

University Physics I · Final Examination

The whole of calculus-based mechanics, weighted to the second half of the course: momentum and collisions, rotational kinematics and dynamics, angular momentum and precession, static equilibrium and elasticity, gravitation and orbits, and oscillations, with fluids as an extension. Kinematics, Newton's laws, work and energy are sampled from the first half, and every answer is expected to name the system, the conservation law and the approximation it rests on.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	116	18	10

SECTION	UPI · Final · University Physics I — Final examination
IN THE FORMAT OF	A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations
ROUTES	Core course only — 17 questions, 115 marks. Core course and Extension — the whole paper, 18 questions, 116 marks. Every question is tagged in the left margin.
DEMAND	Routine 9 · Demanding 6 · Discriminating 3 <i>Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.</i>
COURSE UNITS	UPI-01 · UPI-02 · UPI-03 · UPI-04 · UPI-05 · UPI-06 · UPI-07 · UPI-08 · UPI-09 · UPI-10 · UPI-11 · UPI-12 · UPI-13 · UPI-14 · UPI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Physics I · Final Examination

Answer all questions. 116 marks in 150 minutes.

1 MULTIPLE CHOICE · ROUTINE · UPI-01.02 [1]

CORE
COURSE

Dry sand drains through a circular hole of diameter D in the flat base of a hopper. Experiments show that the mass flow rate $\dot{m}$, in kg/s, does not depend on the depth of sand above the hole. Assume that $\dot{m}$ depends only on the bulk density ρ of the sand, the gravitational field strength g and the diameter D , and use dimensional analysis. If the diameter of the hole is doubled, by what factor does the mass flow rate increase?

- A 2.8
- B 4.0
- C 5.7
- D 8.0

Answer

2 MULTIPLE CHOICE · ROUTINE · UPI-01.06 · UPI-03.01 [1]

CORE
COURSE

A particle moves in the xy-plane with position vector $\mathbf{r}(t) = (2.0t^2 - 8.0t) \hat{i} + (6.0t - 1.0t^2) \hat{j}$, where $\mathbf{r}$ is in metres and t is in seconds. At what time is the speed of the particle least?

- A 2.0 s
- B 2.2 s
- C 2.5 s
- D 3.0 s

Answer

3 MULTIPLE CHOICE · ROUTINE · UPI-04.08 · UPI-05.01 [1]

CORE
COURSE

A 12 kg crate rests on a loading ramp inclined at 20° to the horizontal. The coefficients of friction between the crate and the ramp are $\mu_s = 0.60$ and $\mu_k = 0.45$. Take $g = 9.81 \text{ m/s}^2$. What is the magnitude of the friction force that the ramp exerts on the crate?

- A 40 N
- B 50 N
- C 66 N
- D 71 N

Answer

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-07.04 · UPI-07.05

[1]

An atom adsorbed on a crystal surface can move only along a straight row of binding sites. Along the row its potential energy is $U(x) = -U_0 \cos(2\pi x/a)$, where U_0 is a positive constant and a is the spacing of the sites, so that the sites lie at $x = 0, \pm a, \pm 2a$ and so on. For positions between $x = 0$ and $x = a/2$, where is the force that the surface exerts on the atom along the row greatest in magnitude, and in which direction does that force point there?

- A At $x = a/2$, pointing towards $x = 0$
- B At $x = a/4$, pointing away from $x = 0$
- C At $x = a/2$, pointing away from $x = 0$
- D At $x = a/4$, pointing towards $x = 0$

Answer

5

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPI-08.01 · UPI-08.02

[1]

A 0.170 kg ice-hockey puck slides at 6.00 m/s directly towards a player, who strikes it straight back along the same line. The force of the stick on the puck, directed away from the player, rises linearly from zero to 800 N in 4.0 ms, stays at 800 N for 3.0 ms, then falls linearly to zero in a further 3.0 ms. Friction with the ice is negligible. What is the speed of the puck just after the strike?

- A 17.5 m/s
- B 24.6 m/s
- C 30.6 m/s
- D 36.6 m/s

Answer

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-09.05 · UPI-09.07

[1]

A bicycle wheel of radius 0.35 m rolls without slipping along a level road, its centre moving at 7.0 m/s. Relative to the road, what are the speed of the point at the top of the rim and the speed of a point on the rim level with the axle?

- A 14 m/s at the top; 9.9 m/s level with the axle
- B 14 m/s at the top; 7.0 m/s level with the axle
- C 7.0 m/s at the top; 7.0 m/s level with the axle
- D 14 m/s at the top; 14 m/s level with the axle

Answer

7CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPI-01.06 · UPI-11.05

[1]

A spinning bicycle wheel is mounted at one end of a light horizontal axle whose other end rests on a frictionless pivot at the origin. The axle lies along the x-axis with the wheel's centre at $x = +0.150$ m, and the z-axis points vertically upwards. The wheel has mass 2.00 kg and moment of inertia 0.120 kg m^2 about the axle, and it spins at 40.0 rad/s with its spin angular momentum pointing in the +x direction. Take $g = 9.81 \text{ m/s}^2$. In the fast-spin approximation, at what rate does the axle precess, and in which sense does it turn when viewed from above, looking down the z-axis?

- A 0.61 rad/s, clockwise
- B 1.6 rad/s, counterclockwise
- C 1.6 rad/s, clockwise
- D 0.61 rad/s, counterclockwise

Answer

8CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-13.02 · UPI-13.05

[1]

Transit and radial-velocity observations show that a rocky exoplanet has 4.5 times the mass of Earth and 1.5 times its radius. Treat both planets as spherically symmetric. Compared with the values for Earth, how do the gravitational field strength at the planet's surface and the escape speed from its surface change?

- A Field strength doubles; escape speed increases by a factor of 1.7
- B Field strength doubles; escape speed increases by a factor of 1.4
- C Field strength increases by a factor of 3.0; escape speed increases by a factor of 1.7
- D Field strength doubles; escape speed increases by a factor of 3.0

Answer

9CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-14.03 · UPI-14.04

[1]

A block attached to a horizontal spring slides on a frictionless surface in simple harmonic motion with an amplitude of 0.12 m and a period of 0.80 s. What is the speed of the block when its displacement from equilibrium is 0.060 m?

- A 0.13 m/s
- B 0.47 m/s
- C 0.82 m/s
- D 0.94 m/s

Answer

10

MULTIPLE CHOICE · ROUTINE · UPI-15.06 · UPI-15.07

[1]

EXTENSION

Water of density 1000 kg/m^3 flows steadily through a horizontal pipe whose internal diameter narrows from 0.040 m to 0.020 m . The speed of the water in the wide section is 1.5 m/s . Treating the water as an ideal incompressible fluid, by how much is the pressure in the narrow section lower than the pressure in the wide section?

- A 3.4 kPa
- B 10 kPa
- C 17 kPa
- D 18 kPa

Answer

A laboratory test rig stores braking energy in a flywheel: a uniform steel annular disc of inner radius $R_1 = 0.100$ m, outer radius $R_2 = 0.300$ m and thickness 0.0500 m, which spins about its symmetry axis. The density of the steel is 7.80×10^3 kg/m³. In normal operation the flywheel spins at $\omega_0 = 314$ rad/s.

(a) SHOW THAT

[3]

Starting from the definition $I = \int r^2 dm$, show that the moment of inertia of a uniform annular disc of mass M about its symmetry axis is $I = \frac{1}{2}M(R_1^2 + R_2^2)$.

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(b) CALCULATE

[3]

Calculate the mass of the flywheel, its moment of inertia, and the kinetic energy it stores at ω_0 .

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(c) DETERMINE

[4]

When the flywheel is disconnected at ω_0 it coasts to rest in 1.50×10^3 s under a constant frictional torque from its bearings. Determine the frictional torque, the number of revolutions the flywheel makes while coasting, and the rate at which friction dissipates energy at the instant the flywheel is disconnected.

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(d) COMPARE

[2]

Compare the energy this flywheel stores at ω_0 with the energy stored by a uniform solid disc of the same mass and the same outer radius spinning at the same angular speed, and explain the difference in terms of how the mass is distributed.

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An electric car of mass 1500 kg, driver included, starts from rest on a level road. Its motor controller limits the power delivered to the driven wheels to $P = 60.0$ kW. Ignore air resistance and rolling resistance throughout, and take $g = 9.81$ m/s².

(a) SHOW THAT

[2]

Suppose the full power P is delivered from the instant the car starts. Show that the speed of the car at time t is $v(t) = \sqrt{2Pt/m}$.

(b) CALCULATE

[2]

Using this model, calculate the time the car takes to reach 25.0 m/s and its acceleration at that instant.

(c) DETERMINE

[3]

Determine, by integrating $v(t)$, the distance the car travels in reaching 25.0 m/s under this model.

(d) EVALUATE

[3]

The coefficient of static friction between the tyres and the road is 0.80, and all four wheels are driven, so the car's whole weight is available for traction. Evaluate the constant-power model close to $t = 0$: find the speed below which the car cannot use its full power, and state what limits its motion below that speed.

(e) SKETCH

[2]

The real car accelerates at the traction limit until it can use its full power, then continues at constant power. Sketch a graph of its speed against time from $t = 0$ until it reaches 25.0 m/s, and mark the point where the two stages join.

A uniform horizontal steel beam of length 3.00 m and mass 45.0 kg projects from a vertical wall, to which one end is attached by a frictionless hinge. A steel cable runs from the outer end of the beam to a point on the wall directly above the hinge, making an angle of 30.0° with the beam. A shop sign of mass 80.0 kg hangs from the beam 2.40 m from the wall. The cable is 3.46 m long and 4.00 mm in diameter, and its steel has Young modulus 2.00×10^{11} Pa and yield strength 2.50×10^8 Pa. The weight of the cable is negligible. Take $g = 9.81 \text{ m/s}^2$.

- (a) **EXPLAIN** [2]

Explain why taking torques about the hinge is the most efficient first step in finding the tension in the cable.

- (b) **CALCULATE** [2]

Calculate the tension in the cable.

- (c) **DETERMINE** [3]

Determine the magnitude and direction of the force exerted by the hinge on the beam.

- (d) **CALCULATE** [2]

Calculate the tensile stress in the cable and the extension of the cable.

- (e) **EVALUATE** [2]

The building code requires every load-bearing cable to have a safety factor of at least 2.0 against yield. Evaluate whether this cable complies, and recommend the smallest cable diameter that would.

- (f) **DESCRIBE** [2]

Describe what the stress-strain curve for the cable steel would show if the cable were loaded beyond its yield point and then unloaded, and what this would mean for the sign.

Car A, of mass 1200 kg and travelling due east, and car B, of mass 1800 kg and travelling due north, collide in the middle of an intersection and lock together. Skid marks show that the wreck then slid in a straight line for 8.00 m, directed 35.0° north of east, before stopping. The coefficient of kinetic friction between the tyres and the road is 0.65, and $g = 9.81 \text{ m/s}^2$. The speed limit at the intersection is 13.9 m/s (50 km/h).

(a) JUSTIFY

[2]

Justify treating the two cars as an isolated system for the duration of the collision, even though friction from the road acts on them.

(b) CALCULATE

[2]

Calculate the speed of the wreck immediately after the collision.

(c) DETERMINE

[3]

Determine the speed of each car immediately before the collision.

(d) CALCULATE

[2]

Calculate the fraction of the cars' kinetic energy that was dissipated in the collision itself.

(e) ESTIMATE

[2]

Estimate the magnitude of the average force exerted on a 75 kg occupant of car B, held by a seatbelt, during the collision. State the assumptions you make.

(f) EVALUATE

[3]

The driver of car A claims not to have exceeded the speed limit. The investigator's values are uncertain: μ_k lies between 0.55 and 0.75, and the direction of the skid is $35^\circ \pm 3^\circ$ north of east. Evaluate the driver's claim.

A uniform rod of mass $M = 2.00$ kg and length $L = 1.20$ m hangs vertically at rest from a frictionless horizontal pivot through its upper end. A lump of clay of mass $m = 0.400$ kg, moving horizontally at 6.00 m/s perpendicular to the pivot axis, strikes the lower end of the rod and sticks to it. Treat the clay as a particle and take $g = 9.81$ m/s². The moment of inertia of a uniform rod about a perpendicular axis through its centre is $ML^2/12$.

- (a) **STATE** [1]

State the condition under which the angular momentum of a system about a fixed point is conserved.

- (b) **CALCULATE** [2]

Calculate the moment of inertia of the rod and clay together about the pivot, immediately after the collision.

- (c) **EXPLAIN** [2]

Explain why the linear momentum of the rod-clay system is not conserved in the collision, whereas its angular momentum about the pivot is.

- (d) **DETERMINE** [3]

Determine the angular speed of the rod immediately after the collision and the greatest angle from the vertical through which it then swings.

- (e) **DETERMINE** [2]

Determine the horizontal impulse exerted by the pivot on the rod during the collision, including its direction.

- (f) **DERIVE** [3]

Derive, from conservation of angular momentum about the pivot and the impulse-momentum theorem, the distance x below the pivot at which the clay must strike for the pivot to exert no horizontal impulse during the collision. Show that the result does not depend on the mass of the clay.

A glider of mass 0.250 kg on a level air track is held between two springs and oscillates with a period of 0.800 s. A card sail fixed to the glider increases the air drag. A motion sensor records the amplitude of the oscillation, the greatest displacement from equilibrium, at the end of every fifth oscillation, with an uncertainty of ± 0.3 mm in each reading. Two damping models are proposed. Model V (viscous drag): the resistive force is $-bv$, so the amplitude decays as $A = A_0 e^{-(bt/(2m))}$. Model F (sliding friction): a resistive force of constant magnitude acts, which reduces the amplitude by the same amount in every oscillation.

Amplitude of the glider's oscillation, recorded at the end of every fifth oscillation

Reading	Time t (s)	Amplitude A (mm)
1	0.0	80.0
2	4.0	57.6
3	8.0	40.8
4	12.0	29.6
5	16.0	20.9
6	20.0	15.3
7	24.0	10.8

(a) DESCRIBE

[2]

Describe the pattern that each model predicts in amplitudes recorded at equal intervals of time.

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(b) JUSTIFY

[3]

Justify, using calculations from the table, which model the data support.

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(c) DETERMINE

[3]

Determine the damping constant b by a method that uses all of the readings.

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(d) **CALCULATE** [2]

Calculate the fraction of the glider's mechanical energy that is lost during one oscillation.

(e) **PREDICT** [2]

A 0.250 kg brass block is fixed to the glider, which keeps the same sail and the same springs. Predict the time for the amplitude to fall to half its initial value, before and after the block is added, and give the reason for the change.

17
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION: MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-11.06 · UPI-13.04 · UPI-13.05 · UPI-13.06

[15]

A communications satellite is raised from a circular parking orbit of radius $r_1 = 6.78 \times 10^6$ m to a circular orbit of radius $r_2 = 4.22 \times 10^7$ m by a two-burn transfer. A short engine burn at radius r_1 places it on an elliptical transfer orbit whose nearest point to Earth's centre (perigee) is at r_1 and whose farthest point (apogee) is at r_2 . A second short burn at apogee makes the orbit circular. For Earth, $GM = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2$. Treat both burns as instantaneous, treat Earth as a spherically symmetric mass, and ignore every other body.

(a) **DERIVE** [2]

Derive, from Newton's law of gravitation and Newton's second law, an expression for the speed v of a satellite in a circular orbit of radius r .

(b) **EXPLAIN** [2]

Explain why the satellite's angular momentum about Earth's centre is constant along the transfer orbit, and why its magnitude at perigee and at apogee is simply mr_1v .

(c) **DERIVE** [4]

Using conservation of angular momentum and conservation of mechanical energy between perigee and apogee, derive an expression for the speed v_p at perigee on the transfer orbit, in terms of GM , r_1 and r_2 .

- (d) **SHOW THAT** [2]
 Show that your expression reduces to the circular-orbit speed at r_1 when $r_2 = r_1$, and to the escape speed from r_1 as $r_2 \rightarrow \infty$, and state what each limit means physically.
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- (e) **CALCULATE** [3]
 Calculate the change in speed required at each burn, and the total.
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- (f) **CALCULATE** [2]
 Calculate the time the satellite spends on the transfer orbit, from perigee to apogee.
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18
 CORE
 COURSE

FREE RESPONSE · EXTENDED DERIVATION: FROM A CONSERVATION LAW TO THE EQUATION OF MOTION · DISCRIMINATING
 · UPI-09.07 · UPI-10.08 · UPI-14.02 · UPI-14.06

[15]

A seismic isolation bearing is modelled as a uniform solid cylinder of mass m and radius r that rolls without slipping on the inside of a fixed, concave cylindrical track of radius R , where $R > r$ and both axes are horizontal and parallel. The moment of inertia of the cylinder about its own axis is $\frac{1}{2}mr^2$. Let ϕ be the angle between the vertical and the line from the track's axis of curvature to the cylinder's axis, so that $\phi = 0$ at the lowest position. Take $g = 9.81 \text{ m/s}^2$, and ignore rolling resistance and air resistance.

- (a) **EXPLAIN** [2]
 Explain why the speed of the cylinder's axis is $(R - r)|d\phi/dt|$, and use the rolling condition to obtain the cylinder's angular speed ω about its own axis in terms of $d\phi/dt$.
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- (b) **DERIVE** [5]
 Derive, from conservation of mechanical energy, the equation of motion for ϕ .
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- (c) **SHOW THAT** [2]
Show that, for small oscillations, the motion is simple harmonic with period $T = 2\pi\sqrt{3(R - r)/(2g)}$.
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.....
- (d) **COMPARE** [2]
Compare this period with that of a small block sliding without friction whose centre of mass moves on the same circle of radius $R - r$, and explain the difference.
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.....
- (e) **CALCULATE** [2]
The engineers want an isolation period of 2.50 s. Calculate the value of $R - r$ required.
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.....
- (f) **PREDICT** [2]
Predict the period of the bearing if the solid cylinder is replaced by a thin-walled tube of the same mass and outer radius, whose moment of inertia about its own axis is mr^2 , with $R - r$ unchanged, and give the reason.
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University Physics I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 MULTIPLE CHOICE · ROUTINE · UPI-01.02 **[1]**
CORE COURSE

Dry sand drains through a circular hole of diameter D in the flat base of a hopper. Experiments show that the mass flow rate $\dot{m}$, in kg/s, does not depend on the depth of sand above the hole. Assume that $\dot{m}$ depends only on the bulk density ρ of the sand, the gravitational field strength g and the diameter D , and use dimensional analysis. If the diameter of the hole is doubled, by what factor does the mass flow rate increase?

KEY **C 5.7**

Writing $\dot{m} = k\rho^a g^b D^c$ and matching dimensions, $\text{kg s}^{-1} = (\text{kg m}^{-3})^a (\text{m s}^{-2})^b \text{m}^c$ gives $a = 1$ from mass, $b = \frac{1}{2}$ from time and $c = 3 - \frac{1}{2} = 5/2$ from length, so $\dot{m} \propto \rho g^{1/2} D^{5/2}$ and doubling D multiplies $\dot{m}$ by $2^{5/2} = 5.7$. B is the answer of a candidate who assumes the flow rate scales with the area of the hole, which silently assumes the sand leaves at a speed independent of D ; the dimensional result shows that the exit speed must scale as $\sqrt{gD}$. D comes from forgetting that g carries a dimension of length, so that the length balance reads $c - 3 = 0$ and $c = 3$. A, $2^{3/2}$, combines an exit speed $\sqrt{gD}$ with an opening proportional to D rather than D^2 , as if the hole were a long slot of width D instead of a circle.

U2

2 MULTIPLE CHOICE · ROUTINE · UPI-01.06 · UPI-03.01 **[1]**
CORE COURSE

A particle moves in the xy-plane with position vector $\mathbf{r}(t) = (2.0t^2 - 8.0t) \hat{i} + (6.0t - 1.0t^2) \hat{j}$, where $\mathbf{r}$ is in metres and t is in seconds. At what time is the speed of the particle least?

KEY **B 2.2 s**

The velocity is $\mathbf{v} = (4.0t - 8.0) \hat{i} + (6.0 - 2.0t) \hat{j}$ and the acceleration is the constant $\mathbf{a} = 4.0 \hat{i} - 2.0 \hat{j}$. Because $d(\mathbf{v} \cdot \mathbf{v})/dt = 2\mathbf{v} \cdot \mathbf{a}$, the speed is stationary when $\mathbf{v} \cdot \mathbf{a} = 20t - 44 = 0$, which gives $t = 2.2$ s, where the speed is 1.8 m/s. A is the instant at which the x-component of velocity vanishes and D the instant at which the y-component vanishes; each is tempting because in one-dimensional motion the speed is least where the velocity is zero, but here the other component is still non-zero, and the speeds at those instants are 2.0 m/s and 4.0 m/s. C simply averages those two instants, which has no physical basis: where the minimum falls is set by the components of the acceleration, not by splitting the difference.

U2

3 MULTIPLE CHOICE · ROUTINE · UPI-04.08 · UPI-05.01 **[1]**
CORE COURSE

A 12 kg crate rests on a loading ramp inclined at 20° to the horizontal. The coefficients of friction between the crate and the ramp are $\mu_s = 0.60$ and $\mu_k = 0.45$. Take $g = 9.81 \text{ m/s}^2$. What is the magnitude of the friction force that the ramp exerts on the crate?

KEY **A 40 N**

Because $\tan 20^\circ = 0.36$ is less than μ_s , the crate stays at rest, and static friction takes whatever value equilibrium requires: along the ramp $f = mg \sin 20^\circ = 40 \text{ N}$. C is $\mu_s N = \mu_s mg \cos 20^\circ = 66 \text{ N}$, which is only the largest value static friction can reach; candidates who treat $\mu_s N$ as the friction force rather than as a ceiling choose it. B is $\mu_k mg \cos 20^\circ = 50 \text{ N}$ and applies the kinetic coefficient to a crate that is not sliding. D is $\mu_s mg = 71 \text{ N}$, which compounds the ceiling error by taking the normal force on an incline to be the whole weight.

U1

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-07.04 · UPI-07.05

[1]

An atom adsorbed on a crystal surface can move only along a straight row of binding sites. Along the row its potential energy is $U(x) = -U_0 \cos(2\pi x/a)$, where U_0 is a positive constant and a is the spacing of the sites, so that the sites lie at $x = 0, \pm a, \pm 2a$ and so on. For positions between $x = 0$ and $x = a/2$, where is the force that the surface exerts on the atom along the row greatest in magnitude, and in which direction does that force point there?

KEY D At $x = a/4$, pointing towards $x = 0$

Along the row the force is $F = -dU/dx = -(2\pi U_0/a) \sin(2\pi x/a)$. Between $x = 0$ and $x = a/2$ the sine is positive, so F is negative: the surface pulls the atom back towards the site at $x = 0$. The magnitude of F is greatest where the sine equals 1, at $x = a/4$, which is where U changes most steeply with position; U is zero there, midway between its least value $-U_0$ at the site and its greatest value $+U_0$ at $x = a/2$. B has the right position and the wrong direction: it follows from writing $F = +dU/dx$, which would push the atom up the slope of U , away from the site where its energy is least. A and C tempt candidates who equate a large potential energy with a large force and so pick the top of the barrier at $x = a/2$. But U has zero slope at its maximum, so the force there is zero and the atom is in unstable equilibrium, whichever direction is attached to it. A pairs that position with the direction the force has everywhere else in the range, and C pairs it with the reversed direction that the sign error in B produces.

U1

5

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPI-08.01 · UPI-08.02

[1]

A 0.170 kg ice-hockey puck slides at 6.00 m/s directly towards a player, who strikes it straight back along the same line. The force of the stick on the puck, directed away from the player, rises linearly from zero to 800 N in 4.0 ms, stays at 800 N for 3.0 ms, then falls linearly to zero in a further 3.0 ms. Friction with the ice is negligible. What is the speed of the puck just after the strike?

KEY B 24.6 m/s

The impulse is the area under the force-time profile: $1.6 \text{ N s} + 2.4 \text{ N s} + 1.2 \text{ N s} = 5.2 \text{ N s}$. Taking the direction away from the player as positive, the momentum changes from $(0.170)(-6.00) = -1.02 \text{ kg m/s}$ to $-1.02 + 5.2 = 4.18 \text{ kg m/s}$, so the puck leaves at 24.6 m/s. C is $J/m = 30.6 \text{ m/s}$, the answer of a candidate who treats the puck as starting from rest and forgets that part of the impulse is spent reversing it. D adds the incoming speed instead of subtracting it, a sign error in the initial momentum. A treats the pulse as a single triangle of height 800 N spread over the whole 10 ms, which gives an impulse of only 4.0 N s because it discards the plateau.

U3

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-09.05 · UPI-09.07

[1]

A bicycle wheel of radius 0.35 m rolls without slipping along a level road, its centre moving at 7.0 m/s. Relative to the road, what are the speed of the point at the top of the rim and the speed of a point on the rim level with the axle?

KEY A 14 m/s at the top; 9.9 m/s level with the axle

The velocity of any point on the rim is the velocity of the centre plus the velocity due to rotation about the centre, of magnitude $\omega R = 7.0 \text{ m/s}$ and tangent to the rim. At the top both contributions point forwards, giving 14 m/s; level with the axle the rotational contribution is vertical, perpendicular to the forward 7.0 m/s, so the speed is $7.0\sqrt{2} = 9.9 \text{ m/s}$. C gives every rim point the rim speed ωR , which is correct only in the frame moving with the axle. D adds the two speeds as scalars at the side as well as at the top, ignoring that they are perpendicular there. B keeps only the translational part at the side, as though a vertical contribution could not add to the speed.

U1

7

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPI-01.06 · UPI-11.05

[1]

A spinning bicycle wheel is mounted at one end of a light horizontal axle whose other end rests on a frictionless pivot at the origin. The axle lies along the x-axis with the wheel's centre at $x = +0.150$ m, and the z-axis points vertically upwards. The wheel has mass 2.00 kg and moment of inertia 0.120 kg m^2 about the axle, and it spins at 40.0 rad/s with its spin angular momentum pointing in the +x direction. Take $g = 9.81 \text{ m/s}^2$. In the fast-spin approximation, at what rate does the axle precess, and in which sense does it turn when viewed from above, looking down the z-axis?

KEY D 0.61 rad/s, counterclockwise

About the pivot the weight exerts a torque $\tau = r \times F = (0.150 \text{ m } \hat{i}) \times (-19.6 \text{ N } \hat{k}) = +2.94 \text{ N m } \hat{j}$. That torque changes the angular momentum $L = I\omega \hat{i} = 4.80 \text{ kg m}^2/\text{s } \hat{i}$ in its own direction, so L swings from +x towards +y, which is counterclockwise seen from above, at $\Omega = \tau/L = 2.94/4.80 = 0.61 \text{ rad/s}$. A has the rate right and the sense wrong: it follows from computing $F \times r$ instead of $r \times F$, or from letting L swing opposite to the torque, when dL/dt must point along τ . B and C invert the ratio, but $L/\tau = 1.6 \text{ s}$ is a time, not a rate, and it grows as the spin gets faster, whereas faster spin must mean slower precession because the same torque then turns a larger angular momentum.

U1

8

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-13.02 · UPI-13.05

[1]

Transit and radial-velocity observations show that a rocky exoplanet has 4.5 times the mass of Earth and 1.5 times its radius. Treat both planets as spherically symmetric. Compared with the values for Earth, how do the gravitational field strength at the planet's surface and the escape speed from its surface change?

KEY A Field strength doubles; escape speed increases by a factor of 1.7

The surface field is $g = GM/R^2$, which scales as $4.5/1.5^2 = 2.0$, and the escape speed is $\sqrt{2GM/R}$, which scales as $\sqrt{4.5/1.5} = \sqrt{3.0} = 1.7$. C uses GM/R for the field strength, which gives $4.5/1.5 = 3.0$ and forgets that the inverse-square law squares the radius. D drops the square root from the escape speed: the energy balance $\frac{1}{2}mv^2 = GMm/R$ fixes v^2 , which grows by a factor of 3.0, not v . B writes the escape speed as $\sqrt{2gR}$ and scales g by 2.0 while leaving R unchanged, which gives $\sqrt{2.0} = 1.4$ and misses that the radius has grown by half as well.

U2

9

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPI-14.03 · UPI-14.04

[1]

A block attached to a horizontal spring slides on a frictionless surface in simple harmonic motion with an amplitude of 0.12 m and a period of 0.80 s. What is the speed of the block when its displacement from equilibrium is 0.060 m?

KEY C 0.82 m/s

Energy conservation in simple harmonic motion gives $v = \omega\sqrt{A^2 - x^2}$, with $\omega = 2\pi/T = 7.85 \text{ rad/s}$, so $v = 7.85 \times \sqrt{0.0144 - 0.0036} = 0.82 \text{ m/s}$. D is the maximum speed ωA , which the block has only as it passes through equilibrium. B is half the maximum speed, the answer of a candidate who assumes the speed falls in proportion to the displacement; because the kinetic energy is $\frac{1}{2}k(A^2 - x^2)$, three quarters of the energy is still kinetic at half the amplitude, so the speed is $\sqrt{3/2}$ of its maximum. A uses the frequency $1/T$ in place of the angular frequency $2\pi/T$, which makes it a factor of 2π too small.

U3

10

MULTIPLE CHOICE · ROUTINE · UPI-15.06 · UPI-15.07

[1]

EXTENSION

Water of density 1000 kg/m^3 flows steadily through a horizontal pipe whose internal diameter narrows from 0.040 m to 0.020 m . The speed of the water in the wide section is 1.5 m/s . Treating the water as an ideal incompressible fluid, by how much is the pressure in the narrow section lower than the pressure in the wide section?

KEY **C 17 kPa**

Continuity requires $A_1 v_1 = A_2 v_2$, and halving the diameter divides the cross-sectional area by four, so $v_2 = 6.0 \text{ m/s}$. Bernoulli's equation along a horizontal streamline then gives $p_1 - p_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) = 500 \times (36 - 2.25) = 1.7 \times 10^4 \text{ Pa}$. A scales the speed with the diameter instead of with the area, taking $v_2 = 3.0 \text{ m/s}$. D leaves out the kinetic-energy term for the wide section, which is small here but not negligible. B squares the difference of the speeds, $\frac{1}{2} \rho (v_2 - v_1)^2$, which is not a term that appears in Bernoulli's equation.

U3

A laboratory test rig stores braking energy in a flywheel: a uniform steel annular disc of inner radius $R_1 = 0.100$ m, outer radius $R_2 = 0.300$ m and thickness 0.0500 m, which spins about its symmetry axis. The density of the steel is 7.80×10^3 kg/m³. In normal operation the flywheel spins at $\omega_0 = 314$ rad/s.

(a) SHOW THAT · U2

[3]

Starting from the definition $I = \int r^2 dm$, show that the moment of inertia of a uniform annular disc of mass M about its symmetry axis is $I = \frac{1}{2}M(R_1^2 + R_2^2)$.

- 1 Divides the disc into thin rings of radius r and width dr , each of mass $dm = \sigma(2\pi r dr)$, where $\sigma = M/[\pi(R_2^2 - R_1^2)]$ is the mass per unit area
- 2 Writes $I = \int r^2 \sigma(2\pi r) dr$ from R_1 to $R_2 = (\pi\sigma/2)(R_2^4 - R_1^4)$
- 3 Factorises $R_2^4 - R_1^4 = (R_2^2 - R_1^2)(R_2^2 + R_1^2)$ and substitutes for σ to obtain $I = \frac{1}{2}M(R_1^2 + R_2^2)$

(b) CALCULATE · U3

[3]

Calculate the mass of the flywheel, its moment of inertia, and the kinetic energy it stores at ω_0 .

- 1 $M = \rho\pi(R_2^2 - R_1^2) \times \text{thickness} = 7800 \times \pi \times 0.0800 \times 0.0500 = 98.0$ kg
- 2 $I = \frac{1}{2} \times 98.0 \times (0.0100 + 0.0900) = 4.90$ kg m²
- 3 $K = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 4.90 \times 314^2 = 2.42 \times 10^5$ J

(c) DETERMINE · U3

[4]

When the flywheel is disconnected at ω_0 it coasts to rest in 1.50×10^3 s under a constant frictional torque from its bearings. Determine the frictional torque, the number of revolutions the flywheel makes while coasting, and the rate at which friction dissipates energy at the instant the flywheel is disconnected.

- 1 Constant angular deceleration $\alpha = \omega_0/t = 314/1500 = 0.209$ rad/s²
- 2 Frictional torque $\tau = I\alpha = 4.90 \times 0.2093 = 1.03$ N m (accept 1.02 N m from rounded intermediate values)
- 3 Angle turned $\theta = \frac{1}{2}\omega_0 t = 2.36 \times 10^5$ rad, which is $\theta/(2\pi) = 3.75 \times 10^4$ revolutions
- 4 Initial dissipation rate $P = \tau\omega_0 = 1.026 \times 314 = 322$ W (accept 320 W to 323 W; consistent with $\tau\theta = 2.42 \times 10^5$ J, the whole of the stored energy)

(d) COMPARE · U1

[2]

Compare the energy this flywheel stores at ω_0 with the energy stored by a uniform solid disc of the same mass and the same outer radius spinning at the same angular speed, and explain the difference in terms of how the mass is distributed.

- 1 Solid disc: $I = \frac{1}{2}MR_2^2 = 4.41$ kg m², which stores $\frac{1}{2}I\omega_0^2 = 2.17 \times 10^5$ J, about 10% less than the annular flywheel
- 2 The annulus has all of its mass at 0.100 m or more from the axis, where each kilogram contributes a larger r^2 to I , so for the same mass and the same ω it has the larger moment of inertia and stores more kinetic energy

An electric car of mass 1500 kg, driver included, starts from rest on a level road. Its motor controller limits the power delivered to the driven wheels to $P = 60.0$ kW. Ignore air resistance and rolling resistance throughout, and take $g = 9.81$ m/s².

- (a) **SHOW THAT** · U2 [2]
- Suppose the full power P is delivered from the instant the car starts. Show that the speed of the car at time t is $v(t) = \sqrt{(2Pt/m)}$.
- 1 Applies the work–kinetic energy theorem: the work done on the car in time t is $\int_0^t P \, dt = Pt$, and this equals the gain in kinetic energy $\frac{1}{2}mv^2$ from rest
 - 2 Rearranges $Pt = \frac{1}{2}mv^2$ to $v = \sqrt{(2Pt/m)}$
- Candidates who start from the constant-acceleration equations reach an expression that looks plausible and is wrong. At constant power the driving force P/v falls as the car speeds up, so none of those equations applies.*
- (b) **CALCULATE** · U3 [2]
- Using this model, calculate the time the car takes to reach 25.0 m/s and its acceleration at that instant.
- 1 $t = mv^2/(2P) = (1500 \times 25.0^2)/(1.20 \times 10^5) = 7.81 \, \text{s}$
 - 2 The driving force is $P/v = 2.40 \times 10^3 \, \text{N}$, so $a = P/(mv) = 1.60 \, \text{m/s}^2$
- The usual slip is $a = v/t = 3.20 \, \text{m/s}^2$. That is the average acceleration over the whole run, twice the instantaneous value at 25.0 m/s.*
- (c) **DETERMINE** · U2 [3]
- Determine, by integrating $v(t)$, the distance the car travels in reaching 25.0 m/s under this model.
- 1 Sets up $x = \int_0^t \sqrt{(2P/m)} \, t^{1/2} \, dt$
 - 2 Integrates to $x = (2/3)\sqrt{(2P/m)} \, t^{3/2}$, which equals $(2/3)vt$
 - 3 $x = (2/3) \times 25.0 \times 7.81 = 130 \, \text{m}$
- Candidates who use $x = \frac{1}{2}(u + v)t$, valid only for uniform acceleration, obtain 97.7 m. Here the speed rises steeply at first and then more slowly, so the average speed is two-thirds of the final speed, not half.*
- (d) **EVALUATE** · U1 [3]
- The coefficient of static friction between the tyres and the road is 0.80, and all four wheels are driven, so the car's whole weight is available for traction. Evaluate the constant-power model close to $t = 0$: find the speed below which the car cannot use its full power, and state what limits its motion below that speed.
- 1 The model gives $a = P/(mv)$, which grows without limit as $v \rightarrow 0$, so it demands an arbitrarily large forward friction force from the road and cannot hold at the start
 - 2 The largest traction force is $\mu_s mg = 0.80 \times 1500 \times 9.81 = 1.18 \times 10^4 \, \text{N}$, so full power requires $P/v \leq 1.18 \times 10^4 \, \text{N}$, that is $v \geq P/(\mu_s mg) = 5.10 \, \text{m/s}$
 - 3 Below 5.10 m/s traction, not power, is the limit: the car can accelerate at no more than $\mu_s g = 7.85 \, \text{m/s}^2$, so the constant-power model is valid only above 5.10 m/s
- The judgement expected is conditional: the model fails at low speed and is sound above 5.10 m/s. A candidate who simply calls the model wrong, or simply accepts it, has not evaluated it.*
- (e) **SKETCH** · U1 [2]
- The real car accelerates at the traction limit until it can use its full power, then continues at constant power. Sketch a graph of its speed against time from $t = 0$ until it reaches 25.0 m/s, and mark the point where the two stages join.
- 1 A straight line from the origin with gradient $7.85 \, \text{m/s}^2$, ending at the marked join at about (0.65 s, 5.1 m/s)
 - 2 From the join, a concave-down curve that starts with the same gradient as the line (no corner) and flattens steadily, reaching 25.0 m/s at about 8.1 s, later than the 7.81 s of the full-power model
- Most sketches show a kink at the join. There is none, because 5.10 m/s is by construction the speed at which $P/(mv)$ equals $\mu_s g$. The later arrival should also show: during the traction-limited stage the power rises from zero, so on average only half the available power is used.*

A uniform horizontal steel beam of length 3.00 m and mass 45.0 kg projects from a vertical wall, to which one end is attached by a frictionless hinge. A steel cable runs from the outer end of the beam to a point on the wall directly above the hinge, making an angle of 30.0° with the beam. A shop sign of mass 80.0 kg hangs from the beam 2.40 m from the wall. The cable is 3.46 m long and 4.00 mm in diameter, and its steel has Young modulus 2.00×10^{11} Pa and yield strength 2.50×10^8 Pa. The weight of the cable is negligible. Take $g = 9.81 \text{ m/s}^2$.

(a) EXPLAIN · U1

[2]

Explain why taking torques about the hinge is the most efficient first step in finding the tension in the cable.

- 1 The force exerted by the hinge on the beam is unknown in both magnitude and direction
- 2 That force acts at the hinge, so it has zero lever arm and zero torque about an axis through the hinge, which leaves the cable tension as the only unknown in the torque equation

Candidates who take torques about the centre of the beam lose the horizontal forces, which act along the beam, but bring the vertical component of the hinge force into the torque equation alongside the tension, so they need the vertical force equation as well before either unknown is found. The route is valid but slower, and it is where most sign errors on this question are made.

(b) CALCULATE · U3

[2]

Calculate the tension in the cable.

- 1 Torques about the hinge: $T \sin 30.0^\circ \times 3.00 \text{ m} = (45.0)(9.81)(1.50) + (80.0)(9.81)(2.40) = 662 + 1884 = 2546 \text{ N m}$
- 2 $T = 2546 / (3.00 \times \sin 30.0^\circ) = 1.70 \times 10^3 \text{ N}$

The weight of a uniform beam acts at its centre, 1.50 m from the hinge, not at its end. Using $\cos 30.0^\circ$ in the lever arm, which gives 980 N, shows that the component of the tension perpendicular to the beam was not identified.

(c) DETERMINE · U3

[3]

Determine the magnitude and direction of the force exerted by the hinge on the beam.

- 1 Horizontal equilibrium: $H = T \cos 30.0^\circ = 1.47 \times 10^3 \text{ N}$, directed away from the wall
- 2 Vertical equilibrium: $V = (125.0 \text{ kg})(9.81 \text{ m/s}^2) - T \sin 30.0^\circ = 1226.3 \text{ N} - 848.6 \text{ N} = 378 \text{ N}$, directed upwards
- 3 Magnitude $\sqrt{H^2 + V^2} = 1.52 \times 10^3 \text{ N}$, at 14.4° above the horizontal and pointing away from the wall

The hinge force does not lie along the beam. Candidates who assume it does, as it would for a light strut pinned at both ends, set $V = 0$ and then cannot satisfy vertical equilibrium.

(d) CALCULATE · U3

[2]

Calculate the tensile stress in the cable and the extension of the cable.

- 1 $A = \pi(2.00 \times 10^{-3} \text{ m})^2 = 1.26 \times 10^{-5} \text{ m}^2$, so the stress is $\sigma = T/A = 1.35 \times 10^8 \text{ Pa}$
- 2 Strain $= \sigma/E = 6.75 \times 10^{-4}$, so the extension is $6.75 \times 10^{-4} \times 3.46 \text{ m} = 2.34 \times 10^{-3} \text{ m}$

Using the diameter in place of the radius when finding the area gives a stress four times too small, $3.38 \times 10^7 \text{ Pa}$, which would then pass the check in part (e) that the cable in fact fails.

(e) EVALUATE · U3 [2]

The building code requires every load-bearing cable to have a safety factor of at least 2.0 against yield. Evaluate whether this cable complies, and recommend the smallest cable diameter that would.

1 Safety factor = $(2.50 \times 10^8 \text{ Pa}) / (1.35 \times 10^8 \text{ Pa}) = 1.85$, which is below 2.0, so the cable does not comply

2 Compliance requires $\sigma \leq 1.25 \times 10^8 \text{ Pa}$, so $A \geq 1.36 \times 10^{-5} \text{ m}^2$ and $d \geq 4.16 \text{ mm}$; recommends the next available size above this, for example 4.5 mm or 5 mm

The judgement must follow from the criterion given. A cable working at 54% of its yield stress will not fail today, and candidates who call it safe enough have ignored the code they were asked to apply.

(f) DESCRIBE · U1 [2]

Describe what the stress-strain curve for the cable steel would show if the cable were loaded beyond its yield point and then unloaded, and what this would mean for the sign.

1 The curve is straight only up to the proportional limit; past the elastic limit and the yield point it lies well off that line and the strain increases substantially for little further increase in stress: the steel deforms plastically

2 On unloading, the curve returns along a straight line parallel to the original elastic line and meets the strain axis at a non-zero strain, so the cable keeps a permanent extension and the beam and sign are left sagging

The common error is to say the cable returns to its original length once the load is removed. That holds only below the elastic limit; beyond it the unloading path runs parallel to the loading line rather than retracing the curve.

14 CORE COURSE

STRUCTURED · DEMANDING · UPI-07.07 · UPI-08.02 · UPI-08.03 · UPI-08.05

[14]

Car A, of mass 1200 kg and travelling due east, and car B, of mass 1800 kg and travelling due north, collide in the middle of an intersection and lock together. Skid marks show that the wreck then slid in a straight line for 8.00 m, directed 35.0° north of east, before stopping. The coefficient of kinetic friction between the tyres and the road is 0.65, and $g = 9.81 \text{ m/s}^2$. The speed limit at the intersection is 13.9 m/s (50 km/h).

(a) JUSTIFY · U1 [2]

Justify treating the two cars as an isolated system for the duration of the collision, even though friction from the road acts on them.

1 The collision lasts a very short time, of order 0.1 s, and during it the forces between the cars are of order 10^5 N , several times the largest friction force the road can exert on the wreck, $\mu_k(3000 \text{ kg})g \approx 1.9 \times 10^4 \text{ N}$

2 The external impulse from the road over that brief interval is therefore small compared with the momentum of either car, so the total momentum just after the collision equals the total just before

Candidates often assert that friction is negligible, which is plainly false: friction is what stops the wreck. The argument that earns credit compares impulses over the duration of the collision, not forces over the whole event.

(b) CALCULATE · U3 [2]

Calculate the speed of the wreck immediately after the collision.

1 The work done by kinetic friction removes all the wreck's kinetic energy: $\frac{1}{2}(m_A + m_B)V^2 = \mu_k(m_A + m_B)g d$, so $V = \sqrt{2\mu_k g d}$

2 $V = \sqrt{(2 \times 0.65 \times 9.81 \times 8.00)} = 10.1 \text{ m/s}$

The mass cancels only if the same mass is used in the kinetic energy and in the friction force. Candidates who use one car's mass in one of them and the combined mass in the other obtain a wrong speed.

- (c) DETERMINE · U3 [3]
- Determine the speed of each car immediately before the collision.
- 1 Momentum just after: $(3000 \text{ kg})(10.10 \text{ m/s}) = 3.030 \times 10^4 \text{ kg m/s}$, with components $2.482 \times 10^4 \text{ kg m/s}$ east and $1.738 \times 10^4 \text{ kg m/s}$ north
 - 2 The east component is carried only by car A before the collision: $v_A = (2.482 \times 10^4)/1200 = 20.7 \text{ m/s}$
 - 3 The north component is carried only by car B: $v_B = (1.738 \times 10^4)/1800 = 9.66 \text{ m/s}$ (accept 9.67 m/s from a rounded component)
- This part catches conservation of speed or of kinetic energy in place of vector momentum, and the pairing of $\sin 35.0^\circ$ with car A. Each component equation involves only one car, which is what makes the reconstruction unique.*
- (d) CALCULATE · U3 [2]
- Calculate the fraction of the cars' kinetic energy that was dissipated in the collision itself.
- 1 Just before: $\frac{1}{2}(1200)(20.7)^2 + \frac{1}{2}(1800)(9.66)^2 = 2.57 \times 10^5 \text{ J} + 8.40 \times 10^4 \text{ J} = 3.41 \times 10^5 \text{ J}$; just after: $\frac{1}{2}(3000)(10.1)^2 = 1.53 \times 10^5 \text{ J}$
 - 2 Fraction dissipated = $(3.41 \times 10^5 - 1.53 \times 10^5)/(3.41 \times 10^5) = 0.55$
- Candidates who also subtract the energy removed during the skid count that energy twice. The question concerns only the collision, from just before it to just after it.*
- (e) ESTIMATE · U4 [2]
- Estimate the magnitude of the average force exerted on a 75 kg occupant of car B, held by a seatbelt, during the collision. State the assumptions you make.
- 1 States the assumptions: the occupant moves with car B, whose velocity changes from 9.66 m/s north to the wreck's velocity of 8.27 m/s east and 5.79 m/s north, over a collision time of about 0.1 s
 - 2 $|\Delta v| = \sqrt{(8.27^2 + 3.87^2)} \approx 9.1 \text{ m/s}$, so $F \approx (75 \text{ kg})(9.1 \text{ m/s})/(0.1 \text{ s}) \approx 7 \times 10^3 \text{ N}$, roughly nine times the occupant's weight
- The change in velocity is a vector difference. Subtracting speeds gives $10.1 - 9.66 \approx 0.4 \text{ m/s}$ and a force about twenty times too small, which should be rejected on physical grounds: a collision that locks two cars together is not a gentle nudge.*
- (f) EVALUATE · U4 [3]
- The driver of car A claims not to have exceeded the speed limit. The investigator's values are uncertain: μ_k lies between 0.55 and 0.75, and the direction of the skid is $35^\circ \pm 3^\circ$ north of east. Evaluate the driver's claim.
- 1 Writes $v_A = (m_A + m_B)\sqrt{(2\mu_k g d)} \cos \theta / m_A$ and identifies that the smallest value comes from the smallest μ_k , 0.55, and the largest angle, 38°
 - 2 $v_{A,\min} = (3000/1200) \times \sqrt{(2 \times 0.55 \times 9.81 \times 8.00)} \times \cos 38^\circ = 18.3 \text{ m/s}$ (66 km/h)
 - 3 Even at the least favourable limits of the evidence v_A exceeds 13.9 m/s by about a third, so the evidence contradicts the driver's claim
- An evaluation must test the claim at the edge of the uncertainty, not at the central values. Candidates who recalculate only with $\mu_k = 0.65$ and 35° have not established that the conclusion survives the uncertainty.*

A uniform rod of mass $M = 2.00 \text{ kg}$ and length $L = 1.20 \text{ m}$ hangs vertically at rest from a frictionless horizontal pivot through its upper end. A lump of clay of mass $m = 0.400 \text{ kg}$, moving horizontally at 6.00 m/s perpendicular to the pivot axis, strikes the lower end of the rod and sticks to it. Treat the clay as a particle and take $g = 9.81 \text{ m/s}^2$. The moment of inertia of a uniform rod about a perpendicular axis through its centre is $ML^2/12$.

- (a) STATE · U1 [1]
 State the condition under which the angular momentum of a system about a fixed point is conserved.
- 1 The net external torque on the system about that point is zero (equivalently, the net external angular impulse about it is zero)
Answering that there must be no external force states a stronger condition than is needed, and one that fails here: the pivot exerts a large external force, yet angular momentum about the pivot is still conserved.
- (b) CALCULATE · U3 [2]
 Calculate the moment of inertia of the rod and clay together about the pivot, immediately after the collision.
- 1 Parallel-axis theorem for the rod: $I_{\text{rod}} = ML^2/12 + M(L/2)^2 = ML^2/3 = 0.960 \text{ kg m}^2$
 - 2 Adds the clay as a particle at distance L : $mL^2 = 0.576 \text{ kg m}^2$, giving $I = 1.536 \text{ kg m}^2$ (1.54 kg m²)
Using $ML^2/12$, the value about the centre, for a rod turning about its end gives $I = 0.816 \text{ kg m}^2$ and an exaggerated swing in part (d).
- (c) EXPLAIN · U1 [2]
 Explain why the linear momentum of the rod-clay system is not conserved in the collision, whereas its angular momentum about the pivot is.
- 1 The pivot holds the top of the rod in place, so during the collision it exerts a large horizontal impulsive force on the rod; that external force changes the linear momentum of the system
 - 2 The pivot force acts at the pivot and has zero torque about it, and the rod's weight acts along the vertical line through the pivot while the rod hangs straight down, with any gravitational angular impulse negligible over the brief collision; so the net external angular impulse about the pivot is zero and angular momentum about it is conserved
Candidates who conserve linear momentum, $(0.400)(6.00) = (2.40 \text{ kg})v$, reach a quite different answer in part (d). Choosing the conservation law is the substance of the question.
- (d) DETERMINE · U3 [3]
 Determine the angular speed of the rod immediately after the collision and the greatest angle from the vertical through which it then swings.
- 1 Angular momentum about the pivot: $mvL = (0.400)(6.00)(1.20) = 2.88 \text{ kg m}^2/\text{s} = I\omega$, so $\omega = 2.88/1.536 = 1.875 \text{ rad/s}$ (1.88 rad/s)
 - 2 Mechanical energy is conserved during the swing: $\frac{1}{2}I\omega^2 = 2.70 \text{ J} = (MgL/2 + mgL)(1 - \cos \theta) = (16.5 \text{ J})(1 - \cos \theta)$
 - 3 $1 - \cos \theta = 0.164$, so $\theta = 33.3^\circ$ (accept 33.2° from rounded values)
Energy is conserved during the swing but not during the collision. Candidates who equate the clay's initial 7.20 J to the gain in potential energy predict a swing of 55.7° ; only 37.5% of that kinetic energy survives the collision.
- (e) DETERMINE · U2 [2]
 Determine the horizontal impulse exerted by the pivot on the rod during the collision, including its direction.
- 1 Linear momentum just after is $(ML/2 + mL)\omega = (1.20 + 0.480)(1.875) = 3.15 \text{ kg m/s}$, compared with $mv = 2.40 \text{ kg m/s}$ just before
 - 2 The pivot's impulse is $3.15 - 2.40 = 0.750 \text{ N s}$, directed the same way as the clay was moving
The direction surprises most candidates, who expect the pivot to resist the blow. Struck below its centre of percussion, a free rod's upper end would move backwards, so the pivot has to push it forwards.

(f) DERIVE · U2

[3]

Derive, from conservation of angular momentum about the pivot and the impulse-momentum theorem, the distance x below the pivot at which the clay must strike for the pivot to exert no horizontal impulse during the collision. Show that the result does not depend on the mass of the clay.

- 1 Angular momentum about the pivot: $mvx = (ML^2/3 + mx^2)\omega$
- 2 No pivot impulse means linear momentum is conserved: $mv = (ML/2 + mx)\omega$
- 3 Dividing the two equations eliminates v and ω : $x(ML/2 + mx) = ML^2/3 + mx^2$, so the mx^2 terms cancel and $x = 2L/3 = 0.800$ m whatever the value of m

Candidates who write the linear-momentum condition with the rod alone, $mv = (ML/2)\omega$, lose the cancellation and find an x that depends on m . The point $2L/3$ is the rod's centre of percussion, the sweet spot of a bat.

16
CORE
COURSE

DATA ANALYSIS · DEMANDING · UPI-01.09 · UPI-14.04 · UPI-14.07

[12]

A glider of mass 0.250 kg on a level air track is held between two springs and oscillates with a period of 0.800 s. A card sail fixed to the glider increases the air drag. A motion sensor records the amplitude of the oscillation, the greatest displacement from equilibrium, at the end of every fifth oscillation, with an uncertainty of ± 0.3 mm in each reading. Two damping models are proposed. Model V (viscous drag): the resistive force is $-bv$, so the amplitude decays as $A = A_0 e^{-(bt/(2m))}$. Model F (sliding friction): a resistive force of constant magnitude acts, which reduces the amplitude by the same amount in every oscillation.

(a) DESCRIBE · U1

[2]

Describe the pattern that each model predicts in amplitudes recorded at equal intervals of time.

- 1 Model V: each amplitude is the same fixed fraction of the one before, so successive ratios are constant and $\ln A$ falls linearly with t
- 2 Model F: each amplitude is smaller than the one before by the same fixed amount, so successive differences are constant and A falls linearly with t , reaching zero at a finite time

Many candidates describe both models as 'the amplitude decreases', which is true of both and distinguishes nothing. The marks are for naming the quantity that stays constant.

(b) JUSTIFY · U4

[3]

Justify, using calculations from the table, which model the data support.

- 1 Successive differences are 22.4, 16.8, 11.2, 8.7, 5.6 and 4.5 mm: they fall steadily by a factor of five, far beyond the ± 0.3 mm reading uncertainty, so Model F is rejected
- 2 Successive ratios are 0.720, 0.708, 0.725, 0.706, 0.732 and 0.706: constant at about 0.72, with a scatter of about $\pm 2\%$ that the ± 0.3 mm reading uncertainty can account for (combining fractional uncertainties in quadrature, it is $\pm 2.4\%$ in 15.3/20.9 and $\pm 3.4\%$ in 10.8/15.3)
- 3 Concludes that the data support Model V, viscous drag

An answer that says only that A against t is curved earns part of the credit: curvature rules out Model F but does not by itself show that the decay is exponential. The ratio test, set against the reading uncertainty, does both.

(c) DETERMINE · U4

[3]

Determine the damping constant b by a method that uses all of the readings.

- 1 Uses $\ln A = \ln A_0 - (b/2m)t$: tabulates $\ln(A/\text{mm}) = 4.382, 4.054, 3.709, 3.388, 3.040, 2.728, 2.380$ and plots or fits $\ln A$ against t
- 2 Best-fit gradient = -0.0833 s^{-1} (accept -0.080 s^{-1} to -0.086 s^{-1})
- 3 $b = -2m \times \text{gradient} = 2 \times 0.250 \text{ kg} \times 0.0833 \text{ s}^{-1} = 0.0416 \text{ kg/s}$ (accept 0.040 kg/s to 0.043 kg/s)

A single pair of adjacent readings gives b anywhere between 0.039 kg/s and 0.044 kg/s depending on the pair chosen, which is why the question asks for a method that uses every reading. Forgetting the factor of 2 in $b/(2m)$ halves the answer.

(d) CALCULATE · U3 [2]

Calculate the fraction of the glider's mechanical energy that is lost during one oscillation.

- 1 Mechanical energy is proportional to A^2 , so over one period it falls by the factor $e^{(-bT/m)} = e^{(-0.0416 \times 0.800/0.250)} = e^{(-0.133)} = 0.875$
- 2 Fraction lost per oscillation = $1 - 0.875 = 0.125$, about 12.5% (accept 12% to 13%)

Candidates who apply the amplitude ratio directly, $1 - e^{(-bT/(2m))} = 6.4\%$, have forgotten that the energy of an oscillator scales as the square of its amplitude.

(e) PREDICT · U1 [2]

A 0.250 kg brass block is fixed to the glider, which keeps the same sail and the same springs. Predict the time for the amplitude to fall to half its initial value, before and after the block is added, and give the reason for the change.

- 1 Before: $t_{1/2} = 2m \ln 2/b = (2 \times 0.250 \times 0.693)/0.0416 = 8.33 \text{ s}$
- 2 The sail, and so b , is unchanged while m doubles, so the decay constant $b/(2m)$ halves and $t_{1/2}$ doubles to 16.7 s (the period also lengthens, by $\sqrt{2}$ to 1.13 s, but that does not alter the decay time)

The usual prediction is that a heavier glider is damped faster because more mass means more friction. That reasoning belongs to Model F, which the data rejected; under viscous drag the resistive force depends on speed, not on weight, so extra inertia makes the decay slower.

17 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION: MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPI-11.06 · UPI-13.04 · UPI-13.05 · UPI-13.06

[15]

A communications satellite is raised from a circular parking orbit of radius $r_1 = 6.78 \times 10^6 \text{ m}$ to a circular orbit of radius $r_2 = 4.22 \times 10^7 \text{ m}$ by a two-burn transfer. A short engine burn at radius r_1 places it on an elliptical transfer orbit whose nearest point to Earth's centre (perigee) is at r_1 and whose farthest point (apogee) is at r_2 . A second short burn at apogee makes the orbit circular. For Earth, $GM = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2$. Treat both burns as instantaneous, treat Earth as a spherically symmetric mass, and ignore every other body.

(a) DERIVE · U2 [2]

Derive, from Newton's law of gravitation and Newton's second law, an expression for the speed v of a satellite in a circular orbit of radius r .

- 1 The gravitational force provides the centripetal acceleration: $GMm/r^2 = mv^2/r$
- 2 $v = \sqrt{GM/r}$, which is independent of the satellite's mass

Candidates who quote $v = \sqrt{2GM/r}$ have used the escape condition, which sets the kinetic energy equal to the depth of the potential well rather than balancing force against acceleration; it overstates the orbital speed by a factor of $\sqrt{2}$.

(b) EXPLAIN · U1 [2]

Explain why the satellite's angular momentum about Earth's centre is constant along the transfer orbit, and why its magnitude at perigee and at apogee is simply mr_v .

- 1 The gravitational force is central, directed along the line to Earth's centre, so its torque $\mathbf{r} \times \mathbf{F}$ about the centre is zero and the angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ is constant between the burns
- 2 At perigee and apogee r is at its minimum or maximum, so $dr/dt = 0$ and the velocity is perpendicular to the radius vector; there $|\mathbf{r} \times m\mathbf{v}| = mrv \sin 90^\circ = mrv$, which is not true elsewhere on the ellipse

Candidates who write $L = mrv$ at a general point of the ellipse set up equations that hold only at its two ends. The perpendicularity at perigee and apogee is what makes part (c) tractable.

- (c) **DERIVE** · U2 [4]
- Using conservation of angular momentum and conservation of mechanical energy between perigee and apogee, derive an expression for the speed v_p at perigee on the transfer orbit, in terms of GM , r_1 and r_2 .
- 1 Angular momentum: $r_1 v_p = r_2 v_a$, so $v_a = v_p r_1 / r_2$
 - 2 Energy: $\frac{1}{2} m v_p^2 - GMm/r_1 = \frac{1}{2} m v_a^2 - GMm/r_2$
 - 3 Substituting for v_a : $\frac{1}{2} v_p^2 (1 - r_1^2/r_2^2) = GM(1/r_1 - 1/r_2) = GM(r_2 - r_1)/(r_1 r_2)$
 - 4 Using $1 - r_1^2/r_2^2 = (r_2 - r_1)(r_2 + r_1)/r_2^2$ and cancelling $(r_2 - r_1)$ gives $v_p = \sqrt{(2GM r_2 / [r_1(r_1 + r_2)])}$
- Candidates who use the near-Earth form mgh for the potential energy cannot reach the result: g is far from constant over a sixfold change in radius, and the $-GMm/r$ form is required.*
- (d) **SHOW THAT** · U1 [2]
- Show that your expression reduces to the circular-orbit speed at r_1 when $r_2 = r_1$, and to the escape speed from r_1 as $r_2 \rightarrow \infty$, and state what each limit means physically.
- 1 $r_2 = r_1$ gives $v_p = \sqrt{(2GM r_1 / (2r_1^2))} = \sqrt{(GM/r_1)}$: an ellipse whose perigee and apogee coincide is the circular orbit, and no burn is needed
 - 2 As $r_2 \rightarrow \infty$, $r_2/(r_1 + r_2) \rightarrow 1$ and $v_p \rightarrow \sqrt{(2GM/r_1)}$: an ellipse whose apogee recedes without limit becomes the escape trajectory, with zero total mechanical energy
- Limit checks are where an algebraic slip in part (c) reveals itself. A candidate whose expression fails either limit should return to part (c) rather than carry the error into the numbers.*
- (e) **CALCULATE** · U3 [3]
- Calculate the change in speed required at each burn, and the total.
- 1 First burn: $v_1 = \sqrt{(GM/r_1)} = 7.67 \times 10^3$ m/s and $v_p = 1.007 \times 10^4$ m/s, so $\Delta v_1 = 2.40 \times 10^3$ m/s
 - 2 Second burn: $v_a = v_p r_1 / r_2 = 1.62 \times 10^3$ m/s and $v_2 = \sqrt{(GM/r_2)} = 3.07 \times 10^3$ m/s, so $\Delta v_2 = 1.46 \times 10^3$ m/s
 - 3 Total change in speed = 3.85×10^3 m/s
- The satellite speeds up at both burns yet ends at 3.07×10^3 m/s, well under half its initial 7.67×10^3 m/s. Candidates who find this contradictory have not separated the burns from the coast between them, during which the climb converts kinetic energy into gravitational potential energy.*
- (f) **CALCULATE** · U3 [2]
- Calculate the time the satellite spends on the transfer orbit, from perigee to apogee.
- 1 The semi-major axis is $a = (r_1 + r_2)/2 = 2.45 \times 10^7$ m, and Kepler's third law gives the period $T = 2\pi\sqrt{(a^3/GM)} = 3.81 \times 10^4$ s
 - 2 The transfer is half an orbit: $T/2 = 1.91 \times 10^4$ s, about 5.3 hours
- Using r_2 in Kepler's third law in place of the semi-major axis gives the period of the final orbit, 8.63×10^4 s, and a transfer time more than twice too long.*

18
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION: FROM A CONSERVATION LAW TO THE EQUATION OF MOTION · DISCRIMINATING
· UPI-09.07 · UPI-10.08 · UPI-14.02 · UPI-14.06

A seismic isolation bearing is modelled as a uniform solid cylinder of mass m and radius r that rolls without slipping on the inside of a fixed, concave cylindrical track of radius R , where $R > r$ and both axes are horizontal and parallel. The moment of inertia of the cylinder about its own axis is $\frac{1}{2}mr^2$. Let ϕ be the angle between the vertical and the line from the track's axis of curvature to the cylinder's axis, so that $\phi = 0$ at the lowest position. Take $g = 9.81$ m/s², and ignore rolling resistance and air resistance.

[15]

- (a) **EXPLAIN** · U1 [2]
- Explain why the speed of the cylinder's axis is $(R - r)|d\phi/dt|$, and use the rolling condition to obtain the cylinder's angular speed ω about its own axis in terms of $d\phi/dt$.
- 1 The cylinder's axis stays a fixed distance $R - r$ from the track's axis of curvature, so it moves on a circle of that radius with speed $v = (R - r)|d\phi/dt|$
 - 2 Rolling without slipping means the point of the cylinder in contact with the fixed track is instantaneously at rest, so $v = r\omega$ and $\omega = (R - r)|d\phi/dt|/r$
- Candidates who take the axis to move on a circle of radius R , or who set $\omega = d\phi/dt$, build the wrong kinetic energy into part (b) and cannot recover. The radius of the path is $R - r$, and the spin rate is $(R - r)/r$ times the rate at which the line to the axis turns, not equal to it.*
- (b) **DERIVE** · U2 [5]
- Derive, from conservation of mechanical energy, the equation of motion for ϕ .
- 1 Gravitational potential energy measured from the lowest position: $U = mg(R - r)(1 - \cos \phi)$
 - 2 Kinetic energy is translational plus rotational: $K = \frac{1}{2}m(R - r)^2(d\phi/dt)^2 + \frac{1}{2}(\frac{1}{2}mr^2)\omega^2$
 - 3 Substituting $\omega = (R - r)(d\phi/dt)/r$ gives $K = \frac{3}{4}m(R - r)^2(d\phi/dt)^2$
 - 4 Sets the time derivative of $K + U$ to zero: $(3/2)m(R - r)^2(d\phi/dt)(d^2\phi/dt^2) + mg(R - r)\sin \phi (d\phi/dt) = 0$
 - 5 Divides by $d\phi/dt$, which is non-zero during the motion, to obtain $d^2\phi/dt^2 = -[2g/(3(R - r))]\sin \phi$
- Leaving out the rotational kinetic energy, the error this part is set to catch, gives $d^2\phi/dt^2 = -[g/(R - r)]\sin \phi$, the equation for a particle sliding on a frictionless track. Static friction does no work on a cylinder that rolls without slipping, which is why energy conservation is legitimate here even though friction acts.*
- (c) **SHOW THAT** · U2 [2]
- Show that, for small oscillations, the motion is simple harmonic with period $T = 2\pi\sqrt{(3(R - r)/(2g))}$.
- 1 For small ϕ in radians $\sin \phi \approx \phi$, so $d^2\phi/dt^2 = -[2g/(3(R - r))]\phi$, which has the form of the simple harmonic equation $d^2\phi/dt^2 = -\omega_0^2\phi$
 - 2 $\omega_0 = \sqrt{(2g/(3(R - r)))}$, so $T = 2\pi/\omega_0 = 2\pi\sqrt{(3(R - r)/(2g))}$
- Confusing ω_0 , the angular frequency of the oscillation, with ω , the spin rate of the cylinder, is the error to avoid: the two symbols describe different motions of the same body.*
- (d) **COMPARE** · U1 [2]
- Compare this period with that of a small block sliding without friction whose centre of mass moves on the same circle of radius $R - r$, and explain the difference.
- 1 The sliding block is a simple pendulum of length $R - r$ with $T = 2\pi\sqrt{((R - r)/g)}$; the rolling cylinder's period is longer by a factor of $\sqrt{(3/2)} \approx 1.22$
 - 2 In both cases the component of gravity along the path, $mg \sin \phi$, drives the motion, but the rolling cylinder must also gain rotational kinetic energy: its effective inertia is $3m/2$ rather than m , so it accelerates along the path at only two-thirds the rate
- Candidates who say the cylinder is slower because friction dissipates energy reach the right direction for the wrong reason. Static friction does no work on a cylinder that rolls without slipping: it slows the translation only by supplying the torque that spins the cylinder up, so it diverts energy into rotation without removing any, and that is the whole of the effect.*
- (e) **CALCULATE** · U3 [2]
- The engineers want an isolation period of 2.50 s. Calculate the value of $R - r$ required.
- 1 Rearranges $T = 2\pi\sqrt{(3(R - r)/(2g))}$ to $R - r = gT^2/(6\pi^2)$
 - 2 $R - r = (9.81 \times 2.50^2)/(6\pi^2) = 1.04 \text{ m}$
- Candidates who use the simple-pendulum period here obtain $R - r = 1.55 \text{ m}$, a bearing 50% larger than the specification requires.*

(f) PREDICT · U1

[2]

Predict the period of the bearing if the solid cylinder is replaced by a thin-walled tube of the same mass and outer radius, whose moment of inertia about its own axis is mr^2 , with $R - r$ unchanged, and give the reason.

- 1 For a thin-walled tube $I = mr^2$, so $K = m(R - r)^2(d\phi/dt)^2$ and $\omega_0^2 = g/(2(R - r))$, giving $T = 2\pi\sqrt{2(R - r)/g}$
- 2 $T = 2.50 \text{ s} \times \sqrt{4/3} = 2.89 \text{ s}$: a larger share of the energy is rotational, so the effective inertia rises to $2m$ and the oscillation is slower

Neither the mass nor the radius of the rolling body appears in any period in this question except through $R - r$; only the shape factor $I/(mr^2)$ survives. Candidates who expect a heavier roller to change the period should notice that m has cancelled.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 34 MARKS · 29%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 29 MARKS · 25%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

42 MARKS · 36%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

11 MARKS · 9%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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