increase the weight of the pole.

C At M² = 9m², because the symmetry φ → −φ stops φ from connecting the vacuum to states with an even number of particles; 0 ≤ Z ≤ 1, because ρ is non-negative and the canonical commutator fixes ∫(dM²/2π)ρ(M²) = 1.

D Immediately above M² = m², because a particle can always be accompanied by arbitrarily soft quanta, so the pole sits at the start of a branch cut.

Answer

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4 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-06.01 **[1]**

In d spacetime dimensions a dimensionless action with a canonical kinetic term gives a scalar field the mass dimension (d − 2)/2. Which interaction has a dimensionless coupling, so that the theory is renormalisable but not super-renormalisable?

A gφ³ in d = 6

B gφ³ in d = 4

C λφ⁴ in d = 3

D gφ⁶ in d = 4

Answer

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-07.07 **[1]**

A spin-zero boson φ of mass 2.40 GeV decays to a fermion–antifermion pair, the fermion being a Dirac particle of mass 0.90 GeV. Compare a scalar coupling gφψ̄ψ with a pseudoscalar coupling igφψ̄γ⁵ψ of the same real strength g. In both cases Γ = |p| Σ|M|²/(8πm_φ²), with the sum over final spins and |p| the momentum of either fermion in the rest frame of φ. What is Γ_scalar/Γ_pseudoscalar?

A 0.66

B 1.00

C 2.29

D 0.44

Answer

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-08.04 · UQFTI-08.06 **[1]**

In a general covariant gauge the photon propagator is $-i[g_{\mu\nu} - (1-\xi)k_\mu k_\nu/k^2]/(k^2 + i\varepsilon)$. In the tree amplitude for $e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)$, the virtual photon carries $k = p_1 + p_2 = p_3 + p_4$, with $k^2 = s$. Why does the amplitude not depend on ξ?

A Because the photon is massless, $k^2 = 0$, so the $k_\mu k_\nu$ term is multiplied by zero.

B Because k_μ contracted into the electron current gives $\bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = (m_e - m_e)\bar{v}(p_2)u(p_1) = 0$ by the Dirac equation, and likewise on the muon line, so the $k_\mu k_\nu$ term vanishes for any ξ.

C Because one may choose $\xi = 1$, and a choice of gauge cannot change a physical amplitude.

D Because the $k_\mu k_\nu$ term is suppressed by m_e^2/s, and vanishes once the electron mass is neglected.

Answer

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-09.07 **[1]**

At $\sqrt{s} \gg m_\mu$ the reaction $e^+e^- \to \mu^+\mu^-$ by one-photon exchange proceeds through four helicity combinations, because the vector coupling joins a right-handed fermion only to a left-handed antifermion, and vice versa. The e^- moves along $+z$, and θ is the angle between the outgoing μ^- and the incoming e^-. How does the amplitude for $e^-_R\, e^+_L \to \mu^-_L\, \mu^+_R$ depend on θ, and why?

A As $(1 - \cos\theta)$: the initial pair has angular momentum $+1$ along the beam and the final pair -1 along the μ^- direction, so the amplitude is proportional to the rotation matrix element $d^1_{1,-1}(\theta)$, which vanishes in the forward direction.

B As $(1 + \cos\theta)$: helicity is conserved at each vertex, so the μ^- prefers to follow the e^-.

C It vanishes at every angle: helicity is conserved along a fermion line, so a right-handed electron cannot produce a left-handed muon.

D As $\sin\theta$: the virtual photon carries one unit of angular momentum, so every helicity amplitude has the angular dependence of a dipole.

Answer

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-11.08 · UQFTI-11.09 **[1]**

A positronium level with orbital angular momentum L and total spin S has charge-conjugation parity C = (−1)^(L+S), and a state of n photons has C = (−1)^n. Landau and Yang's theorem, quoted, forbids a state of total angular momentum J = 1 from decaying into two photons. Which level can annihilate directly into two photons?

A 1^3S_1

B 2^1P_1

C 2^3P_1

D 2^3P_2

Answer

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9 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-12.06 · UQFTI-12.07 **[1]**

In renormalised QED with on-shell conditions the physical charge is $e = Z_2 Z_3^{1/2} e_0 / Z_1$, where Z_1 renormalises the vertex, Z_2 the charged field and Z_3 the photon field. A proton's coupling to the photon is dressed by strong interactions that the electron's is not, yet the hydrogen atom is neutral to better than one part in 10^{20}. Why are the renormalised charges of the electron and the proton exactly opposite, given opposite bare charges?

A The Ward–Takahashi identity gives $Z_1 = Z_2$ for every charged field, whatever its other interactions, so $e = Z_3^{1/2} e_0$, and Z_3 comes from the photon's own self-energy, which is common to every particle the photon meets.

B Strong interactions conserve charge, so they leave the proton's vertex unrenormalised.

C The electron and the proton both have spin ½, so their vertex corrections are equal.

D The on-shell condition is imposed separately for each particle, so each renormalised charge is set equal to the measured one by definition.

Answer

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10 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-13.01 · UQFTI-13.02 **[1]**

At one loop the electron loop gives the photon a self-energy $i\Pi^{\mu\nu}(q)$. Why is it computed in dimensional regularisation rather than with a sharp cutoff Λ on the Euclidean loop momentum?

A The sharp cutoff leaves $\Pi(q^2)$ quadratically divergent, which no counterterm can absorb, whereas dimensional regularisation makes the loop finite.

B The electron loop diverges in the infrared at $q^2 = 0$, which a cutoff on large momenta cannot regulate.

C A sharp cutoff is not invariant under shifts of the loop momentum and breaks gauge invariance: $\Pi^{\mu\nu}$ acquires a term proportional to $g^{\mu\nu}\Lambda^2$ that is not of the transverse form $(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ and would give the photon a mass, whereas dimensional regularisation keeps $\Pi^{\mu\nu}$ transverse.

D Both regulators keep $\Pi^{\mu\nu}$ transverse; dimensional regularisation is preferred only because its integrals are easier.

Answer

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11 CORE COURSE

STRUCTURED · ROUTINE · UQFTI-02.01 · UQFTI-04.09 · UQFTI-05.03 · UQFTI-05.06 **[9]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model a complex scalar ψ of mass m_ψ and a real scalar φ of mass m_φ interact through $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\varphi$. Take $m_\varphi = 0.80$ GeV, $m_\psi = 0.25$ GeV and $g = 0.40$ GeV, with $\hbar = 6.582 \times 10^{-25}$ GeV s and $\hbar c = 0.1973$ GeV fm. For a particle of mass m_φ at rest decaying to two distinguishable spinless products, $\Gamma = |\mathbf{p}||\mathcal{M}|^2/(8\pi m_\varphi^2)$, where $\mathcal{M}$ is the invariant amplitude and $|\mathbf{p}|$ the momentum of either product.

(a) STATE **[1]**

State the mass dimension of g, and whether the interaction is renormalisable, super-renormalisable or non-renormalisable.

(b) SHOW THAT **[2]**

Show that, at tree level, $\Gamma(\varphi \to \psi\bar{\psi}) = (g^2/16\pi m_\varphi)\sqrt{(1 - 4m_\psi^2/m_\varphi^2)}$.

(c) CALCULATE **[2]**

Calculate Γ in MeV and the mean lifetime of φ in seconds.

(d) DETERMINE **[3]**

Two ψ particles scatter at low speed. From the t-channel φ-exchange diagram, determine the potential $V(r)$ between them, allowing for the relativistic normalisation of the states, and state whether it attracts and what its range is in femtometres.

(e) PREDICT **[1]**

Predict what becomes of φ, and of the use of the LSZ formula with φ as an external particle, if m_φ were lowered to 0.45 GeV with m_ψ unchanged.

12 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-03.02 · UQFTI-05.04 · UQFTI-05.07 · UQFTI-05.08 · UQFTI-06.09 **[11]**

Work in natural units with the metric (+, −, −, −). A real scalar φ of mass m = 0.40 GeV has L_int = −(λ/4!)φ⁴ with λ = 0.80. Two φ particles collide head-on with centre-of-mass energy √s = 2.0 GeV. For 2 → 2 scattering of equal masses in the centre-of-mass frame, dσ/dΩ = |M|²/(64π²s). Two-body Lorentz-invariant phase space for distinguishable particles is ∫dΠ₂ = |p_cm|/(4π√s), with |p_cm| the momentum of either particle. Use (ħc)² = 0.3894 GeV² mb. In minimal subtraction the one-loop β function of the theory is β(λ) = μ dλ/dμ = 3λ²/(16π²).

(a) CALCULATE **[3]**

Calculate the tree-level amplitude M and the total cross-section for φφ → φφ in microbarns.

(b) SHOW THAT **[2]**

At order λ² the s-channel one-loop diagram joins p_1 and p_2 at one vertex and p_3 and p_4 at the other, the two vertices being linked by two internal lines. Show by counting Wick contractions that it carries a symmetry factor ½, so that iM_s = ½(−iλ)²∫d⁴l/(2π)⁴ [i/(l² − m² + iε)][i/((k − l)² − m² + iε)], with k = p_1 + p_2.

(c) SHOW THAT **[3]**

Show that the imaginary part of the forward amplitude at order λ², which comes from this diagram alone, is Im M = λ²β/(32π) with β = √(1 − 4m²/s), and that it satisfies the optical theorem Im M(forward) = 2E_cm|p_cm|σ_tot at √s = 2.0 GeV.

(d) EXPLAIN **[1]**

Explain why Im M is finite although the loop integral itself diverges logarithmically.

(e) CALCULATE **[2]**

Taking λ(μ_0) = 0.80 at μ_0 = 0.40 GeV in minimal subtraction, calculate λ at μ = 100 GeV from the one-loop β function, and the scale at which the one-loop coupling would diverge.

13 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-09.02 · UQFTI-09.05 · UQFTI-09.06 · UQFTI-09.08 **[12]**

Work in natural units with the metric (+, −, −, −). In QED with an electron and a muon, $e^+e^- \to \mu^+\mu^-$ proceeds at lowest order through one-photon exchange in the s channel, with vertex $-ie\gamma^\mu$ and Feynman-gauge photon propagator $-ig_{\mu\nu}/(q^2 + i\varepsilon)$. Label the momenta $e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)$, neglect the electron mass, keep the muon mass $m_\mu = 0.10566$ GeV, and take $\alpha = e^2/4\pi = 1/137.036$ and $(\hbar c)^2 = 0.3894$ GeV^2 mb. The tree amplitude is $M = (e^2/s)[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)]$, with $s = (p_1 + p_2)^2$. Use $\Sigma\, u\,\bar{u} = \not{p} + m$ and $\Sigma\, v\,\bar{v} = \not{p} - m$ over spins, $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$; traces of an odd number of gamma matrices vanish.

(a) SHOW THAT **[3]**

Show that the spin-averaged squared amplitude is $\tfrac{1}{4}\Sigma|M|^2 = (8e^4/s^2)[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mu^2(p_1\cdot p_2)]$.

(b) SHOW THAT **[3]**

In the centre-of-mass frame, with θ the angle between p_3 and p_1 and $\beta = \sqrt{(1 - 4m_\mu^2/s)}$ the muon speed, show that $d\sigma/d\Omega = (\alpha^2\beta/4s)(2 - \beta^2\sin^2\theta)$, and hence that $\sigma = (4\pi\alpha^2/3s) \times \beta(3 - \beta^2)/2$.

(c) CALCULATE **[2]**

Calculate σ in nanobarns at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, and, at the lower energy, the ratio of σ to the value $4\pi\alpha^2/3s$ that massless muons would give.

(d) SKETCH **[2]**

Sketch on one set of axes the angular distribution at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, each normalised to its own value at $\cos\theta = \pm 1$, that is $(d\sigma/d\Omega)/(d\sigma/d\Omega$ at $\cos\theta = 1)$ against $\cos\theta$ from -1 to $+1$, marking the value of each curve at $\cos\theta = 0$.

(e) DETERMINE **[2]**

Determine by crossing the spin-averaged $|M|^2$ for $e^-\mu^- \to e^-\mu^-$ at energies where both masses can be neglected, in the Mandelstam variables of that process, and state the sign that crossing introduces.

14 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-10.03 · UQFTI-10.04 · UQFTI-10.07 · UQFTI-10.08 · UQFTI-10.09 **[10]**

Work in natural units with the metric (+, −, −, −). Compton scattering $\gamma(k)e^-(p) \to \gamma(k')e^-(p')$ has, at tree level, an s-channel diagram, in which the electron absorbs k before emitting k′, and a u-channel diagram, in which it emits k′ first. For an electron initially at rest and a photon of energy ω, the spin- and polarisation-summed result integrates to the Klein–Nishina total cross-section $\sigma = 2\pi r_e^2\{[(1+\kappa)/\kappa^2][2(1+\kappa)/(1+2\kappa) - \ln(1+2\kappa)/\kappa] + \ln(1+2\kappa)/(2\kappa) - (1+3\kappa)/(1+2\kappa)^2\}$, with $\kappa = \omega/m$ and $r_e = \alpha/m$. Crossing the same amplitude into $e^+e^- \to \gamma\gamma$, with the factor ½ for identical photons, gives for a slow pair the spin-averaged $\sigma v_{rel} \to \pi\alpha^2/m^2$ as the relative speed $v_{rel} \to 0$. Take $m_e = 0.51100$ MeV, $m_\mu = 105.658$ MeV, $\alpha = 1/137.036$, $\hbar c = 197.327$ MeV fm, $\hbar = 6.582 \times 10^{-22}$ MeV s and 1 b = 100 fm².

(a) EXPLAIN **[2]**

Replacing the polarisation vector of the incoming photon by its momentum k, explain why neither Compton diagram alone satisfies the Ward identity $k_\mu M^{\mu\nu} = 0$, while their sum does.

(b) CALCULATE **[1]**

Calculate the Thomson cross-section $(8\pi/3)r_e^2$ in barns.

(c) CALCULATE **[3]**

A 511 keV photon from positron annihilation enters the scintillator of a positron-emission tomograph. Calculate the Klein–Nishina cross-section per electron in barns, its ratio to the Thomson value, and the largest kinetic energy the photon can give to a free electron at rest.

(d) DETERMINE **[3]**

True muonium is the atom formed by a μ^+ and a μ^-. Treating it as a hydrogen-like atom bound by the Coulomb potential, determine the lifetime of its para ground state, 1^1S_0, against annihilation into two photons, which is its only significant decay.

(e) EXPLAIN **[1]**

Explain why the two Compton diagrams enter with the same sign, whereas the two diagrams of Bhabha scattering, $e^+e^- \to e^+e^-$, enter with a relative minus sign.

15 CORE COURSE

STRUCTURED · DEMANDING · UQFTI-11.02 · UQFTI-11.04 · UQFTI-11.05 · UQFTI-11.06 · UQFTI-11.07 **[10]**

Work in natural units $\hbar = c = 1$ with the metric $(+,-,-,-)$. Charge conjugation is a unitary operator C on the Fock space of QED. On the Dirac field it acts as $C\psi(x)C^{-1} = M_C\bar{\psi}^{\mathsf{T}}(x)$ and $C\bar{\psi}(x)C^{-1} = -\psi^{\mathsf{T}}(x)M_C^{-1}$, where $^{\mathsf{T}}$ is the transpose in spinor indices and the 4×4 matrix M_C satisfies $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^{\mathsf{T}}$; on the photon field $CA_\mu(x)C^{-1} = -A_\mu(x)$. Write $(-1)^\mu = +1$ for $\mu = 0$ and -1 for $\mu = 1, 2, 3$. Under parity the Hermitian bilinear $i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi$ is multiplied by $-(-1)^\mu(-1)^\nu$ and $F_{\mu\nu}$ by $(-1)^\mu(-1)^\nu$; under time reversal, which is antiunitary, they are multiplied by $(-1)^\mu(-1)^\nu$ and $-(-1)^\mu(-1)^\nu$ respectively; under C both are multiplied by -1. Each is evaluated at the transformed point. An electric dipole moment d of the electron would add $\mathcal{L}_{\mathrm{EDM}} = -(d/2)(i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi)F_{\mu\nu}$ to the Lagrangian, with d real, whose non-relativistic limit is the energy $-d\boldsymbol{\sigma}\cdot\mathbf{E}$. An experiment on polar molecules has found $|d| < 4.1\times 10^{-30}$ e cm. Take $m_e = 0.511$ MeV and $\hbar c = 1.973\times 10^{-11}$ MeV cm.

(a) SHOW THAT **[3]**

Show that the current $\bar{\psi}\gamma^\mu\psi$ is odd under C while $\bar{\psi}\psi$ is even, identifying the step at which the anticommutation of the fields supplies a sign, and say what would become of the electron's coupling to the photon if the Dirac field were quantised with commutators.

(b) DETERMINE **[2]**

Determine whether $\mathcal{L}_{\mathrm{EDM}}$ is even or odd under each of P, C, T and CP, and check that it is even under CPT.

(c) EXPLAIN **[2]**

Explain, both from C invariance and from the diagrams, why the sum of the one-loop electron-loop diagrams with three external photons vanishes in QED, and what this means for the counterterms QED needs.

(d) ESTIMATE **[2]**

A new interaction at a mass scale Λ could generate d at one loop. Estimate the lowest Λ consistent with the experimental bound, stating your assumptions.

(e) PREDICT **[1]**

Predict, from CPT invariance alone, the electric dipole moment of the positron if the electron's were found to be d.

16 CORE COURSE

STRUCTURED · DISCRIMINATING · UQFTI-13.08 · UQFTI-15.02 · UQFTI-15.03 · UQFTI-15.04 · UQFTI-15.05 · UQFTI-15.07 · UQFTI-15.08 **[12]**

Work in natural units with the metric (+, −, −, −). An electron of mass m bound to a nucleus of charge Ze, treated as infinitely heavy, feels at lowest order the Coulomb potential $V(r) = -Z\alpha/r$, whose Fourier transform is $\tilde{V}(q) = -4\pi Z\alpha/|q|^2$. The renormalised one-loop vertex, computed with a photon mass μ_γ, has to first order in the spacelike momentum transfer $q^2 = -|q|^2$ the form factors $F_1(q^2) = 1 + (\alpha/3\pi)(q^2/m^2)[\ln(m/\mu_\gamma) - 3/8]$ and $F_2(0) = \alpha/2\pi$. For an ns state $|\psi_{ns}(0)|^2 = (Z\alpha m)^3/(\pi n^3)$, and states with $l \geq 1$ vanish at the origin. Write $U = 4\alpha(Z\alpha)^4 m/(3\pi n^3)$. In these units, at order $\alpha(Z\alpha)^4$: the F_2 (Pauli) term shifts ns levels by $+(3/8)U$ and $2P_{1/2}$ by $-(1/8)U$; the Uehling (vacuum-polarisation) term shifts ns levels by $-(1/5)U$; Bethe's non-relativistic treatment of photons of energy below a cutoff K shifts ns levels by $U \ln(K/\langle E\rangle_{ns})$ and $2P_{1/2}$ by $-U \ln(\langle E\rangle_{2p}/(Z^2\mathrm{Ry}))$, where $Z^2\mathrm{Ry} = (Z\alpha)^2 m/2$ and the Bethe logarithms are $\ln(\langle E\rangle_{2s}/(Z^2\mathrm{Ry})) = 2.8118$ and $\ln(\langle E\rangle_{2p}/(Z^2\mathrm{Ry})) = -0.0300$. Matching across $(Z\alpha)^2 m \ll K \ll m$ replaces $\ln(m/\mu_\gamma)$ in the high-energy part by $\ln(m/2K) + 5/6$. For hydrogen-like helium, $^4He^+$ ($Z = 2$), take $\alpha = 1/137.036$, $m = 0.511$ MeV and, for $n = 2$, $U = 2170$ MHz, reduced-mass corrections neglected. The same assembly for hydrogen gives 1051.9 MHz for the $2S_{1/2}$–$2P_{1/2}$ splitting against the measured 1057.8 MHz; for He^+ the measured splitting is 14 041 MHz.

(a) SHOW THAT **[3]**

Show that the q^2 term of F_1 adds to the Coulomb potential a contact term $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, and that this raises an ns level by $U[\ln(m/\mu_\gamma) - 3/8]$ while leaving P levels unshifted.

(b) EXPLAIN **[2]**

Explain why the result of (a) cannot be used on its own, and why a cutoff K with $(Z\alpha)^2 m \ll K \ll m$ can be found for He^+, giving the two ends of the window in electronvolts.

(c) DETERMINE **[3]**

Using the matching relation, determine the total order-$\alpha(Z\alpha)^4$ shift of the He^+ $2S_{1/2}$ level in units of U and in megahertz, showing that K cancels.

(d) CALCULATE **[2]**

Calculate the shift of $2P_{1/2}$ and hence the $2S_{1/2}$–$2P_{1/2}$ splitting of He^+ in megahertz.

(e) EVALUATE **[2]**

Evaluate the agreement with the measured splittings of He^+ and of hydrogen, and say what the comparison suggests about the largest term omitted.

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17 CORE COURSE

DATA ANALYSIS · DEMANDING · UQFTI-09.06 · UQFTI-13.04 · UQFTI-13.05 · UQFTI-13.06 **[12]**

An electron–positron collider measures the total cross-section for $e^+e^- \to \mu^+\mu^-$ at six centre-of-mass energies well below the Z boson. Z exchange and initial- and final-state radiation have been corrected for, so that what remains is one-photon exchange through the vacuum-polarisation-corrected photon propagator, for which $\sigma = 4\pi\alpha(s)^2/(3s)$, with $\alpha(s)$ the effective coupling at momentum transfer squared s; the muon mass is negligible at these energies. Work in natural units with $\alpha(0) = 1/137.036$ and $(\hbar c)^2 = 0.3894$ GeV² mb, so that $4\pi\alpha(0)^2/3 = 86.86$ nb GeV². At one loop a fermion of charge Q_f e with N_c colours makes α run as $\mu\, d\alpha/d\mu = 2N_c Q_f^2 \alpha^2/(3\pi)$ while μ is well above its mass, with $\mu^2 = s$. Every quoted uncertainty is 0.2% of σ, statistical and systematic combined. Lepton loops alone, run from $q^2 = 0$ with the measured lepton masses, give $1/\alpha = 134.02$ at $\sqrt{s} = 12.0$ GeV and 132.73 at $\sqrt{s} = 91.19$ GeV.

Total cross-section for $e^+e^- \to \mu^+\mu^-$ after radiative and Z-exchange corrections

Run	$\sqrt{s}$ / GeV	σ / pb	Uncertainty in σ / pb
1	12.0	652.5	1.3
2	18.0	291.8	0.6
3	25.0	152.7	0.3
4	35.0	78.51	0.16
5	45.0	47.66	0.10
6	58.0	28.86	0.06

(a) SHOW THAT **[2]**

Show that if every fermion lighter than $\sqrt{s}$ contributes at one loop, $1/\alpha(s)$ is a straight line in $\ln s$ with gradient $-\Sigma_f N_c Q_f^2/(3\pi)$, and that $\sigma/(86.86\text{ nb GeV}^2) = [\alpha(s)/\alpha(0)]^2$.

(b) DESCRIBE **[2]**

Describe how you would process the readings to test this prediction and extract the gradient, including how the uncertainty in each σ becomes an uncertainty in $1/\alpha$.

(c) DETERMINE **[3]**

Determine 1/α at each energy, the gradient and intercept of the best straight line with an estimate of the uncertainty in the gradient, and hence Σ_f N_cQ_f².

(d) EVALUATE **[3]**

Evaluate which of these the data support: the three charged leptons alone; the leptons with the five quarks u, d, s, c and b each counted once; or the leptons with those quarks in three colours.

(e) PREDICT **[2]**

Predict 1/α at √s = 91.19 GeV by extrapolating your line, and say what its difference from the lepton-only running requires of hadrons, already at 12 GeV and at the Z.

18

CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQFTI-12.01 · UQFTI-12.03 · UQFTI-12.04 · UQFTI-12.06 · UQFTI-12.07 **[15]**

Work in natural units with the metric (+, −, −, −), in QED with an electron of physical mass m and bare mass m₀, in Feynman gauge with a small photon mass μ_γ. At one loop the electron self-energy is $-i\Sigma_2(p) = (-ie)^2\int d^dk/(2\pi)^d\, \gamma^\mu[i(\not{k} + m)/(k^2 - m^2 + i\varepsilon)]\gamma_\mu[-i/((p-k)^2 - \mu_\gamma^2 + i\varepsilon)]$, continued to d = 4 − ε dimensions with $e^2 \to e^2\mu^\varepsilon$ to keep e dimensionless. You may use: in d dimensions $\gamma^\mu\gamma_\mu = d$ and $\gamma^\mu\not{k}\gamma_\mu = -(d-2)\not{k}$; Feynman's parametrisation $1/(AB) = \int_0^1 dx\, 1/[xA + (1-x)B]^2$; the Minkowski master integral $\int d^dl/(2\pi)^d\, 1/(l^2 - \Delta + i\varepsilon)^2 = i\Gamma(2 - d/2)\Delta^{(d/2-2)}/(4\pi)^{(d/2)}$, integrals odd in l vanishing; and, as ε → 0, $\Gamma(\varepsilon/2)(4\pi\mu^2/\Delta)^{(\varepsilon/2)} = 2/\varepsilon - \gamma_E + \ln(4\pi\mu^2/\Delta) + O(\varepsilon)$. The physical mass is the pole of the full propagator, $m = m_0 + \Sigma_2(\not{p} = m)$ at this order. With the full vertex Γ^μ (its factor −ie removed) and the full inverse propagator $\tilde{S}^{-1}(p) = \not{p} - m_0 - \Sigma(p)$, the Ward–Takahashi identity reads $q_\mu\Gamma^\mu(p+q, p) = \tilde{S}^{-1}(p+q) - \tilde{S}^{-1}(p)$. Take α = 1/137.036, m = 0.511 MeV and the Planck mass 1.22 × 10¹⁹ GeV.

(a) DERIVE **[2]**

Derive, by counting powers of loop momentum, the superficial degree of divergence D of a QED diagram in terms of the number B of its external photons and the number F of its external electron lines, and hence the value of D for the electron self-energy.

(b) EXPLAIN **[2]**

Explain why, although D = 1, the mass shift can diverge only logarithmically and must be proportional to m, and contrast this with what happens to a scalar's mass.

(c) SHOW THAT **[4]**

Show that $\Sigma_2(p) = [e^2\mu^{\varepsilon}/(4\pi)^{(d/2)}]\int_0^1 dx[(4-\varepsilon)m - (2-\varepsilon)x\not{p}]\Gamma(2-d/2)\Delta^{(d/2-2)}$, with $\Delta = (1-x)m^2 + x\mu_\gamma^2 - x(1-x)p^2$.

(d) DERIVE **[3]**

Derive the self-mass $\delta m = m - m_0$ at one loop, showing that it is proportional to m, logarithmic and finite as $\mu_\gamma \to 0$, and find its finite part.

(e) COMPARE **[2]**

Interpreting the pole as $\ln(\Lambda^2/m^2)$ for a cutoff Λ, so that $\delta m \approx (3\alpha/4\pi)m\ln(\Lambda^2/m^2)$, compare $\delta m/m$ for Λ equal to the Planck mass with the shift of order $\alpha\Lambda$ that a linearly divergent electromagnetic mass would give.

(f) SHOW THAT **[2]**

From the Ward–Takahashi identity show that $Z_1 = Z_2$, where near the mass shell $\tilde{S}^{-1}(p) \to Z_2^{-1}(\not{p} - m)$ and, between on-shell spinors as $q \to 0$, $\Gamma^{\mu}(p, p) \to Z_1^{-1}\gamma^{\mu}$.

19 CORE COURSE

FREE RESPONSE · DERIVE, CONFRONT WITH MEASUREMENT AND CANCEL THE INFRARED · DISCRIMINATING · UQFTI-14.01 · UQFTI-14.02 · UQFTI-14.03 · UQFTI-14.04 · UQFTI-14.07 · UQFTI-14.08 · UQFTI-14.09 **[16]**

Work in natural units with the metric (+, −, −, −). Between on-shell spinors the full electron–photon vertex is ū(p′)Γ^μu(p), with q = p′ − p and σ^μν = (i/2)[γ^μ, γ^ν]; the Gordon identity reads ū(p′)γ^μu(p) = ū(p′)[(p′ + p)^μ/2m + iσ^μνq_ν/2m]u(p). At one loop in Feynman gauge, after Feynman parameters, the shift $\ell = k + yq - zp$ and use of the Dirac equation, the correction is ū(p′)δΓ^μu(p) = $2ie^2\int d^4\ell/(2\pi)^4 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)(2/D^3)$ū(p′)N^μu(p), with $D = \ell^2 - \Delta + i\varepsilon$, $\Delta = -xyq^2 + (1-z)^2m^2$ and N^μ = $\gamma^\mu[-\ell^2/2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2] + (i\sigma^{\mu\nu}q_\nu/2m)[2m^2z(1-z)]$. Use $\int d^4\ell/(2\pi)^4\, 1/(\ell^2-\Delta)^3 = -i/(2(4\pi)^2\Delta)$. The measured electron anomaly is $a_e = 1.159\,652\,180\,59(13) \times 10^{-3}$, and the QED series is $a_e = \alpha/2\pi + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \ldots$, with $C_2 = -0.328\,478\,965$, $C_3 = 1.181\,241\,456$ and $C_4 = -1.912\,245\,764$; in this question take $\alpha^{-1} = 137.035\,999$, from atom-recoil measurements. For elastic scattering of an electron of energy E through angle θ by a recoil-free nucleus, $-q^2 = 4E^2\sin^2(\theta/2)$; the order-α virtual correction multiplies the Born cross-section by $1 - (\alpha/\pi) f_{IR}(q^2)\ln(-q^2/\mu_\gamma^2)$, μ_γ being a photon mass, and the probability of emitting one soft photon with energy between μ_γ and E_l is $(\alpha/\pi) f_{IR}(q^2)\ln(E_l^2/\mu_\gamma^2)$, where for $-q^2 \gg m^2$, $f_{IR}(q^2) = \ln(-q^2/m^2) - 1$. Take m = 0.511 MeV.

(a) JUSTIFY **[2]**

Justify writing the on-shell vertex as $\Gamma^\mu = \gamma^\mu F_1(q^2) + (i\sigma^{\mu\nu}q_\nu/2m)F_2(q^2)$.

(b) SHOW THAT **[2]**

Show that in a slowly varying magnetic field the vertex gives the electron a magnetic moment with $g = 2[F_1(0) + F_2(0)]$.

(c) DERIVE **[3]**

Derive $F_2(q^2)$ as an integral over Feynman parameters, evaluate $F_2(0)$, and explain why F_2 needs neither an ultraviolet nor an infrared regulator.

(d) CALCULATE **[3]**

Calculate the residual, the measured a_e minus the QED series truncated after one, two and three terms, and compare each residual with the size of the next term.

(e) ESTIMATE **[1]**

Estimate the fractional precision with which this measurement of a_e would determine α if the QED series were known exactly, stating your assumption.

(f) SHOW THAT **[2]**

Show that in the cross-section measured by a detector that cannot register photons below E_l the photon mass cancels at order α, leaving the factor $1 - (\alpha/\pi) f_{IR}(q^2) \ln(-q^2/E_l^2)$.

(g) CALCULATE **[2]**

Calculate this correction factor for 4.0 GeV electrons scattered through 20° with $E_l = 20$ MeV, and compare it with its exponentiated form $\exp[-(\alpha/\pi) f_{IR} \ln(-q^2/E_l^2)]$.

(h) PREDICT **[1]**

Predict what happens, as $E_l \to 0$, to the rate of scattering with no photon above E_l and to the inclusive rate.

MARK SCHEME

University Quantum Field Theory I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

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1 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07 **[1]**

For a free real scalar field of mass m, in natural units $\hbar = c = 1$ with the metric (+, −, −, −), the Feynman propagator is $D_F(x - y) = \int d^4p/(2\pi)^4\, i\, e^{-ip\cdot(x-y)}/(p^2 - m^2 + i\varepsilon)$, with $E_p = \sqrt{(|p|^2 + m^2)}$ for the three-momentum p and $\varepsilon \to 0^+$. The p^0 integral is done first, by closing the contour in whichever half of the complex p^0 plane makes the exponential decay. Which statement is correct?

KEY **B** **The poles sit at $p^0 = \pm(E_p - i\varepsilon')$; for $x^0 > y^0$ the contour closes in the lower half-plane around $p^0 = +E_p$ and gives $\langle 0|\varphi(x)\varphi(y)|0\rangle$, and for $x^0 < y^0$ it closes in the upper half-plane around $p^0 = -E_p$ and gives $\langle 0|\varphi(y)\varphi(x)|0\rangle$, so the placement of the iε is the time ordering.**

B is correct. Writing $p^2 - m^2 + i\varepsilon = (p^0)^2 - E_p^2 + i\varepsilon$, the zeros are at $p^0 = \pm\sqrt{(E_p^2 - i\varepsilon)} = \pm(E_p - i\varepsilon')$ with $\varepsilon' = \varepsilon/2E_p$: the positive-energy pole lies just below the real axis and the negative-energy pole just above it. For $x^0 > y^0$ the factor $e^{-ip^0(x^0 - y^0)}$ dies away where $\mathrm{Im}\, p^0 < 0$, so the contour closes below, encloses $p^0 = E_p - i\varepsilon'$, and the residue gives $\int d^3p/((2\pi)^3 2E_p)\, e^{-ip\cdot(x-y)} = \langle 0|\varphi(x)\varphi(y)|0\rangle$; for $x^0 < y^0$ it closes above and gives the other order. A describes the retarded propagator, whose poles are both pushed below the axis by $p^0 \to p^0 + i\varepsilon$; closing above for $x^0 < y^0$ then encloses nothing, and it is D_R, not D_F, that vanishes before the source acts. C is the commonest misconception about D_F: outside the light cone it falls as e^{-mr} but does not vanish. Causality lives in the commutator $[\varphi(x), \varphi(y)] = D(x - y) - D(y - x)$, which is zero at spacelike separation because the two orderings there are related by a Lorentz transformation. D closes the contour on the wrong side: for $x^0 > y^0$ the exponential grows in the upper half-plane, and the pole that contributes for that time order is the positive-energy one, a particle created at y and absorbed at x; the negative-energy pole contributes only for $x^0 < y^0$.

U1

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2 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-03.02 · UQFTI-03.06 **[1]**

Two theories are expanded to second order in their coupling, in natural units. In theory (i) a real scalar φ has $L_{int} = -(g/3!)\varphi^3$. In theory (ii), the scalar Yukawa model, a complex scalar ψ and a real scalar φ have $L_{int} = -g\psi^\dagger\psi\varphi$. In each, the order-$g^2$ correction to the φ two-point function $\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle$ contains a one-loop diagram in which x_1 joins one vertex, x_2 joins the other, and the two vertices are linked by two internal lines; in (ii) these form a closed ψ loop. If each diagram is written with its vertex factors and propagators and an overall 1/S, what are the symmetry factors S in (i) and (ii)?

KEY **D** **S = 2 in (i) and S = 1 in (ii)**

In (i) the second-order term carries $(1/2!)(1/3!)^2$. The external points can be attached to the two vertices in 2 ways, x_1 can join any of the 3 fields at its vertex and x_2 any of the 3 at the other, and the two remaining fields at one vertex pair with the two at the other in 2 ways: $2 \times 3 \times 3 \times 2 = 36$ contractions against $2! \times 3! \times 3! = 72$, leaving ½, so S = 2, the automorphism that swaps the two internal lines. In (ii) the 1/2! is cancelled by the 2 ways of assigning x_1 and x_2 to the vertices, there is no 1/3! to cancel, and ψ at each vertex must contract with $\psi^\dagger$ at the other, which can be done in only one way, so S = 1 (D). A treats the two ψ propagators like the two φ lines of (i), but exchanging them reverses the arrows of ψ number, so it is not a symmetry of the diagram: what were two pairings of identical fields in (i) is here a single $\psi\psi^\dagger$ and $\psi^\dagger\psi$ contraction. B forgets that x_2, like x_1, can join any of the 3 fields at its vertex: $2 \times 3 \times 2 = 12$ contractions against 72 leaves 1/6, as though one vertex kept its whole 1/3!; counting the 3 choices for x_2 as well gives 36 and leaves only ½. C assumes that the 1/n! of a φ^n coupling always cancels completely, which it does for tree diagrams; a loop whose two lines can be exchanged keeps a ½.

U2

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3 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-04.02 · UQFTI-04.03 **[1]**

In φ⁴ theory, L = ½∂_μφ∂^μφ − ½m₀²φ² − (λ/4!)φ⁴ with 0 < λ ≪ 1, the single particle has physical mass m. The Källén–Lehmann representation writes the exact two-point function as ⟨Ω|Tφ(x)φ(y)|Ω⟩ = ∫₀^∞ (dM²/2π) ρ(M²) D_F(x − y; M²), with ρ(M²) = 2πZδ(M² − m²) plus a continuum. Where does the continuum begin, and what bounds Z?

KEY **C** **At M² = 9m², because the symmetry φ → −φ stops φ from connecting the vacuum to states with an even number of particles; 0 ≤ Z ≤ 1, because ρ is non-negative and the canonical commutator fixes ∫(dM²/2π)ρ(M²) = 1.**

C is correct. ρ(M²) collects $|\langle\lambda_0|\varphi(0)|\Omega\rangle|^2$ over the states of invariant mass M. The Lagrangian is invariant under φ → −φ, and so is the vacuum, so φ connects the vacuum only to states of odd particle number: the one-particle pole, then three particles from M² = 9m². The canonical equal-time commutator, applied to the spectral form, gives ∫(dM²/2π)ρ(M²) = 1, that is Z plus the weight of the continuum equals 1, and since every contribution to ρ is a squared modulus, 0 ≤ Z ≤ 1. A takes the generic two-particle threshold, which is right for a single scalar without the symmetry, such as φ³ theory, where φ can create two of its own quanta; in φ⁴ the two-particle matrix element $\langle p_1p_2|\varphi(0)|\Omega\rangle$ vanishes identically. B inverts the sum rule: the continuum takes weight away from the pole, so Z falls below 1, and a Z above 1 would need states of negative norm. D describes QED, where massless photons put the electron pole at the end of a cut and the residue needs an infrared regulator; in φ⁴ every quantum has mass m, so a gap separates the pole from the continuum.

U1

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4 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-06.01 **[1]**

In d spacetime dimensions a dimensionless action with a canonical kinetic term gives a scalar field the mass dimension (d − 2)/2. Which interaction has a dimensionless coupling, so that the theory is renormalisable but not super-renormalisable?

KEY **A** **gφ³ in d = 6**

The coupling of φ^n in d dimensions has mass dimension d − n(d − 2)/2. For φ³ in d = 6 this is 6 − 3 × 2 = 0, so the superficial degree of divergence of a diagram depends only on its external legs and a finite set of counterterms serves at every order: renormalisable (A). B gives 4 − 3 = 1 and C gives 3 − 4 × ½ = 1, positive mass dimensions, so the degree of divergence falls with every extra vertex and only finitely many diagrams diverge at all: super-renormalisable, a stronger property than the one asked for. D gives 4 − 6 = −2, so each vertex raises the degree of divergence by 2, functions with ever more external legs diverge at higher orders, and ever more counterterms are needed: non-renormalisable. φ⁶ is renormalisable in d = 3, where its coupling is dimensionless, as φ⁴ is in d = 4.

U1

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5 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-07.07 **[1]**

A spin-zero boson φ of mass 2.40 GeV decays to a fermion–antifermion pair, the fermion being a Dirac particle of mass 0.90 GeV. Compare a scalar coupling gφψ̄ψ with a pseudoscalar coupling igφψ̄γ⁵ψ of the same real strength g. In both cases Γ = |p| Σ|M|²/(8πm_φ²), with the sum over final spins and |p| the momentum of either fermion in the rest frame of φ. What is Γ_scalar/Γ_pseudoscalar?

KEY **D** **0.44**

With β = √(1 − 4m²/m_φ²) = √(1 − 3.24/5.76) = 0.661, the traces give $\Sigma|M|^2 = g^2\mathrm{Tr}[(\not p_1 + m)(\not p_2 - m)] = 4g^2(p_1\cdot p_2 - m^2) = 2g^2m_\varphi^2\beta^2$ for the scalar coupling and $g^2\mathrm{Tr}[(\not p_1 + m)(\not p_2 + m)] = 4g^2(p_1\cdot p_2 + m^2) = 2g^2m_\varphi^2$ for the pseudoscalar, the γ⁵ on each side of $\not p_2 - m$ turning it into $-\not p_2 - m$. The phase space is the same, so the ratio is β² = 0.4375 ≈ 0.44 (D): the scalar rate goes as β³ and the pseudoscalar rate as β. A fermion pair has parity (−1)^(L+1), so a 0⁺ boson must produce it in a p wave, suppressed near threshold, and a 0⁻ boson in an s wave. A stops at one power of β, as though the p-wave suppression entered the rate linearly rather than through the square of an amplitude proportional to β. B supposes that γ⁵ changes only a phase of the amplitude; it changes the relative sign of $\not p_2$ and m inside the trace, which is the difference between $p_1\cdot p_2 - m^2$ and $p_1\cdot p_2 + m^2$. C gives the pseudoscalar the p wave and the scalar the s wave, the parity assignment reversed.

U3

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6 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-08.04 · UQFTI-08.06 **[1]**

In a general covariant gauge the photon propagator is $-i[g_{\mu\nu} - (1-\xi)k_\mu k_\nu/k^2]/(k^2 + i\varepsilon)$. In the tree amplitude for $e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)$, the virtual photon carries $k = p_1 + p_2 = p_3 + p_4$, with $k^2 = s$. Why does the amplitude not depend on ξ?

KEY **B** **Because k_μ contracted into the electron current gives $\bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = (m_e - m_e)\bar{v}(p_2)u(p_1) = 0$ by the Dirac equation, and likewise on the muon line, so the $k_\mu k_\nu$ term vanishes for any ξ.**

B is correct. Between on-shell spinors $\not{p}_1 u(p_1) = m_e\, u(p_1)$ and $\bar{v}(p_2)\not{p}_2 = -m_e\, \bar{v}(p_2)$, so $k_\mu\, \bar{v}(p_2)\gamma^\mu u(p_1) = 0$ exactly, with the masses kept; on the muon line $\bar{u}(p_3)(\not{p}_3 + \not{p}_4)v(p_4) = (m_\mu - m_\mu)\bar{u}(p_3)v(p_4) = 0$ likewise. Each current is conserved, and any term proportional to k_μ or k_ν drops out whatever ξ multiplies it. A confuses the internal photon with an external one: here $k^2 = s$, far from zero, and it is the conservation of the currents, not the mass shell, that removes the term. C argues in a circle: that no physical amplitude depends on ξ is what has to be shown, and it is current conservation that shows it, here at tree level and, through the Ward–Takahashi identity, to all orders. D gets the mechanism wrong: the term is not suppressed by m_e^2/s but vanishes exactly for any masses, because each current is conserved on its own, so nothing has to be neglected.

U1

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7 CORE COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-09.07 **[1]**

At $\sqrt{s} \gg m_\mu$ the reaction $e^+e^- \to \mu^+\mu^-$ by one-photon exchange proceeds through four helicity combinations, because the vector coupling joins a right-handed fermion only to a left-handed antifermion, and vice versa. The e^- moves along $+z$, and θ is the angle between the outgoing μ^- and the incoming e^-. How does the amplitude for $e^-_R\, e^+_L \to \mu^-_L\, \mu^+_R$ depend on θ, and why?

KEY **A** **As $(1 - \cos\theta)$: the initial pair has angular momentum +1 along the beam and the final pair −1 along the μ^- direction, so the amplitude is proportional to the rotation matrix element $d^1_{1,-1}(\theta)$, which vanishes in the forward direction.**

A is correct. With the e^- right-handed along $+z$ and the e^+ left-handed along $-z$, both spins point along $+z$, so $J_z = +1$. The μ^- is left-handed along its direction $\hat{n}$ and the μ^+ right-handed along $-\hat{n}$, so both spins point along $-\hat{n}$, a projection -1 on $\hat{n}$. The amplitude is therefore proportional to $d^1_{1,-1}(\theta) = (1 - \cos\theta)/2$, which vanishes at $\theta = 0$, where a projection $+1$ cannot become -1 along the same axis, and is largest backwards. Squaring and summing the four combinations gives $2[(1 + \cos\theta)^2 + (1 - \cos\theta)^2] \propto 1 + \cos^2\theta$. B gives the dependence of $e^-_R\, e^+_L \to \mu^-_R\, \mu^+_L$, in which the final spins also point along $+\hat{n}$; it credits the vertices with a correlation that comes from angular momentum. C misplaces the chirality rule: it holds along each fermion line, so it fixes the e^+ helicity once the e^-'s is chosen, but the muons are on a different line, and the photon couples to $\mu^-_L\, \mu^+_R$ as readily as to $\mu^-_R\, \mu^+_L$. D forgets that the initial and final pairs have definite projections ± 1 along their own axes, so the amplitudes are d^1 functions with $m = \pm 1$, $(1 \pm \cos\theta)/2$; a $\sin\theta$ dependence would need a projection 0, which helicity conservation at high energy forbids.

U1

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8 CORE COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-11.08 · UQFTI-11.09 **[1]**

A positronium level with orbital angular momentum L and total spin S has charge-conjugation parity C = (−1)^(L+S), and a state of n photons has C = (−1)^n. Landau and Yang's theorem, quoted, forbids a state of total angular momentum J = 1 from decaying into two photons. Which level can annihilate directly into two photons?

KEY **D** **2^3P_2**

Two photons have C = +1, so the level needs L + S even and, by Landau and Yang, J ≠ 1. 2^3P_2 has L = 1, S = 1, C = +1 and J = 2, so J^PC = 2^{++} and two-photon annihilation is allowed (D), though in practice this level decays radiatively to lower levels far faster than it annihilates, because annihilation needs the pair at contact, where a P-wave function vanishes. A is ortho-positronium, C = (−1)^(0+1) = −1, which needs an odd number of photons and decays into three; that is why its lifetime, 142 ns, is about a thousand times longer than para-positronium's 125 ps. B has C = (−1)^(1+0) = −1, like ortho-positronium, although S = 0: the orbital angular momentum contributes its own sign. C has C = +1 and passes the charge-conjugation test, but J = 1, and Landau and Yang's theorem, which follows from Bose symmetry and rotational invariance for two real photons, forbids it.

U1

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9 CORE COURSE

MULTIPLE CHOICE · RECALL · UQFTI-12.06 · UQFTI-12.07 **[1]**

In renormalised QED with on-shell conditions the physical charge is $e = Z_2 Z_3^{1/2} e_0 / Z_1$, where Z_1 renormalises the vertex, Z_2 the charged field and Z_3 the photon field. A proton's coupling to the photon is dressed by strong interactions that the electron's is not, yet the hydrogen atom is neutral to better than one part in 10^{20}. Why are the renormalised charges of the electron and the proton exactly opposite, given opposite bare charges?

KEY **A** **The Ward–Takahashi identity gives $Z_1 = Z_2$ for every charged field, whatever its other interactions, so $e = Z_3^{1/2} e_0$, and Z_3 comes from the photon's own self-energy, which is common to every particle the photon meets.**

A is correct. The identity follows from current conservation alone and forces the vertex and field renormalisations of any charged particle to cancel, so charge renormalisation is carried entirely by Z_3, the vacuum polarisation of the photon, which acts in the same way on every charge. B is wrong in fact: strong interactions dress the proton's vertex heavily, giving it a charge radius near 0.84 fm and an anomalous moment of 1.79 nuclear magnetons; what they cannot change is $F_1(0) = 1$, and that is the content of $Z_1 = Z_2$. C confuses the form of the coupling with its size: the electron's anomalous moment is 0.00116 and the proton's 1.79, so the vertex corrections are wildly different, and still the charges agree. D puts the conclusion into the definition: QED has one charge parameter, fixed by Thomson scattering, and that every other particle's charge then comes out as an exact multiple of it is a prediction, not a convention.

U1

10
CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-13.01 · UQFTI-13.02 **[1]**

At one loop the electron loop gives the photon a self-energy $i\Pi^{\mu\nu}(q)$. Why is it computed in dimensional regularisation rather than with a sharp cutoff Λ on the Euclidean loop momentum?

KEY **C** **A sharp cutoff is not invariant under shifts of the loop momentum and breaks gauge invariance: $\Pi^{\mu\nu}$ acquires a term proportional to $g^{\mu\nu}\Lambda^2$ that is not of the transverse form $(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ and would give the photon a mass, whereas dimensional regularisation keeps $\Pi^{\mu\nu}$ transverse.**

C is correct. With a sharp cutoff the shift of loop momentum used to complete the square changes the integral by a surface term, and the result contains a piece of order $(\alpha/\pi)\Lambda^2 g^{\mu\nu}$. Contracting it with q_μ does not give zero, so the Ward identity fails, and in the resummed propagator that piece acts as a photon mass squared of order $\alpha\Lambda^2$. Dimensional regularisation has no surface terms, respects the shift and the gauge symmetry, and gives $\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ exactly, so the photon stays massless. A has the regulators backwards: dimensional regularisation does not make the loop finite but turns its logarithmic divergence into a $1/\varepsilon$ pole, which Z_3 absorbs, and the transverse part of the cutoff result diverges only logarithmically too; the trouble with the cutoff is the extra term that is not transverse. B forgets that the electron in the loop is massive: $\Pi(q^2)$ is finite as $q^2 \to 0$, and QED's infrared problems come from massless photons on external or soft lines, not from this loop. D is exactly what fails for the sharp cutoff.

U1

11

CORE COURSE

STRUCTURED · ROUTINE · UQFTI-02.01 · UQFTI-04.09 · UQFTI-05.03 · UQFTI-05.06 **[9]**

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model a complex scalar ψ of mass m_ψ and a real scalar φ of mass m_φ interact through $L_{int} = -g\psi\dagger\psi\varphi$. Take $m_\varphi = 0.80$ GeV, $m_\psi = 0.25$ GeV and $g = 0.40$ GeV, with $\hbar = 6.582 \times 10^{-25}$ GeV s and $\hbar c = 0.1973$ GeV fm. For a particle of mass m_φ at rest decaying to two distinguishable spinless products, $\Gamma = |p||M|^2/(8\pi m_\varphi^2)$, where M is the invariant amplitude and $|p|$ the momentum of either product.

(a) STATE · U1 **[1]**

State the mass dimension of g, and whether the interaction is renormalisable, super-renormalisable or non-renormalisable.

1 In four dimensions ψ and φ each have mass dimension 1, so $[g] = 4 - 3 = 1$; a coupling of positive mass dimension makes the interaction super-renormalisable

(b) SHOW THAT · U2 **[2]**

Show that, at tree level, $\Gamma(\varphi \to \psi\bar{\psi}) = (g^2/16\pi m_\varphi)\sqrt{(1 - 4m_\psi^2/m_\varphi^2)}$.

1 One vertex and no propagator: contracting φ with the initial particle and ψ, $\psi\dagger$ with the outgoing particle and antiparticle gives $iM = -ig$, so $|M|^2 = g^2$, with no factor ½ because ψ and $\bar{\psi}$ are distinguishable

2 $|p| = \tfrac{1}{2}\sqrt{(m_\varphi^2 - 4m_\psi^2)} = (m_\varphi/2)\sqrt{(1 - 4m_\psi^2/m_\varphi^2)}$, so $\Gamma = |p|g^2/(8\pi m_\varphi^2) = (g^2/16\pi m_\varphi)\sqrt{(1 - 4m_\psi^2/m_\varphi^2)}$

(c) CALCULATE · U3 **[2]**

Calculate Γ in MeV and the mean lifetime of φ in seconds.

1 $\sqrt{(1 - 4m_\psi^2/m_\varphi^2)} = \sqrt{(1 - 0.25/0.64)} = 0.781$, so $\Gamma = (0.16\ \text{GeV}^2 \times 0.781)/(16\pi \times 0.80\ \text{GeV}) = 3.11 \times 10^{-3}$ GeV $= 3.11$ MeV

2 $\tau = \hbar/\Gamma = (6.582 \times 10^{-25}\ \text{GeV s})/(3.11 \times 10^{-3}\ \text{GeV}) = 2.12 \times 10^{-22}$ s

Candidates who divide $\hbar$ by Γ in MeV while keeping $\hbar$ in GeV s are out by a factor of 1000; the check is that $\Gamma/m_\varphi = 0.004$, a narrow resonance, so τ must be a few hundred times $1/m_\varphi = 8.2 \times 10^{-25}$ s.

(d) DETERMINE · U2 **[3]**

Two ψ particles scatter at low speed. From the t-channel φ-exchange diagram, determine the potential V(r) between them, allowing for the relativistic normalisation of the states, and state whether it attracts and what its range is in femtometres.

1 The t-channel diagram gives $iM = (-ig)^2\, i/(q^2 - m_\varphi^2)$; for slow particles $q^0 \approx 0$ and $q^2 \approx -|q|^2$, so $M = g^2/(|q|^2 + m_\varphi^2)$, the u-channel diagram supplying the exchange term that identical bosons require

2 Relativistic normalisation puts a factor $2m_\psi$ on each of the two particles, so the Born amplitude $M/(2m_\psi)^2$ equals $-\tilde{V}(q)$, giving $\tilde{V}(q) = -(g^2/4m_\psi^2)/(|q|^2 + m_\varphi^2)$ and $V(r) = -(g^2/4m_\psi^2)e\hat{}(-m_\varphi r)/(4\pi r)$

3 V is negative, so the force attracts, with dimensionless strength $g^2/(16\pi m_\psi^2) = 0.16/(16\pi \times 0.0625) = 0.051$ in the place of α, and range $1/m_\varphi = 0.1973$ GeV fm$/0.80$ GeV $= 0.25$ fm

The factor $1/(2m_\psi)^2$ is the part most often lost: without it $\tilde{V}(q)$ is dimensionless, V(r) comes out with the dimension of an energy cubed, too large by the factor $(2m_\psi)^2$, and cannot be compared with the Born amplitude of quantum mechanics.

(e) PREDICT · U1 **[1]**

Predict what becomes of φ, and of the use of the LSZ formula with φ as an external particle, if m_φ were lowered to 0.45 GeV with m_ψ unchanged.

1 With m_φ below $2m_\psi = 0.50$ GeV the decay is kinematically forbidden, so φ becomes stable and its pole is an isolated one-particle pole below the $\psi\bar{\psi}$ continuum at $4m_\psi^2$, as LSZ requires; at $m_\varphi = 0.80$ GeV the pole lies on the continuum and moves off the real axis by $m_\varphi\Gamma$, so φ is not an asymptotic state

12

CORE COURSE

STRUCTURED · DEMANDING · UQFTI-03.02 · UQFTI-05.04 · UQFTI-05.07 · UQFTI-05.08 · UQFTI-06.09 **[11]**

Work in natural units with the metric (+, −, −, −). A real scalar φ of mass m = 0.40 GeV has L_int = −(λ/4!)φ⁴ with λ = 0.80. Two φ particles collide head-on with centre-of-mass energy √s = 2.0 GeV. For 2 → 2 scattering of equal masses in the centre-of-mass frame, dσ/dΩ = |M|²/(64π²s). Two-body Lorentz-invariant phase space for distinguishable particles is ∫dΠ₂ = |p_cm|/(4π√s), with |p_cm| the momentum of either particle. Use (ħc)² = 0.3894 GeV² mb. In minimal subtraction the one-loop β function of the theory is β(λ) = μ dλ/dμ = 3λ²/(16π²).

(a) CALCULATE · U3 **[3]**

Calculate the tree-level amplitude M and the total cross-section for φφ → φφ in microbarns.

1. The 4! ways of contracting the four external particles with the four fields at the vertex cancel the 1/4!, so iM = −iλ and M = −λ = −0.80
2. dσ/dΩ = λ²/(64π²s) is isotropic, and the two final particles are identical, so σ = ½ × 4π × λ²/(64π²s) = λ²/(32πs)
3. σ = 0.64/(32π × 4.0 GeV²) = 1.59 × 10⁻³ GeV⁻², and multiplying by 0.3894 GeV² mb gives 6.20 × 10⁻⁴ mb = 0.620 μb

   Candidates who integrate over the full 4π without the factor ½ count every final configuration twice and report 1.24 μb.

(b) SHOW THAT · U2 **[2]**

At order λ² the s-channel one-loop diagram joins p_1 and p_2 at one vertex and p_3 and p_4 at the other, the two vertices being linked by two internal lines. Show by counting Wick contractions that it carries a symmetry factor ½, so that iM_s = ½(−iλ)²∫d⁴l/(2π)⁴ [i/(l² − m² + iε)][i/((k − l)² − m² + iε)], with k = p_1 + p_2.

1. The second-order term carries (1/2!)(1/4!)². The incoming pair can sit at either vertex (2 ways), p_1 and p_2 can take 4 × 3 = 12 choices of field at theirs, p_3 and p_4 12 at the other, and the two remaining fields at one vertex pair with the two at the other in 2 ways: 2 × 12 × 12 × 2 = 576 contractions
2. 576 × (1/2!)(1/4!)² = 576/1152 = ½, so S = 2, the automorphism that exchanges the two internal lines, and each vertex carries −iλ

   Candidates who stop at 2 × 12 × 12 forget that the two internal lines can be formed in two ways and find ¼, which the optical theorem in (c) would then show to be wrong.

(c) SHOW THAT · U2 **[3]**

Show that the imaginary part of the forward amplitude at order λ², which comes from this diagram alone, is Im M = λ²β/(32π) with β = √(1 − 4m²/s), and that it satisfies the optical theorem Im M(forward) = 2E_cm|p_cm|σ_tot at √s = 2.0 GeV.

1. For s > 4m² the two internal lines can go on shell together, and the iε prescription gives the imaginary part by cutting the diagram, replacing each 1/(l² − m² + iε) by −2πiδ(l² − m²) with positive energy; in the forward direction the t- and u-channel loops sit at t = 0 and u = 4m² − s = −3.36 GeV², below their thresholds, and stay real
2. The cut gives 2 Im M = ½λ²∫dΠ₂ = ½λ²|p_cm|/(4π√s), and |p_cm| = β√s/2, so Im M = λ²β/(32π); with β = √(1 − 0.64/4.0) = 0.9165, Im M = 0.64 × 0.9165/(32π) = 5.83 × 10⁻³
3. 2E_cm|p_cm|σ_tot = 2 × 2.0 GeV × 0.9165 GeV × 1.59 × 10⁻³ GeV⁻² = 5.83 × 10⁻³, equal to Im M: the loop's symmetry factor ½ does the work of the ½ for identical particles in σ_tot

   The factor ½ appears twice and must be matched: once as the loop's symmetry factor and once in the identical-particle phase space. A candidate who drops either finds the optical theorem out by a factor of 2 and should read that as a counting error, not as a failure of unitarity.

(d) EXPLAIN · U1 **[1]**

Explain why Im M is finite although the loop integral itself diverges logarithmically.

1 The imaginary part comes only from the region in which both internal particles are on shell with total four-momentum k, a compact two-body phase space; the ultraviolet divergence comes from large loop momenta far off shell, is a real constant independent of s, and is absorbed into δ_λ

A candidate who says the divergence cancels in Im M has the right conclusion for the wrong reason: nothing cancels, the divergence is simply real.

(e) CALCULATE · U3 **[2]**

Taking $\lambda(\mu_0) = 0.80$ at $\mu_0 = 0.40$ GeV in minimal subtraction, calculate λ at μ = 100 GeV from the one-loop β function, and the scale at which the one-loop coupling would diverge.

1 Integrating $\mu\, d\lambda/d\mu = 3\lambda^2/(16\pi^2)$ gives $1/\lambda(\mu) = 1/\lambda(\mu_0) - (3/16\pi^2)\ln(\mu/\mu_0) = 1.250 - 0.01900 \times \ln 250 = 1.250 - 0.105 = 1.145$, so $\lambda(100\ \text{GeV}) = 0.873$

2 $1/\lambda$ reaches zero when $\ln(\mu/\mu_0) = 16\pi^2/(3\lambda(\mu_0)) = 65.8$, at $\mu = 0.40\ \text{GeV} \times e^{65.8} \approx 1.5 \times 10^{28}$ GeV, the Landau pole: far beyond the Planck mass, and long before it the coupling is too large for one loop to be trusted

Candidates who use $\log_{10}$ for the natural logarithm, or $\ln(\mu^2/\mu_0^2)$ with the coefficient for d/d ln μ, are out by factors of 2.3 or 2 in the running.

13

CORE COURSE

STRUCTURED · DEMANDING · UQFTI-09.02 · UQFTI-09.05 · UQFTI-09.06 · UQFTI-09.08 **[12]**

Work in natural units with the metric (+, −, −, −). In QED with an electron and a muon, $e^+e^- \to \mu^+\mu^-$ proceeds at lowest order through one-photon exchange in the s channel, with vertex $-ie\gamma^\mu$ and Feynman-gauge photon propagator $-ig_{\mu\nu}/(q^2 + i\varepsilon)$. Label the momenta $e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)$, neglect the electron mass, keep the muon mass $m_\mu = 0.10566$ GeV, and take $\alpha = e^2/4\pi = 1/137.036$ and $(\hbar c)^2 = 0.3894\ \text{GeV}^2\ \text{mb}$. The tree amplitude is $M = (e^2/s)[\bar v(p_2)\gamma^\mu u(p_1)][\bar u(p_3)\gamma_\mu v(p_4)]$, with $s = (p_1 + p_2)^2$. Use $\Sigma\, u\bar u = \not p + m$ and $\Sigma\, v\bar v = \not p - m$ over spins, $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$; traces of an odd number of gamma matrices vanish.

(a) SHOW THAT · U2 **[3]**

Show that the spin-averaged squared amplitude is $\tfrac14\Sigma|M|^2 = (8e^4/s^2)[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mu^2(p_1\cdot p_2)]$.

1 Casimir's trick turns the spin sums into traces: $\tfrac14\Sigma|M|^2 = (e^4/4s^2)\mathrm{Tr}[\not p_2\gamma^\mu\not p_1\gamma^\nu]\mathrm{Tr}[(\not p_3 + m_\mu)\gamma_\mu(\not p_4 - m_\mu)\gamma_\nu]$

2 The electron trace is $4[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu}(p_1\cdot p_2)]$ and the muon trace $4[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}(p_3\cdot p_4 + m_\mu^2)]$, the mass term coming from $-m_\mu^2\mathrm{Tr}(\gamma_\mu\gamma_\nu)$ and the terms linear in m_μ vanishing as odd traces

3 Contracting, with $g^{\mu\nu}g_{\mu\nu} = 4$, the $(p_1\cdot p_2)(p_3\cdot p_4)$ terms cancel and the product of traces is $32[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mu^2(p_1\cdot p_2)]$, so $\tfrac14\Sigma|M|^2 = (e^4/4s^2) \times 32[\ldots] = (8e^4/s^2)[\ldots]$

Candidates who write the antiparticle spin sum as $\not p + m$ reverse the sign of the mass term, and those who drop $-m_\mu^2\mathrm{Tr}(\gamma_\mu\gamma_\nu)$ lose it. The massless limit, $(s^2/8)(1 + \cos^2\theta)$, cannot catch either slip, because both errors vanish with m_μ; threshold can: at $\beta = 0$ the correct bracket is $s^2/4$ at every angle, whereas the reversed sign makes it vanish and the dropped term halves it.

(b) SHOW THAT · U2 **[3]**

In the centre-of-mass frame, with θ the angle between p_3 and p_1 and $\beta = \sqrt{(1 - 4m_\mu^2/s)}$ the muon speed, show that $d\sigma/d\Omega = (\alpha^2\beta/4s)(2 - \beta^2\sin^2\theta)$, and hence that $\sigma = (4\pi\alpha^2/3s) \times \beta(3 - \beta^2)/2$.

1. With every energy $\sqrt{s}/2$ and muon momentum $\beta\sqrt{s}/2$, $p_1 \cdot p_2 = s/2$, $p_1 \cdot p_3 = p_2 \cdot p_4 = (s/4)(1 - \beta\cos\theta)$ and $p_1 \cdot p_4 = p_2 \cdot p_3 = (s/4)(1 + \beta\cos\theta)$, so the bracket is $(s^2/8)(1 + \beta^2\cos^2\theta) + m_\mu^2 s/2 = (s^2/8)(2 - \beta^2\sin^2\theta)$, using $4m_\mu^2/s = 1 - \beta^2$
2. $\frac{1}{4}\Sigma|M|^2 = e^4(2 - \beta^2\sin^2\theta) = 16\pi^2\alpha^2(2 - \beta^2\sin^2\theta)$, and $d\sigma/d\Omega = (|p_f|/|p_i|)\frac{1}{4}\Sigma|M|^2/(64\pi^2 s)$ with $|p_f|/|p_i| = \beta$ gives $(\alpha^2\beta/4s)(2 - \beta^2\sin^2\theta)$
3. Integrating, $\int(2 - \beta^2 + \beta^2\cos^2\theta)d\Omega = 2\pi[2(2 - \beta^2) + (2/3)\beta^2] = (8\pi/3)(3 - \beta^2)$, so $\sigma = (\alpha^2\beta/4s)(8\pi/3)(3 - \beta^2) = (4\pi\alpha^2/3s)\beta(3 - \beta^2)/2$, which tends to $4\pi\alpha^2/3s$ as $\beta \to 1$

   The ratio $|p_f|/|p_i| = \beta$ is the factor most often left out above threshold, and with it the cross-section's rise from zero at threshold.

(c) CALCULATE · U3 **[2]**

Calculate σ in nanobarns at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, and, at the lower energy, the ratio of σ to the value $4\pi\alpha^2/3s$ that massless muons would give.

1. $4\pi\alpha^2/3 = 2.2306 \times 10^{-4}$, which with 0.3894 GeV² mb is 86.86 nb GeV²; at $\sqrt{s} = 0.300$ GeV, $s = 0.0900$ GeV² and $\beta = \sqrt{(1 - 0.04466/0.0900)} = 0.7098$, so $\beta(3 - \beta^2)/2 = 0.886$ and $\sigma = (86.86/0.0900)$ nb $\times$ 0.886 = 965 nb $\times$ 0.886 = 855 nb
2. At $\sqrt{s} = 3.00$ GeV, $\beta = 0.9975$ and $\beta(3 - \beta^2)/2 = 1.0000$ to four figures, so $\sigma = 86.86/9.00$ nb = 9.65 nb; the ratio at 0.300 GeV is 0.886, an 11% suppression by the muon mass that has disappeared by 3 GeV

   Candidates who forget $(\hbar c)^2$, or use it upside down, are wrong by factors of order 10^6; 86.9 nb divided by s in GeV² is the number to check against.

(d) SKETCH · U1 **[2]**

Sketch on one set of axes the angular distribution at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, each normalised to its own value at $\cos\theta = \pm 1$, that is $(d\sigma/d\Omega)/(d\sigma/d\Omega$ at $\cos\theta = 1)$ against $\cos\theta$ from -1 to $+1$, marking the value of each curve at $\cos\theta = 0$.

1. Both normalised curves equal 1 at $\cos\theta = \pm 1$, are symmetric about $\cos\theta = 0$ and have their minimum there; at 3.00 GeV the curve is $(1 + \cos^2\theta)/2$ to within 0.5%, marked 0.50 at $\cos\theta = 0$
2. At 0.300 GeV the curve is $(2 - \beta^2 + \beta^2\cos^2\theta)/2$ with $\beta^2 = 0.504$, flatter and lying above the 3.00 GeV curve at every angle between the ends, marked $(2 - \beta^2)/2 = 0.75$ at $\cos\theta = 0$; at threshold it would become flat, the pair being produced in an s wave

   A sketch that shows the $1 + \cos^2\theta$ shape at both energies has missed what the mass term does: near threshold the muons are produced mostly in an s wave and the distribution flattens.

(e) DETERMINE · U2 **[2]**

Determine by crossing the spin-averaged $|M|^2$ for $e^-\mu^- \to e^-\mu^-$ at energies where both masses can be neglected, in the Mandelstam variables of that process, and state the sign that crossing introduces.

1. With the masses neglected, $p_1 \cdot p_3 = p_2 \cdot p_4 = -t/2$ and $p_1 \cdot p_4 = p_2 \cdot p_3 = -u/2$, so $\frac{1}{4}\Sigma|M|^2$ for $e^+e^- \to \mu^+\mu^-$ is $2e^4(t^2 + u^2)/s^2$
2. Crossing the incoming e^+ into an outgoing e^- and the outgoing μ^+ into an incoming μ^- turns the old s, t and u into the new t, u and s; each crossed fermion multiplies $\Sigma|M|^2$ by -1 and two are crossed, so the sign is $(-1)^2 = +1$ and $\frac{1}{4}\Sigma|M|^2 = 2e^4(s^2 + u^2)/t^2$, positive as it must be and rising as $1/t^2$ in the forward direction

   Crossing one fermion only would give a negative squared amplitude: the sign rule is what keeps the crossed result physical, and checking the sign of the answer catches the error.

14

CORE
COURSE

STRUCTURED · DEMANDING · UQFTI-10.03 · UQFTI-10.04 · UQFTI-10.07 · UQFTI-10.08 · UQFTI-10.09 **[10]**

Work in natural units with the metric (+, −, −, −). Compton scattering γ(k)e⁻(p) → γ(k′)e⁻(p′) has, at tree level, an s-channel diagram, in which the electron absorbs k before emitting k′, and a u-channel diagram, in which it emits k′ first. For an electron initially at rest and a photon of energy ω, the spin- and polarisation-summed result integrates to the Klein–Nishina total cross-section σ = 2πr_e²{[(1 + κ)/κ²][2(1 + κ)/(1 + 2κ) − ln(1 + 2κ)/κ] + ln(1 + 2κ)/(2κ) − (1 + 3κ)/(1 + 2κ)²}, with κ = ω/m and r_e = α/m. Crossing the same amplitude into e⁺e⁻ → γγ, with the factor ½ for identical photons, gives for a slow pair the spin-averaged σv_rel → πα²/m² as the relative speed v_rel → 0. Take m_e = 0.51100 MeV, m_μ = 105.658 MeV, α = 1/137.036, ħc = 197.327 MeV fm, ħ = 6.582 × 10⁻²² MeV s and 1 b = 100 fm².

(a) EXPLAIN · U1 **[2]**

Replacing the polarisation vector of the incoming photon by its momentum k, explain why neither Compton diagram alone satisfies the Ward identity k_μM^μν = 0, while their sum does.

1. In the s-channel diagram $\not{k}$ stands next to u(p): $(\not{p} + m)\not{k}u(p) = [2p\cdot k - \not{k}(\not{p} - m)]u(p) = 2p\cdot k\, u(p)$ and $k^2 = 0$, so the numerator cancels the denominator $(p + k)^2 - m^2 = 2p\cdot k$ and leaves a term proportional to ū(p′)γ^νu(p); in the u-channel diagram $\not{k}$ stands next to ū(p′), and the same steps with $(p' - k)^2 - m^2 = -2p'\cdot k$ leave the same term with the opposite sign
2. So each diagram alone gives ±e²ū(p′)γ^νu(p), which does not vanish, and only their sum is zero: the Ward identity holds for the gauge-invariant set, and only for the sum can both photon polarisation sums be replaced by −g_μν

   Candidates who check one diagram, find a non-zero result and conclude that the Ward identity fails have missed that it is a statement about the complete amplitude; the two diagrams are not separately gauge invariant.

(b) CALCULATE · U3 **[1]**

Calculate the Thomson cross-section (8π/3)r_e² in barns.

1. r_e = αħc/m_e = 197.327 MeV fm/(137.036 × 0.51100 MeV) = 2.818 fm, so (8π/3)r_e² = 66.52 fm² = 0.6652 b

   The classical radius is α times the reduced Compton wavelength, 386 fm; candidates who put that wavelength itself in place of r_e are out by 137² in the cross-section.

(c) CALCULATE · U3 **[3]**

A 511 keV photon from positron annihilation enters the scintillator of a positron-emission tomograph. Calculate the Klein–Nishina cross-section per electron in barns, its ratio to the Thomson value, and the largest kinetic energy the photon can give to a free electron at rest.

1. κ = 0.511/0.51100 = 1.000, so the bracket is 2(4/3 − ln 3) + (ln 3)/2 − 4/9 = 0.4694 + 0.5493 − 0.4444 = 0.5743, and σ = 2π(2.818 fm)² × 0.5743 = 28.65 fm² = 0.287 b
2. σ/σ_T = 0.287/0.665 = 0.431: at κ = 1 the cross-section has already fallen to less than half its classical value
3. The largest transfer is in back-scattering, θ = π: ω′ = ω/(1 + 2κ) = 170.3 keV, so T_max = ω − ω′ = 2κω/(1 + 2κ) = 340.7 keV, the Compton edge of the 511 keV line

   Candidates who evaluate the bracket with $\log_{10}$ obtain a cross-section larger than Thomson's, which should stop them at once: Klein–Nishina never exceeds the Thomson value.

(d) DETERMINE · U3 **[3]**

True muonium is the atom formed by a μ^+ and a μ^-. Treating it as a hydrogen-like atom bound by the Coulomb potential, determine the lifetime of its para ground state, 1^1S_0, against annihilation into two photons, which is its only significant decay.

1. Only the spin singlet, one of the four spin states, has C = +1 and can annihilate into two photons, so it carries the whole of the spin-averaged rate and its σv_rel is four times the average: $4\pi\alpha^2/m_\mu^2$
2. The rate is $\Gamma = |\psi(0)|^2 \sigma v_{rel}$; with the reduced mass $m_\mu/2$ the Bohr radius is $a = 2/(m_\mu \alpha)$, so $|\psi(0)|^2 = 1/(\pi a^3) = (m_\mu\alpha)^3/(8\pi)$ and $\Gamma = (m_\mu\alpha)^3/(8\pi) \times 4\pi\alpha^2/m_\mu^2 = \alpha^5 m_\mu/2$
3. $\Gamma = (1/137.036)^5 \times 105.658\ \text{MeV}/2 = 2.069 \times 10^{-11} \times 52.83\ \text{MeV} = 1.093 \times 10^{-9}\ \text{MeV}$, so $\tau = \hbar/\Gamma = (6.582 \times 10^{-22}\ \text{MeV s})/(1.093 \times 10^{-9}\ \text{MeV}) = 6.02 \times 10^{-13}\ \text{s} = 0.602\ \text{ps}$, para-positronium's 124.5 ps scaled by m_e/m_μ

   Two factors are lost most often: the 4 that comes from the spin singlet carrying the whole two-photon rate, and the reduced mass, which makes the atom's radius 2/(mα) rather than 1/(mα) and changes $|\psi(0)|^2$ by a factor of 8.

(e) EXPLAIN · U1 **[1]**

Explain why the two Compton diagrams enter with the same sign, whereas the two diagrams of Bhabha scattering, $e^+e^- \to e^+e^-$, enter with a relative minus sign.

1. The Compton diagrams differ only in which photon field is contracted with which external photon, and photon operators commute, so no sign arises; Bhabha's annihilation and exchange diagrams differ in the order in which the fermion operators are contracted with the external states, and moving one anticommuting operator past another brings a factor −1

   Candidates who attribute the sign to the s- and t-channel propagators having opposite signs confuse a kinematic factor with the statistics of the fields.

15

CORE COURSE

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Work in natural units $\hbar = c = 1$ with the metric (+, −, −, −). Charge conjugation is a unitary operator C on the Fock space of QED. On the Dirac field it acts as $C\psi(x)C^{-1} = M_C\bar{\psi}^T(x)$ and $C\bar{\psi}(x)C^{-1} = -\psi^T(x)M_C^{-1}$, where ^{T} is the transpose in spinor indices and the 4 × 4 matrix M_C satisfies $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^T$; on the photon field $CA_\mu(x)C^{-1} = -A_\mu(x)$. Write $(-1)^\mu = +1$ for $\mu = 0$ and −1 for $\mu = 1, 2, 3$. Under parity the Hermitian bilinear $i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi$ is multiplied by $-(-1)^\mu(-1)^\nu$ and $F_{\mu\nu}$ by $(-1)^\mu(-1)^\nu$; under time reversal, which is antiunitary, they are multiplied by $(-1)^\mu(-1)^\nu$ and $-(-1)^\mu(-1)^\nu$ respectively; under C both are multiplied by −1. Each is evaluated at the transformed point. An electric dipole moment d of the electron would add $L_{EDM} = -(d/2)(i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi)F_{\mu\nu}$ to the Lagrangian, with d real, whose non-relativistic limit is the energy $-d\sigma\cdot E$. An experiment on polar molecules has found $|d| < 4.1 \times 10^{-30}$ e cm. Take $m_e = 0.511$ MeV and $\hbar c = 1.973 \times 10^{-11}$ MeV cm.

(a) SHOW THAT · U2 **[3]**

Show that the current $\bar{\psi}\gamma^\mu\psi$ is odd under C while $\bar{\psi}\psi$ is even, identifying the step at which the anticommutation of the fields supplies a sign, and say what would become of the electron's coupling to the photon if the Dirac field were quantised with commutators.

1. $C(\bar{\psi}\gamma^\mu\psi)C^{-1} = -\psi^T M_C^{-1}\gamma^\mu M_C\bar{\psi}^T = +\psi^T(\gamma^\mu)^T\bar{\psi}^T = \psi_a(\gamma^\mu)_{ba}\bar{\psi}_b$, using $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^T$
2. Reordering $\psi_a\bar{\psi}_b = -\bar{\psi}_b\psi_a$, by anticommutation up to a c-number that normal ordering removes, gives $-\bar{\psi}_b(\gamma^\mu)_{ba}\psi_a = -\bar{\psi}\gamma^\mu\psi$, odd; for $\bar{\psi}\psi$ the same steps give $-\psi^T M_C^{-1} M_C\bar{\psi}^T = -\psi_a\bar{\psi}_a = +\bar{\psi}\psi$, even
3. With commuting fields the reordering would bring no sign, the current would be C-even, particle and antiparticle would carry the same charge, and the coupling $-e\bar{\psi}\gamma^\mu\psi A_\mu$ to the C-odd photon could not be C-invariant: the C-odd current, and with it the positron's opposite charge, rests on anticommutation

   Candidates who move ψ past $\bar{\psi}$ without a sign find the current even and must then conclude, wrongly, that QED violates C. The sign from anticommutation and the sign in $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^T$ are two different signs, and both are needed.

(b) DETERMINE · U2 **[2]**

Determine whether L_EDM is even or odd under each of P, C, T and CP, and check that it is even under CPT.

1. Under P the signs multiply to $-(-1)^{\mu}(-1)^{\nu} \times (-1)^{\mu}(-1)^{\nu} = -1$, since $[(-1)^{\mu}(-1)^{\nu}]^2 = 1$; under T to $(-1)^{\mu}(-1)^{\nu} \times [-(-1)^{\mu}(-1)^{\nu}] = -1$, the real coefficient $-d/2$ being untouched by the antiunitary T; under C to $(-1)(-1) = +1$: L_EDM is P-odd, T-odd and C-even
2. Hence it is CP-odd, and under CPT the signs multiply to $(-1)(+1)(-1) = +1$, even, as every Hermitian, Lorentz-invariant local term must be; the non-relativistic form $-d\sigma\cdot E$ agrees, the spin being P-even and T-odd and E being P-odd and T-even

Candidates who let T complex-conjugate the coefficient as well count the i twice: it is already inside the Hermitian bilinear whose sign is given, and d is real.

(c) EXPLAIN · U1 **[2]**

Explain, both from C invariance and from the diagrams, why the sum of the one-loop electron-loop diagrams with three external photons vanishes in QED, and what this means for the counterterms QED needs.

1. QED is C-invariant and A_μ is C-odd, so $\langle\Omega|TA_\mu A_\nu A_\rho|\Omega\rangle = (-1)^3\langle\Omega|TA_\mu A_\nu A_\rho|\Omega\rangle = 0$; diagrammatically, the two orientations of the electron loop are related by transposing the trace with M_C, which turns each γ^{μ} into $-(\gamma^{\mu})^{T}$ and reverses each propagator, so they differ by $(-1)^3$ and cancel in pairs (Furry's theorem)
2. The three-photon function, superficially divergent with $D = 4 - B = 1$, therefore needs no counterterm and no A^3 term is generated; with gauge invariance this leaves the electron self-energy, the photon self-energy and the vertex as QED's divergent functions

A common slip is to say that each loop vanishes separately; only the two orientations together cancel, and for an even number of photons the same pairing doubles the result instead.

(d) ESTIMATE · U4 **[2]**

A new interaction at a mass scale Λ could generate d at one loop. Estimate the lowest Λ consistent with the experimental bound, stating your assumptions.

1. Assume new particles of mass Λ coupling to the electron with couplings g of order 1, a CP-violating phase of order 1, the photon attached with charge e, a loop factor $g^2/16\pi^2$ and one power of m_e for the chirality flip that $\sigma^{\mu\nu}\gamma^5$ requires, so that $d \approx e\, m_e/(16\pi^2\Lambda^2)$
2. In natural units $d/e < 4.1 \times 10^{-30}$ cm$/(1.973 \times 10^{-11}$ MeV cm$) = 2.08 \times 10^{-19}$ MeV^{-1}, so $\Lambda^2 > 0.511$ MeV$/(16\pi^2 \times 2.08 \times 10^{-19}$ MeV$^{-1}) = 1.6 \times 10^{16}$ MeV^2 and $\Lambda > 1.2 \times 10^8$ MeV $\approx 10^5$ GeV, about 100 TeV, beyond the direct reach of any collider

An estimate earns nothing without its assumptions. A candidate who writes $d \sim e/\Lambda$, without the chirality flip and the loop factor, finds about 5×10^{15} GeV, a scale that means nothing here.

(e) PREDICT · U1 **[1]**

Predict, from CPT invariance alone, the electric dipole moment of the positron if the electron's were found to be d.

1. CPT relates the electron to the positron with every charge and moment reversed, so the positron's dipole moment is $-d$, equal in size and opposite in sign, just as its magnetic moment is opposite to the electron's; a measured difference in size would violate CPT

Candidates who reason from C alone predict the same relation but cannot defend it: C is not a symmetry of nature, being broken maximally by the weak interaction. Only CPT, which holds for every local, Lorentz-invariant, Hermitian theory, is guaranteed to survive, and it alone fixes the positron's moment.

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16 CORE COURSE

STRUCTURED · DISCRIMINATING · UQFTI-13.08 · UQFTI-15.02 · UQFTI-15.03 · UQFTI-15.04 · UQFTI-15.05 · UQFTI-15.07 · UQFTI-15.08 **[12]**

Work in natural units with the metric (+, −, −, −). An electron of mass m bound to a nucleus of charge Ze, treated as infinitely heavy, feels at lowest order the Coulomb potential $V(r) = -Z\alpha/r$, whose Fourier transform is $\tilde{V}(q) = -4\pi Z\alpha/|q|^2$. The renormalised one-loop vertex, computed with a photon mass μ_γ, has to first order in the spacelike momentum transfer $q^2 = -|q|^2$ the form factors $F_1(q^2) = 1 + (\alpha/3\pi)(q^2/m^2)[\ln(m/\mu_\gamma) - 3/8]$ and $F_2(0) = \alpha/2\pi$. For an ns state $|\psi_{ns}(0)|^2 = (Z\alpha m)^3/(\pi n^3)$, and states with $l \geq 1$ vanish at the origin. Write $U = 4\alpha(Z\alpha)^4 m/(3\pi n^3)$. In these units, at order $\alpha(Z\alpha)^4$: the F_2 (Pauli) term shifts ns levels by $+(3/8)U$ and $2P_{1/2}$ by $-(1/8)U$; the Uehling (vacuum-polarisation) term shifts ns levels by $-(1/5)U$; Bethe's non-relativistic treatment of photons of energy below a cutoff K shifts ns levels by $U \ln(K/\langle E\rangle_{ns})$ and $2P_{1/2}$ by $-U \ln(\langle E\rangle_{2p}/(Z^2\mathrm{Ry}))$, where $Z^2\mathrm{Ry} = (Z\alpha)^2 m/2$ and the Bethe logarithms are $\ln(\langle E\rangle_{2s}/(Z^2\mathrm{Ry})) = 2.8118$ and $\ln(\langle E\rangle_{2p}/(Z^2\mathrm{Ry})) = -0.0300$. Matching across $(Z\alpha)^2 m \ll K \ll m$ replaces $\ln(m/\mu_\gamma)$ in the high-energy part by $\ln(m/2K) + 5/6$. For hydrogen-like helium, $^4He^+$ ($Z = 2$), take $\alpha = 1/137.036$, $m = 0.511$ MeV and, for $n = 2$, $U = 2170$ MHz, reduced-mass corrections neglected. The same assembly for hydrogen gives 1051.9 MHz for the $2S_{1/2}$–$2P_{1/2}$ splitting against the measured 1057.8 MHz; for He^+ the measured splitting is 14 041 MHz.

(a) SHOW THAT · U2 **[3]**

Show that the q^2 term of F_1 adds to the Coulomb potential a contact term $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, and that this raises an ns level by $U[\ln(m/\mu_\gamma) - 3/8]$ while leaving P levels unshifted.

1. In a static field the vertex multiplies the Coulomb amplitude by $F_1(q^2)$, F_2 giving the separate Pauli term, so $\tilde{V}(q) = -(4\pi Z\alpha/|q|^2)F_1(-|q|^2) = -4\pi Z\alpha/|q|^2 + (4\pi Z\alpha)(\alpha/3\pi m^2)[\ln(m/\mu_\gamma) - 3/8]$
2. The correction does not depend on q, so it transforms to a delta function, $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, positive, a repulsive contact term that raises the level
3. $\Delta E_{ns} = \langle\delta V\rangle = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8](Z\alpha m)^3/(\pi n^3) = (4\alpha(Z\alpha)^4 m/3\pi n^3)[\ln(m/\mu_\gamma) - 3/8] = U[\ln(m/\mu_\gamma) - 3/8]$, and zero for $l \geq 1$ because those wavefunctions vanish at the origin

   The sign is where candidates go wrong: q^2 is negative for a spacelike transfer, so the q^2 term of F_1 weakens the attraction at large $|q|$, which is a repulsive contact term. A candidate who finds the s levels lowered has lost that sign.

(b) EXPLAIN · U1 **[2]**

Explain why the result of (a) cannot be used on its own, and why a cutoff K with $(Z\alpha)^2 m \ll K \ll m$ can be found for He^+, giving the two ends of the window in electronvolts.

1. The result depends on the photon mass μ_γ, an infrared regulator: the free-electron vertex treats virtual photons of every energy as though the electron were free, but photons with energies of order the binding, $(Z\alpha)^2 m$, see a bound electron whose intermediate states are atomic levels, and there the free vertex is wrong; Bethe's dipole treatment is right there but fails for photon energies approaching m, where the electron becomes relativistic
2. The two descriptions overlap where $(Z\alpha)^2 m \ll K \ll m$: for He^+, $(Z\alpha)^2 m = (2/137.036)^2 \times 0.511$ MeV $= 109$ eV and $m = 511\,000$ eV, a window of more than three decades, so both hold at some K and the matching is controlled

   Candidates who say that the μ_γ dependence is a divergence to be renormalised away have confused the infrared with the ultraviolet: no counterterm removes μ_γ here; the binding does, by cutting off the soft photons at atomic energies.

(c) DETERMINE · U3 **[3]**

Using the matching relation, determine the total order-α(Zα)⁴ shift of the He⁺ 2S_(1/2) level in units of U and in megahertz, showing that K cancels.

1. The high-energy part with the matching is U[ln(m/2K) + 5/6 − 3/8] and the low-energy part U ln(K/⟨E⟩_2s); their sum, U[ln(m/(2⟨E⟩_2s)) + 5/6 − 3/8], is free of K, and since 2⟨E⟩_2s = (Zα)²m e^(2.8118) the logarithm is ln(1/(Zα)²) − 2.8118 = 8.4542 − 2.8118 = 5.6424
2. Adding the Pauli term +3/8 and the Uehling term −1/5 gives 5.6424 + 5/6 − 3/8 + 3/8 − 1/5 = 6.2757 in units of U
3. ΔE(2S_(1/2)) = 6.2757 × 2170 MHz = 13 618 MHz, of which the Uehling term contributes −434 MHz and the Pauli term +814 MHz

Candidates who keep K or μ_γ in a final answer have not finished the matching. One who measures ⟨E⟩_2s in Ry rather than Z²Ry makes an error of ln 4 = 1.39 U, about 3000 MHz.

(d) CALCULATE · U3 **[2]**

Calculate the shift of 2P_(1/2) and hence the 2S_(1/2)–2P_(1/2) splitting of He⁺ in megahertz.

1. 2P_(1/2) receives no contact terms, so its shift is −U ln(⟨E⟩_2p/(Z²Ry)) − U/8 = (0.0300 − 0.1250)U = −0.0950U = −206 MHz
2. The splitting is (6.2757 + 0.0950)U = 6.3707 × 2170 MHz = 13 824 MHz

The 2P_(1/2) level moves down, so its shift adds to the splitting; candidates who subtract its magnitude find 13 412 MHz.

(e) EVALUATE · U4 **[2]**

Evaluate the agreement with the measured splittings of He⁺ and of hydrogen, and say what the comparison suggests about the largest term omitted.

1. For He⁺, 14 041 − 13 824 = 217 MHz, 1.5% of the measured value; for hydrogen, 1057.8 − 1051.9 = 5.9 MHz, 0.56%: in both the order-α(Zα)⁴ assembly falls short, by a fraction that grows by a factor of about 2.8 from Z = 1 to Z = 2
2. A fractional shortfall that grows at least in proportion to Z points to an omitted term one power of Zα higher, of order α(Zα)⁵m, from a further Coulomb exchange inside the loop, about +7 MHz in hydrogen and +230 MHz in He⁺; two-loop terms are suppressed by a further α/π and do not grow with Z, and the neglected reduced-mass correction (about −1.7 MHz in hydrogen, which widens the gap) and the nuclear size (about +0.1 MHz in hydrogen) are too small to account for it

An evaluation that stops at 'agrees to 1.5%' earns one mark. The second is for using the two atoms together: the Z dependence of the shortfall identifies the order of the missing term.

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CORE COURSE

DATA ANALYSIS · DEMANDING · UQFTI-09.06 · UQFTI-13.04 · UQFTI-13.05 · UQFTI-13.06 **[12]**

An electron–positron collider measures the total cross-section for e⁺e⁻ → μ⁺μ⁻ at six centre-of-mass energies well below the Z boson. Z exchange and initial- and final-state radiation have been corrected for, so that what remains is one-photon exchange through the vacuum-polarisation-corrected photon propagator, for which σ = 4πα(s)²/(3s), with α(s) the effective coupling at momentum transfer squared s; the muon mass is negligible at these energies. Work in natural units with α(0) = 1/137.036 and (ħc)² = 0.3894 GeV² mb, so that 4πα(0)²/3 = 86.86 nb GeV². At one loop a fermion of charge Q_f e with N_c colours makes α run as μ dα/dμ = 2N_cQ_f²α²/(3π) while μ is well above its mass, with μ² = s. Every quoted uncertainty is 0.2% of σ, statistical and systematic combined. Lepton loops alone, run from q² = 0 with the measured lepton masses, give 1/α = 134.02 at √s = 12.0 GeV and 132.73 at √s = 91.19 GeV.

(a) SHOW THAT · U2 **[2]**

Show that if every fermion lighter than $\sqrt{s}$ contributes at one loop, $1/\alpha(s)$ is a straight line in $\ln s$ with gradient $-\Sigma_f N_c Q_f^2/(3\pi)$, and that $s\sigma/(86.86\ \text{nb GeV}^2) = [\alpha(s)/\alpha(0)]^2$.

1. $d(1/\alpha)/d\ln\mu = -(1/\alpha^2)\mu\, d\alpha/d\mu = -2\Sigma_f N_c Q_f^2/(3\pi)$, and since $\ln\mu = \tfrac{1}{2}\ln s$, $d(1/\alpha)/d\ln s = -\Sigma_f N_c Q_f^2/(3\pi)$, constant while no threshold is crossed, so $1/\alpha(s) = 1/\alpha(s_0) - [\Sigma_f N_c Q_f^2/(3\pi)]\ln(s/s_0)$
2. $\sigma = 4\pi\alpha(s)^2/(3s)$ gives $s\sigma = [4\pi\alpha(0)^2/3][\alpha(s)/\alpha(0)]^2$, and $4\pi\alpha(0)^2/3 = 2.2306 \times 10^{-4} \times 0.3894\ \text{GeV}^2\ \text{mb} = 8.686 \times 10^{-5}\ \text{GeV}^2\ \text{mb} = 86.86\ \text{nb GeV}^2$

   Candidates who differentiate with respect to $\ln\mu$ and then plot against $\ln s$ are out by a factor of 2, and would read the data as requiring only half as many charged species, $\Sigma \approx 3.3$, which looks like the leptons alone.

(b) DESCRIBE · U4 **[2]**

Describe how you would process the readings to test this prediction and extract the gradient, including how the uncertainty in each σ becomes an uncertainty in $1/\alpha$.

1. For each run form $R = s\sigma/(86.86\ \text{nb GeV}^2)$ and $1/\alpha(s) = 137.036/\sqrt{R}$, then plot $1/\alpha$ against $\ln(s/\text{GeV}^2)$ and fit a straight line, whose gradient is $-\Sigma_f N_c Q_f^2/(3\pi)$; a straight line tests one-loop running with a fixed set of active species
2. Since $1/\alpha \propto \sigma^{\wedge}(-1/2)$, $\delta(1/\alpha) = \tfrac{1}{2}(1/\alpha)(\delta\sigma/\sigma) = \tfrac{1}{2} \times 131 \times 0.002 \approx 0.13$ at every point, equal error bars, so an unweighted fit is appropriate; the intercept at $\ln s = 0$ has no direct meaning, since it absorbs all the running below 12 GeV

   Fitting σ itself against $1/s$, or plotting α rather than $1/\alpha$, gives curves rather than a line; the linear variable is $1/\alpha$, because one-loop running sums a geometric series in the logarithm.

(c) DETERMINE · U4 **[3]**

Determine $1/\alpha$ at each energy, the gradient and intercept of the best straight line with an estimate of the uncertainty in the gradient, and hence $\Sigma_f N_c Q_f^2$.

1. $s\sigma = 93.96$, 94.54, 95.44, 96.17, 96.51 and 97.09 nb GeV², so $1/\alpha = 131.76$, 131.35, 130.73, 130.23, 130.00 and 129.62 at $\ln(s/\text{GeV}^2) = 4.970$, 5.781, 6.438, 7.111, 7.613 and 8.121
2. Gradient $= -0.695$ (accept -0.66 to -0.73), with an uncertainty of about 0.05 from 0.13 error bars over the spread of $\ln s$ (accept 0.03 to 0.06), and intercept 135.25 at $\ln(s/\text{GeV}^2) = 0$ (accept 135.0 to 135.5)
3. $\Sigma_f N_c Q_f^2 = -3\pi \times \text{gradient} = 6.6 \pm 0.5$ (accept 6.2 to 6.9)

   Candidates who take the gradient from the first and last points alone get -0.68, which is acceptable, but must then estimate its uncertainty from the error bars; a gradient quoted without one cannot be used in (d).

(d) EVALUATE · U4 **[3]**

Evaluate which of these the data support: the three charged leptons alone; the leptons with the five quarks u, d, s, c and b each counted once; or the leptons with those quarks in three colours.

1. Leptons alone give $\Sigma = 3$ and gradient $-3/(3\pi) = -0.318$; adding u and c ($Q = 2/3$) and d, s and b ($Q = -1/3$) once each gives $\Sigma = 3 + 8/9 + 3/9 = 4.22$ and gradient -0.448; with three colours $\Sigma = 3 + 11/3 = 20/3$ and gradient -0.707
2. The measured -0.695 ± 0.05 lies about 7.5 standard deviations from the leptons-only value and 5 from uncoloured quarks, and within 0.3 standard deviations of the three-colour value, so the data require quarks in three colours
3. The verdict survives what was neglected: the strong-interaction correction multiplies the quark term by about $1 + \alpha_s/\pi \approx 1.05$, moving the prediction to about -0.73, within the error, and the top quark, at 173 GeV, is not yet active

   A comparison that does not use the uncertainty in the gradient is not an evaluation; the point of (c) was to have one.

(e) PREDICT · U3 **[2]**

Predict $1/\alpha$ at $\sqrt{s} = 91.19$ GeV by extrapolating your line, and say what its difference from the lepton-only running requires of hadrons, already at 12 GeV and at the Z.

1. $\ln(s/\text{GeV}^2) = 9.026$ at 91.19 GeV, so $1/\alpha \approx 135.25 - 0.695 \times 9.026 = 128.98$, about 129.0 (accept 128.8 to 129.2), against 132.73 from leptons alone
2. Hadrons must supply $\Delta(1/\alpha) \approx 3.7$ at the Z, of which 2.3 ($134.02 - 131.76$) is already present at 12 GeV and comes from hadronic states below 12 GeV, resonances included, where quark loops with ill-defined masses cannot be trusted; it is obtained instead from a dispersion relation over the measured $e^+e^- \to$ hadrons cross-section, which the optical theorem ties to $\text{Im}\,\Pi$

   Candidates who extrapolate from the 58 GeV point with the lepton-only gradient reach about 129.3 and have discarded the slope they measured; the extrapolation must use the fitted line.

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FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQFTI-12.01 · UQFTI-12.03 · UQFTI-12.04 · UQFTI-12.06 · UQFTI-12.07 **[15]**

Work in natural units with the metric $(+, -, -, -)$, in QED with an electron of physical mass m and bare mass m_0, in Feynman gauge with a small photon mass μ_γ. At one loop the electron self-energy is $-i\Sigma_2(p) = (-ie)^2\int d^dk/(2\pi)^d\, \gamma^\mu[i(\not{k} + m)/(k^2 - m^2 + i\varepsilon)]\gamma_\mu[-i/((p-k)^2 - \mu_\gamma^2 + i\varepsilon)]$, continued to $d = 4 - \varepsilon$ dimensions with $e^2 \to e^2\mu^\varepsilon$ to keep e dimensionless. You may use: in d dimensions $\gamma^\mu\gamma_\mu = d$ and $\gamma^\mu\not{k}\gamma_\mu = -(d-2)\not{k}$; Feynman's parametrisation $1/(AB) = \int_0^1 dx\, 1/[xA + (1-x)B]^2$; the Minkowski master integral $\int d^dl/(2\pi)^d\, 1/(l^2 - \Delta + i\varepsilon)^2 = i\Gamma(2 - d/2)\Delta^{(d/2-2)}/(4\pi)^{(d/2)}$, integrals odd in l vanishing; and, as $\varepsilon \to 0$, $\Gamma(\varepsilon/2)(4\pi\mu^2/\Delta)^{(\varepsilon/2)} = 2/\varepsilon - \gamma_E + \ln(4\pi\mu^2/\Delta) + O(\varepsilon)$. The physical mass is the pole of the full propagator, $m = m_0 + \Sigma_2(\not{p} = m)$ at this order. With the full vertex Γ^μ (its factor $-ie$ removed) and the full inverse propagator $\tilde{S}^{-1}(p) = \not{p} - m_0 - \Sigma(p)$, the Ward–Takahashi identity reads $q_\mu\Gamma^\mu(p+q, p) = \tilde{S}^{-1}(p+q) - \tilde{S}^{-1}(p)$. Take $\alpha = 1/137.036$, $m = 0.511$ MeV and the Planck mass 1.22×10^{19} GeV.

(a) DERIVE · U2 **[2]**

Derive, by counting powers of loop momentum, the superficial degree of divergence D of a QED diagram in terms of the number B of its external photons and the number F of its external electron lines, and hence the value of D for the electron self-energy.

1. With L loops, P_e internal electron and P_γ internal photon lines and V vertices, $D = 4L - P_e - 2P_\gamma$ and $L = P_e + P_\gamma - V + 1$, and each vertex ends two electron lines and one photon line: $2V = 2P_e + F$ and $V = 2P_\gamma + B$
2. Substituting, $D = 3P_e + 2P_\gamma - 4V + 4 = 3(V - F/2) + (V - B) - 4V + 4 = 4 - B - (3/2)F$, independent of V, so finitely many functions diverge; the self-energy, with $B = 0$ and $F = 2$, has $D = 1$

   Candidates who count external lines as propagators, or forget that every vertex ends two electron lines, lose the independence of D from V, which is the whole point.

(b) EXPLAIN · U1 **[2]**

Explain why, although $D = 1$, the mass shift can diverge only logarithmically and must be proportional to m, and contrast this with what happens to a scalar's mass.

1. At $m_0 = 0$ the QED Lagrangian is invariant under the chiral rotation $\psi \to e^{(i\theta\gamma^5)}\psi$, which forbids a $\bar{\psi}\psi$ term, so Σ at $\not{p} = m$ must vanish as $m \to 0$: δm is proportional to m, which takes one power of the cutoff out of $D = 1$ and leaves a logarithm
2. No symmetry protects a scalar's mass, so its shift is of order $(\text{coupling}/16\pi^2)\Lambda^2$, quadratically sensitive to high scales; the electron mass is natural because a small m is protected by the symmetry that appears when it vanishes

   The answer 'because the integral happens to be logarithmic' earns nothing: a symmetry forces the power of m, and the power of m forces the logarithm.

(c) SHOW THAT · U2 **[4]**

Show that $\Sigma_2(p) = [e^2\mu^{\varepsilon}/(4\pi)^{(d/2)}]\int_0^1 dx[(4-\varepsilon)m - (2-\varepsilon)x\not{p}]\Gamma(2-d/2)\Delta^{(d/2-2)}$, with $\Delta = (1-x)m^2 + x\mu_\gamma^2 - x(1-x)p^2$.

1. In d dimensions the numerator is $\gamma^\mu(\not{k}+m)\gamma_\mu = -(d-2)\not{k} + dm = -(2-\varepsilon)\not{k} + (4-\varepsilon)m$
2. With x multiplying the photon denominator, $x[(p-k)^2 - \mu_\gamma^2] + (1-x)(k^2-m^2) = (k-xp)^2 - \Delta$, with $\Delta = (1-x)m^2 + x\mu_\gamma^2 - x(1-x)p^2$
3. Shifting to $l = k - xp$, the term linear in l vanishes and the numerator becomes $(4-\varepsilon)m - (2-\varepsilon)x\not{p}$
4. The master integral gives $i\Gamma(2-d/2)\Delta^{(d/2-2)}/(4\pi)^{(d/2)}$, and the vertices and propagator numerators give $(-ie)^2 \times i \times (-i) = -e^2$, with $e^2 \to e^2\mu^{\varepsilon}$, so $-i\Sigma_2 = -e^2\mu^{\varepsilon} \times i[\ldots]/(4\pi)^{(d/2)}$ and $\Sigma_2 = [e^2\mu^{\varepsilon}/(4\pi)^{(d/2)}]\int_0^1 dx[(4-\varepsilon)m - (2-\varepsilon)x\not{p}]\Gamma(2-d/2)\Delta^{(d/2-2)}$

   Using the four-dimensional identities $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu\not{k}\gamma_\mu = -2\not{k}$ here loses the $O(\varepsilon)$ terms that, multiplied by the $2/\varepsilon$ pole, supply finite parts of δm; the result would then depend on the order in which the limits are taken.

(d) DERIVE · U2 **[3]**

Derive the self-mass $\delta m = m - m_0$ at one loop, showing that it is proportional to m, logarithmic and finite as $\mu_\gamma \to 0$, and find its finite part.

1. On shell $\not{p} = m$ and $p^2 = m^2$, and as $\mu_\gamma \to 0$, $\Delta = (1-x)^2m^2$, whose logarithm is integrable at $x = 1$, so $\delta m = \Sigma_2(\not{p} = m)$ needs no photon mass; with $e^2\mu^{\varepsilon}(4\pi)^{(-d/2)}\Gamma(\varepsilon/2)\Delta^{(-\varepsilon/2)} = (\alpha/4\pi)\Gamma(\varepsilon/2)(4\pi\mu^2/\Delta)^{(\varepsilon/2)}$, $\delta m = (\alpha/4\pi)m\int_0^1 dx[(4-\varepsilon) - (2-\varepsilon)x][2/\varepsilon - \gamma_E + \ln(4\pi\mu^2/((1-x)^2m^2))]$
2. The pole term gives $(2/\varepsilon)\int(4-2x)dx = 6/\varepsilon$; the $O(\varepsilon)$ part of the numerator, $-\varepsilon(1-x)$, times $2/\varepsilon$ gives $-2\int(1-x)dx = -1$; and $-\int(4-2x)\ln[(1-x)^2]dx = -2\int_0^1(2+2y)\ln y\, dy = +5$
3. So $\delta m = (3\alpha/4\pi)m[2/\varepsilon - \gamma_E + \ln(4\pi\mu^2/m^2) + 4/3]$: proportional to m and only logarithmically dependent on the regulator, the $1/\varepsilon$ pole standing for $\ln \Lambda^2$

   The -1 from the $O(\varepsilon)$ numerator times the pole is the finite part the d-dimensional Dirac algebra supplies; candidates who drop it get 5/3 for the constant instead of 4/3.

(e) COMPARE · U3 **[2]**

Interpreting the pole as $\ln(\Lambda^2/m^2)$ for a cutoff Λ, so that $\delta m \approx (3\alpha/4\pi)m\ln(\Lambda^2/m^2)$, compare $\delta m/m$ for Λ equal to the Planck mass with the shift of order $\alpha\Lambda$ that a linearly divergent electromagnetic mass would give.

1. $\ln(\Lambda^2/m^2) = 2\ln(1.22\times10^{19}\ \text{GeV}/5.11\times10^{-4}\ \text{GeV}) = 103.1$, so $\delta m/m = [3/(4\pi \times 137.036)] \times 103.1 = 0.0017421 \times 103.1 = 0.18$: an 18% shift even with the Planck mass as cutoff
2. A linear shift $\alpha\Lambda = 1.22\times10^{19}\ \text{GeV}/137.036 = 8.9\times10^{16}\ \text{GeV}$ would be 1.7×10^{20} times m, so the observed mass would need a cancellation to twenty decimal places; the relativistic theory, with its antiparticles, turns the linear divergence into a logarithm and makes the electron's small mass natural

   Candidates who report $\delta m/m \approx \alpha$ are close in size but have missed the comparison: it is the dependence on Λ, logarithmic against linear, that separates a natural mass from a fine-tuned one.

(f) SHOW THAT · U2 **[2]**

From the Ward–Takahashi identity show that $Z_1 = Z_2$, where near the mass shell $\tilde{S}^{-1}(p) \to Z_2^{-1}(\not{p} - m)$ and, between on-shell spinors as $q \to 0$, $\Gamma^\mu(p, p) \to Z_1^{-1}\gamma^\mu$.

1. Expanding the identity to first order in q, $q_\mu\Gamma^\mu(p + q, p) = q_\mu\partial\tilde{S}^{-1}(p)/\partial p_\mu + O(q^2)$, so $\Gamma^\mu(p, p) = \partial\tilde{S}^{-1}(p)/\partial p_\mu$, Ward's original identity
2. Near the mass shell $\partial\tilde{S}^{-1}/\partial p_\mu \to Z_2^{-1}\gamma^\mu$, so $Z_1^{-1}\gamma^\mu = Z_2^{-1}\gamma^\mu$ and $Z_1 = Z_2$: the vertex and field renormalisations cancel in $e = Z_2Z_3^{1/2}e_0/Z_1$, and the charge is renormalised by Z_3 alone

Candidates who argue from the one-loop expressions show only $\delta_1 = \delta_2$ at this order; the identity holds to all orders and for any charged particle, which is what makes charge universal.

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CORE COURSE

FREE RESPONSE · DERIVE, CONFRONT WITH MEASUREMENT AND CANCEL THE INFRARED · DISCRIMINATING · UQFTI-14.01 · UQFTI-14.02 · UQFTI-14.03 · UQFTI-14.04 · UQFTI-14.07 · UQFTI-14.08 · UQFTI-14.09 **[16]**

Work in natural units with the metric $(+, -, -, -)$. Between on-shell spinors the full electron–photon vertex is $\bar{u}(p')\Gamma^\mu u(p)$, with $q = p' - p$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$; the Gordon identity reads $\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')[(p' + p)^\mu/2m + i\sigma^{\mu\nu}q_\nu/2m]u(p)$. At one loop in Feynman gauge, after Feynman parameters, the shift $\ell = k + yq - zp$ and use of the Dirac equation, the correction is $\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2\int d^4\ell/(2\pi)^4 \int_0^1 dx\,dy\,dz\,\delta(x + y + z - 1)(2/D^3)\bar{u}(p')N^\mu u(p)$, with $D = \ell^2 - \Delta + i\varepsilon$, $\Delta = -xyq^2 + (1 - z)^2m^2$ and $N^\mu = \gamma^\mu[-\ell^2/2 + (1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2] + (i\sigma^{\mu\nu}q_\nu/2m)[2m^2z(1 - z)]$. Use $\int d^4\ell/(2\pi)^4\, 1/(\ell^2 - \Delta)^3 = -i/(2(4\pi)^2\Delta)$. The measured electron anomaly is $a_e = 1.159\,652\,180\,59(13) \times 10^{-3}$, and the QED series is $a_e = \alpha/2\pi + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \ldots$, with $C_2 = -0.328\,478\,965$, $C_3 = 1.181\,241\,456$ and $C_4 = -1.912\,245\,764$; in this question take $\alpha^{-1} = 137.035\,999$, from atom-recoil measurements. For elastic scattering of an electron of energy E through angle θ by a recoil-free nucleus, $-q^2 = 4E^2\sin^2(\theta/2)$; the order-$\alpha$ virtual correction multiplies the Born cross-section by $1 - (\alpha/\pi)f_{IR}(q^2)\ln(-q^2/\mu_\gamma^2)$, μ_γ being a photon mass, and the probability of emitting one soft photon with energy between μ_γ and E_l is $(\alpha/\pi)f_{IR}(q^2)\ln(E_l^2/\mu_\gamma^2)$, where for $-q^2 \gg m^2$, $f_{IR}(q^2) = \ln(-q^2/m^2) - 1$. Take $m = 0.511$ MeV.

(a) JUSTIFY · U1 **[2]**

Justify writing the on-shell vertex as $\Gamma^\mu = \gamma^\mu F_1(q^2) + (i\sigma^{\mu\nu}q_\nu/2m)F_2(q^2)$.

1. Between on-shell spinors the vertex can be built only from γ^μ, p^μ and p'^μ, that is γ^μ, $(p' + p)^\mu$ and q^μ, multiplied by functions of the one invariant q^2, since $p^2 = p'^2 = m^2$; structures with γ^5 are excluded because QED conserves parity, and the Gordon identity trades $(p' + p)^\mu$ for γ^μ and $i\sigma^{\mu\nu}q_\nu$
2. A q^μ term is excluded by the Ward identity $q_\mu\Gamma^\mu = 0$: $q_\mu\gamma^\mu$ vanishes between on-shell spinors by the Dirac equation and $q_\mu\sigma^{\mu\nu}q_\nu = 0$ by antisymmetry, but $q_\mu q^\mu = q^2$ does not, so its coefficient must vanish and two form factors remain

Candidates who list structures without using current conservation are left with three form factors; it is the Ward identity, not Lorentz invariance, that removes q^μ.

(b) SHOW THAT · U2 **[2]**

Show that in a slowly varying magnetic field the vertex gives the electron a magnetic moment with $g = 2[F_1(0) + F_2(0)]$.

1. In a static vector potential with Fourier transform $A^i(q)$ the amplitude contains $\bar{u}(p')\Gamma^i u(p)A^i(q)$; in the non-relativistic expansion of the spinors the spin-dependent part of $\bar{u}(p')\gamma^i u(p)$ is $2m\,\xi'^\dagger(-i\varepsilon^{ijk}q^j\sigma^k/2m)\xi$, through the lower components, and $\bar{u}(p')(i\sigma^{i\nu}q_\nu/2m)u(p)$ gives the same structure, so the spin term is weighted by $F_1(0) + F_2(0)$
2. Since $\varepsilon^{ijk}q^jA^k(q)$ is, up to a factor $\pm i$ set by the Fourier convention, the transform of the magnetic field $B = \nabla \times A$, the spin term is the Born amplitude of the energy $-\mu\cdot B$ with $\mu = (e/m)[F_1(0) + F_2(0)]S$, that is $g = 2[F_1(0) + F_2(0)]$; $F_1(0) = 1$ defines the charge, so $g - 2 = 2F_2(0)$ and $a_e = F_2(0)$

Candidates who take the moment from the $\sigma^{\mu\nu}$ term alone obtain $g = 2F_2(0)$; the Dirac moment $g = 2$ is already in γ^μ, through the lower components of the spinors.

(c) DERIVE · U2 **[3]**

Derive $F_2(q^2)$ as an integral over Feynman parameters, evaluate $F_2(0)$, and explain why F_2 needs neither an ultraviolet nor an infrared regulator.

1. Only the σ^μν term of N^μ contributes to F_2, giving $2ie^2 \times 2 \times 2m^2z(1-z) \times [-i/(2(4\pi)^2\Delta)]$ under the parameter integrals, that is $F_2(q^2) = (\alpha/2\pi)\int dx\, dy\, dz\, \delta(x+y+z-1)\, 2m^2z(1-z)/[(1-z)^2m^2 - xyq^2]$
2. At $q^2 = 0$ the integrand depends only on z, and $\int dx\, dy\, \delta(x+y+z-1) = 1-z$, so $F_2(0) = (\alpha/2\pi)\int_0^1 (1-z) \times 2z(1-z)/(1-z)^2\, dz = (\alpha/2\pi)\int_0^1 2z\, dz = \alpha/2\pi$
3. The σ^μν term carries no ℓ^2 in its numerator, so $\int d^4\ell/D^3$ converges in the ultraviolet and F_2 has no counterterm; at $z \to 1$ the factor $z(1-z)$ and the measure's $1-z$ cancel the $(1-z)^2$ of Δ, so the integral is finite without a photon mass: $F_2(0)$ is a prediction

The factor 2 in $2/D^3$ from the Feynman parametrisation is the one most often dropped, giving $\alpha/4\pi$. A second check is that a prediction cannot depend on a counterterm, and F_2 has none to depend on.

(d) CALCULATE · U3 **[3]**

Calculate the residual, the measured a_e minus the QED series truncated after one, two and three terms, and compare each residual with the size of the next term.

1. $\alpha/2\pi = 1.161\,409\,73 \times 10^{-3}$, and $a_e - \alpha/2\pi = -1.7576 \times 10^{-6}$; divided by $(\alpha/\pi)^2 = 5.3955 \times 10^{-6}$ this is -0.326, close to C_2
2. $C_2(\alpha/\pi)^2 = -1.7723 \times 10^{-6}$, leaving $+1.475 \times 10^{-8}$; divided by $(\alpha/\pi)^3 = 1.2533 \times 10^{-8}$ this is 1.18, close to C_3
3. $C_3(\alpha/\pi)^3 = 1.4804 \times 10^{-8}$, leaving -5.2×10^{-11}, the size of the four-loop term $C_4(\alpha/\pi)^4 = -5.57 \times 10^{-11}$: each order shrinks the residual by a factor of order α/π and confirms the next coefficient

The subtraction must be carried to ten significant figures; candidates who round $\alpha/2\pi$ to 1.1614×10^{-3} find a residual dominated by their own rounding.

(e) ESTIMATE · U4 **[1]**

Estimate the fractional precision with which this measurement of a_e would determine α if the QED series were known exactly, stating your assumption.

1. Assuming the theoretical series, hadronic and weak terms included, is exact, $a_e \approx \alpha/2\pi$ to leading order, so $\delta\alpha/\alpha \approx \delta a_e/(\alpha/2\pi) = 1.3 \times 10^{-13}/1.1614 \times 10^{-3} = 1.1 \times 10^{-10}$, about 0.1 parts per billion

The assumption is what matters: the theory must add no uncertainty of its own, since dropping even the three-loop term would shift α^{-1} by $2\pi \times 1.48 \times 10^{-8} \times 137.036^2 = 0.0017$, some 10^5 times this precision.

(f) SHOW THAT · U2 **[2]**

Show that in the cross-section measured by a detector that cannot register photons below E_l the photon mass cancels at order α, leaving the factor $1 - (\alpha/\pi) f_{IR}(q^2) \ln(-q^2/E_l^2)$.

1. The detector counts elastic events and events with one undetected photon below E_l together, so at order α the measured cross-section is the Born value times $[1 - (\alpha/\pi) f_{IR} \ln(-q^2/\mu_\gamma^2)] + (\alpha/\pi) f_{IR} \ln(E_l^2/\mu_\gamma^2)$
2. Since $\ln(-q^2/\mu_\gamma^2) - \ln(E_l^2/\mu_\gamma^2) = \ln(-q^2/E_l^2)$, μ_γ cancels and the factor is $1 - (\alpha/\pi) f_{IR}(q^2) \ln(-q^2/E_l^2)$: the infrared divergences of the virtual and real corrections cancel in the measured rate, as Bloch and Nordsieck showed

Candidates who try to remove μ_γ with a counterterm have confused the infrared with the ultraviolet; the cancellation is between final states that the detector cannot tell apart.

(g) CALCULATE · U3 **[2]**

Calculate this correction factor for 4.0 GeV electrons scattered through 20° with E_l = 20 MeV, and compare it with its exponentiated form exp[−(α/π)f_IR ln(−q²/E_l²)].

1 $-q^2 = 4 \times (4.0\ \text{GeV})^2 \times \sin^2 10° = 1.930\ \text{GeV}^2$, so $\ln(-q^2/m^2) = \ln[1.930/(5.11 \times 10^{-4})^2] = 15.82$ and f_IR = 14.82, and $\ln(-q^2/E_l^2) = \ln(1.930/0.0004) = 8.48$

2 $(\alpha/\pi) f_{IR} \ln(-q^2/E_l^2) = 0.0023228 \times 14.82 \times 8.48 = 0.292$, so the order-α factor is 0.708 while the exponentiated form gives e^(−0.292) = 0.747: the correction is so large that the difference of about 0.04 between them, the α^2 terms, matters at this resolution

A correction of 29% is no sign of error but the double logarithm doing its work; candidates who stop at the order-α factor have ignored α^2 terms that exponentiation shows to be about 0.04.

(h) PREDICT · U1 **[1]**

Predict what happens, as E_l → 0, to the rate of scattering with no photon above E_l and to the inclusive rate.

1 The exponentiated factor $(-q^2/E_l^2)^{-(\alpha/\pi) f_{IR}}$ tends to zero as E_l → 0, so scattering with no photon at all is impossible, every scattered electron radiating a cloud of soft photons; the inclusive rate, summed over all emissions, stays finite and close to the Born value

Candidates who conclude from (g) that the correction becomes infinitely negative as E_l → 0 have kept only the first order of a series that sums to a power tending to zero.

THE CHEAPEST MARKS ON ANY PAPER

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

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Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 411 MARKS · 24%	**Conceptual understanding** Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 616 MARKS · 36%	**Mathematical modelling and derivation** Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3
467 MARKS · 27%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4
219 MARKS · 13%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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