

University Physics IV · Final Examination

The whole of modern physics and introductory quantum mechanics, weighted towards the second half of the course: angular momentum and spin, the hydrogen atom, many-electron atoms, solids, nuclei, particles and the frontier, with the evidence for quanta, special relativity and the one-dimensional Schrödinger equation sampled throughout. Every result is expected to survive its classical or non-relativistic limit.

SUGGESTED TIME 150 minutes	TOTAL MARKS 116	QUESTIONS 19	MULTIPLE CHOICE 10
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SECTION

UPIV · Final · University Physics IV — Final examination

IN THE FORMAT OF

A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND

Recall 1 · Routine 5 · Demanding 9 · Discriminating 4

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS

UPIV-01 · UPIV-02 · UPIV-03 · UPIV-04 · UPIV-05 · UPIV-06 · UPIV-07 · UPIV-08 · UPIV-09 · UPIV-10 · UPIV-11 · UPIV-12 · UPIV-13 · UPIV-14 · UPIV-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Physics IV · Final Examination

Answer all questions. 116 marks in 150 minutes.

1CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-01.06 · UPIV-01.08

[1]

A clean tungsten surface of work function $\phi = 4.50$ eV is illuminated in vacuum by femtosecond laser pulses of wavelength 800 nm, and electrons are emitted even though each photon carries less energy than the work function. An electron spectrometer shows that the most energetic electrons leave with 0.15 eV at every pulse intensity used, and doubling the peak intensity I multiplies the number of electrons emitted per pulse by 7.9 ± 0.4 . Take $hc = 1240$ eV nm. Which conclusion do these measurements justify?

- A** The intense pulses heat the surface and the electrons are emitted thermally, so these data say nothing about how light exchanges energy with electrons.
- B** Einstein's photoelectric equation is refuted, because it forbids emission whenever the photon energy is below the work function, however intense the light.
- C** Each emitted electron absorbs three quanta of 1.55 eV, since $3h\nu - \phi = 0.15$ eV and an emission rate proportional to I^3 grows by $2^3 = 8$ when I doubles; energy is exchanged with the light in quanta of $h\nu$, but these data do not by themselves prove that the light field is quantised.
- D** Each emitted electron absorbs three quanta, which proves that the light field consists of photons, because a classical wave cannot deliver energy to an electron in whole multiples of $h\nu$.

Answer

2CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-04.05 · UPIV-04.06

[1]

At a pulsed neutron source, each pulse releases neutrons with a wide spread of speeds from the moderator at the same instant. They travel a total flight path of 12.0 m, by way of a powder sample, to a detector that records neutrons scattered elastically through 90.0° by the sample. A Bragg peak reaches the detector 6.00 ms after the pulse. Treating it as a first-order reflection, what is the spacing of the lattice planes that produce it? Take $h = 6.626 \times 10^{-34}$ J s and $m_n = 1.675 \times 10^{-27}$ kg.

- A** 0.140 nm
- B** 0.0989 nm
- C** 0.280 nm
- D** 0.0223 nm

Answer

3CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04

[1]

An electron is held in a one-dimensional trap whose potential energy is not symmetric about any point. It is in the trap's ground state, described by a real, normalised wavefunction $u(x)$ whose probability density is lopsided. A uniform force then acts for a time so short that $u(x)$ does not change while it acts, delivering an impulse $\hbar k_0$ in the $+x$ direction, so that immediately afterwards the wavefunction is $u(x)e^{ik_0x}$. With $\phi(p)$ the momentum-space wavefunction, which statement is correct?

- A** Before the impulse $\langle p \rangle$ is not zero, because the lopsided density $|u(x)|^2$ makes the momentum distribution lopsided too; the impulse then adds $\hbar k_0$ to $\langle p \rangle$.
- B** The factor e^{ik_0x} has modulus 1 and leaves the position density unchanged, so no measurement made immediately afterwards, of position or of momentum, can detect the impulse.
- C** Before the impulse $|\phi(p)|^2$ is symmetric in p only if $u(x)$ is an even function; immediately afterwards the momentum distribution is shifted by $\hbar k_0$ and also broadened, because the factor e^{ik_0x} adds momenta near $\hbar k_0$ to those already present.
- D** Before the impulse $|\phi(p)|^2 = |\phi(-p)|^2$, so $\langle p \rangle = 0$ even though $|u(x)|^2$ is lopsided; immediately afterwards the momentum distribution is the same curve shifted by $\hbar k_0$, while the position density is unchanged.

Answer

4CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.04 · UPIV-08.09

[1]

A single ion held in a radio-frequency trap oscillates along the trap axis as a quantum harmonic oscillator of frequency $f = 4.00$ MHz, with energy levels $E_n = (n + \frac{1}{2})hf$. After laser cooling, its axial motion is in thermal equilibrium at $T = 0.600$ mK. What is the probability that the axial motion is found in its ground state, $n = 0$? Take $h = 6.626 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K $^{-1}$.

- A** 0.726
- B** 0.274
- C** 0.579
- D** 0.852

Answer

5CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-09.01 · UPIV-09.02

[1]

A particle is in the simultaneous eigenstate of J^2 and J_z with quantum numbers $j = 3/2$ and $m = 3/2$. What is the expectation value $\langle J_x^2 \rangle$?

- A** $3\hbar^2/4$
- B** 0
- C** $3\hbar^2/2$
- D** $15\hbar^2/4$

Answer

6CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-11.04 · UPIV-11.06

[1]

A free ion has the ground configuration $[\text{Ar}]3d^3$. Applying all three of Hund's rules, what is its ground-state term symbol?

- A $^4F_{9/2}$
- B $^2H_{9/2}$
- C $^4P_{1/2}$
- D $^4F_{3/2}$

Answer

7CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-12.06

[1]

In a one-dimensional tight-binding model an electron band has dispersion $E(k) = E_0 - 2t \cos(ka)$, with hopping energy $t = 0.50$ eV and lattice spacing $a = 0.30$ nm. What is the effective mass of an electron near the bottom of the band, at $k = 0$? Take $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C and $m_e = 9.109 \times 10^{-31}$ kg.

- A $0.42m_e$
- B $0.85m_e$
- C $1.69m_e$
- D $-0.85m_e$

Answer

8CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-13.03

[1]

In the spin-orbit shell model the proton single-particle levels fill in the order $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, each holding $2j + 1$ protons, and neutrons fill the same sequence. What spin and parity does the model predict for the ground state of ^{39}K , with $Z = 19$ and $N = 20$?

- A $3/2^+$
- B $1/2^+$
- C $3/2^-$
- D $7/2^-$

Answer

9CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-14.01

[1]

A neutral meson decays in flight into two photons, which an electromagnetic calorimeter records with energies 1.50 GeV and 0.500 GeV and an angle of 36.9° between their directions of travel. Treating the photons as massless, what is the rest energy of the meson?

- A 2.00 GeV
- B 1.64 GeV
- C 0.388 GeV
- D 0.548 GeV

Answer

10CORE
COURSE

MULTIPLE CHOICE · RECALL · UPIV-15.02 · UPIV-15.09

[1]

Entangled photon pairs are measured at two stations whose analyser settings are chosen randomly and independently, with the detection events at the two stations spacelike separated and the detection efficiency high enough that no fair-sampling assumption is needed. The measured CHSH combination is $S = 2.70 \pm 0.02$. Which statement correctly describes what this result establishes?

- A The choice of setting at one station changes the outcomes recorded at the other, so a message could be sent faster than light.
- B Every hidden-variable theory, local or non-local, is ruled out, which proves that quantum mechanics is complete.
- C All models in which each outcome is fixed by variables carried locally from the source, independent of the settings, are excluded; the correlations still cannot be used to send a signal faster than light, and non-local hidden-variable models are not excluded.
- D The source emitted a statistical mixture, since a pure entangled state would give S exactly equal to 2.

Answer

A handheld X-ray fluorescence analyser is aimed at a corroded brass fitting. Its primary X-rays eject inner-shell electrons from atoms in the sample, and the detector records two strong lines at 8.04 keV and 8.63 keV, together with weaker lines, one of which lies at 8.90 keV. Model each K line with hydrogen-like levels and a screening constant σ : a transition from shell n to the K shell ($n = 1$) emits a photon of energy $E = (13.6 \text{ eV})(Z - \sigma)^2(1 - 1/n^2)$, so that the $K\alpha$ line ($n = 2$) follows Moseley's law, $E = (3/4)(13.6 \text{ eV})(Z - \sigma)^2$.

(a) DESCRIBE

[2]

Describe the sequence of events, beginning with the absorption of a primary X-ray photon, that leads to the emission of a $K\alpha$ photon, and say why the primary photons must exceed a threshold energy.

(b) CALCULATE

[3]

Taking $\sigma = 1$, calculate the atomic numbers of the elements that produce the 8.04 keV and 8.63 keV lines, and identify the elements.

(c) EXPLAIN

[2]

Explain why $\sigma \approx 1$ is a reasonable screening constant for a $K\alpha$ line.

(d) EVALUATE

[3]

A colleague suggests that the 8.90 keV line is the $K\alpha$ line of a third element. Evaluate this suggestion against the alternative that it is the $K\beta$ line ($n = 3 \rightarrow n = 1$) of copper, and state what your conclusion implies about σ for a $K\beta$ line.

A single lithium-7 atom, of mass $m = 1.165 \times 10^{-26}$ kg, is confined along one direction between two sheets of laser light that act as impenetrable walls at $x = 0$ and $x = L$, with $L = 10.0 \mu\text{m}$. Its stationary states are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ for $0 \leq x \leq L$, with energies $E_n = n^2\pi^2\hbar^2/(2mL^2)$, and it is prepared in the ground state ψ_1 . At $t = 0$ a pulse of light lasting far less than any timescale of the atom's motion illuminates only the half of the box $0 < x < L/2$, multiplying the wavefunction there by $e^{i\pi} = -1$ and leaving the half $L/2 < x < L$ untouched. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K⁻¹. You may use $\int_0^{L/2} \sin(n\pi x/L) \sin(\pi x/L) dx = (L/(2\pi))[\sin((n-1)\pi/2)/(n-1) - \sin((n+1)\pi/2)/(n+1)]$ for $n \geq 2$.

(a) CALCULATE

[2]

Calculate E_1 , and the temperature T at which $k_B T$ equals $E_2 - E_1$, the temperature scale that the atom must be cooled well below for its ground state to be prepared reliably.

(b) DERIVE

[4]

Starting from the expansion of $\Psi(x, 0)$ in the stationary states and the Born rule, derive the probability P_n that a measurement of the energy made just after the pulse returns E_n . Find every n for which $P_n = 0$, and evaluate P_2 and P_4 .

(c) EXPLAIN

[3]

The pulse leaves the probability density $|\Psi(x, 0)|^2$ equal to $|\psi_1(x)|^2$. Explain why the density nevertheless changes after the pulse, why it remains symmetric about the centre of the box at all later times, and why it is zero at $x = L/2$ at every $t > 0$.

(d) PREDICT

[2]

In the real trap the light sheets form barriers of finite height $V_0 = 400E_1$, with E_1 the value found in (a), and the same pulse, which reverses the sign of the wavefunction everywhere to the left of $x = L/2$, is applied to the ground state of this finite well. Predict, with reasons, whether a measurement of the energy just after the pulse could return the ground-state energy of the finite well, and whether the atom is certain to remain in the trap.

The resonance lines of the alkali atoms come from an $np \rightarrow ns$ transition that spin-orbit coupling splits into D_1 ($^2P_{1/2} \rightarrow ^2S_{1/2}$) and D_2 ($^2P_{3/2} \rightarrow ^2S_{1/2}$). The fine-structure splitting between $^2P_{3/2}$ and $^2P_{1/2}$ is 7.16×10^{-3} eV for the 4p level of potassium and 4.16×10^{-5} eV for the 2p level of lithium. Both vapours are placed in a uniform magnetic field of 0.500 T. Ignore hyperfine structure throughout. Take $\mu_B = 5.788 \times 10^{-5}$ eV T $^{-1}$ and $g_s = 2$, so that $g_J = 1 + [j(j+1) + s(s+1) - l(l+1)]/[2j(j+1)]$.

(a) CALCULATE

[2]

Calculate g_J for the $^2S_{1/2}$, $^2P_{1/2}$ and $^2P_{3/2}$ levels.

(b) DETERMINE

[3]

In the weak-field limit each sublevel shifts by $g_J m_J \mu_B B$. Using the selection rule $\Delta m_J = 0, \pm 1$, determine the number of components into which the D_1 line splits and the shift of each component from the zero-field line, in units of $\mu_B B$.

(c) COMPARE

[2]

Compare the fine-structure splitting with $\mu_B B$ at 0.500 T for each atom.

(d) JUSTIFY

[2]

Justify, for each atom, whether the D_1 pattern you found in (b) describes its spectrum at 0.500 T.

(e) PREDICT

[2]

Predict the appearance of the lithium resonance line at 5.00 T, naming the good quantum numbers.

The muon decays by $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ with a branching fraction of essentially 1 and a mean lifetime $\tau_\mu = 2.197 \times 10^{-6}$ s. The τ^- lepton decays through the same charged weak current into $e^- + \bar{\nu}_e + \nu_\tau$ with branching fraction $B_e = 0.178$, and also into $\mu^- + \bar{\nu}_\mu + \nu_\tau$ and into hadrons. Take $m_\mu c^2 = 105.66$ MeV, $m_\tau c^2 = 1776.9$ MeV and the Fermi constant $G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV⁻². In natural units, with $\hbar = c = 1$, a decay width $\Gamma = \hbar/\tau$ has the dimensions of energy.

(a) STATE

[2]

State the values of L_e , L_μ and L_τ before and after the decay $\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$, and state which conservation law forbids $\tau^- \rightarrow e^- + \gamma$.

(b) DESCRIBE

[2]

Describe the lowest-order Feynman diagram for $\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$, and give the power of the weak coupling g_W in its amplitude and in its rate.

(c) SHOW THAT

[2]

Show that if the width depends only on G_F and on the mass m of the decaying lepton, the result of (b) requires $\Gamma \propto G_F^2 m^5$.

(d) ESTIMATE

[3]

Estimate the mean lifetime of the τ , stating the assumptions you make.

(e) EVALUATE

[3]

The measured τ lifetime is 2.903×10^{-13} s. Evaluate what the agreement with your estimate tests, and use a count of the final states available to the virtual W^- to explain why B_e is close to 0.2.

^{210}Po decays by α emission to ^{206}Pb with a half-life of 138.4 days. The atomic masses are ^{210}Po 209.982874 u, ^{206}Pb 205.974465 u and ^4He 4.002603 u, with $1\text{ u} = 931.494\text{ MeV}/c^2$. In the Gamow model the α particle, of reduced mass μ , meets a Coulomb barrier $V(r) = Z_d Z_\alpha \hbar c / r$ that begins at the nuclear radius R , and tunnels out to the classical turning point b at which $V(b) = Q$. The WKB exponent is $2G = 2\alpha Z_d Z_\alpha \sqrt{(2\mu c^2/Q)} [\arccos\sqrt{R/b} - \sqrt{(R/b)(1 - R/b)}]$, and the half-life is proportional to $e^{(2G)}$ divided by an assault frequency. Take $\alpha \hbar c = 1.440\text{ MeV fm}$, $\alpha = 1/137.0$, $\mu c^2 = 3.66 \times 10^3\text{ MeV}$, $R = 9.0\text{ fm}$, $Z_d = 82$ and $Z_\alpha = 2$.

(a) CALCULATE

[2]

Calculate the Q value of the decay.

(b) CALCULATE

[1]

Calculate the outer turning point b .

(c) SHOW THAT

[3]

Using $\arccos\sqrt{x} \approx \pi/2 - \sqrt{x}$ and $\sqrt{x(1-x)} \approx \sqrt{x}$ for $x = R/b$ much less than 1, show that $2G \approx \pi\alpha Z_d Z_\alpha \sqrt{(2\mu c^2/Q)} - 4\sqrt{(2\alpha Z_d Z_\alpha \mu c^2 R/\hbar c)}$, and hence that the second term does not depend on Q .

(d) PREDICT

[3]

^{214}Po also decays by α emission to a lead isotope ($Z_d = 82$), with $Q = 7.833\text{ MeV}$. Assuming the same R , μ and assault frequency for both nuclei, predict the half-life of ^{214}Po from that of ^{210}Po using the result of (c).

(e) EVALUATE

[2]

The measured half-life of ^{214}Po is $164\text{ }\mu\text{s}$. Evaluate the model in the light of this comparison.

A sample-return probe leaves a space station, which defines an inertial frame S , and travels in a straight line at $0.600c$ to a relay beacon at rest in S at a distance of 3.00 light-years. At the beacon it reverses in a negligibly short time and returns at $0.800c$. Work in units in which $c = 1$ light-year per year, label events by their S coordinates (t, x) , and take the departure as the event $(0, 0)$. Let S' be the inertial frame moving with the probe on the outbound leg and S'' the inertial frame moving with it on the return leg.

(a) CALCULATE

[3]

Calculate the time that elapses on the station clock, and the proper time that elapses on the probe's clock, for the round trip.

(b) DETERMINE

[4]

Determine the reading of the station clock that is simultaneous with the turnaround event, first in S' and then in S'' .

(c) SHOW THAT

[2]

Show that the probe's account, with the station clock running slow on each leg plus the jump found in (b), reproduces the station's total of 8.75 y.

(d) EVALUATE

[2]

Evaluate the claim: "The probe ages less because it accelerated, and the harder it accelerates at the turnaround, the more time it loses."

The intrinsic carrier density n_i of an undoped semiconductor bar is found from Hall-effect and conductivity measurements at six temperatures, and each value of n_i carries a random uncertainty of $\pm 3\%$. For a non-degenerate intrinsic semiconductor with a temperature-independent band gap E_g , $n_i = CT^{3/2}\exp(-E_g/2kT)$, where C is a constant. Take $k = 8.617 \times 10^{-5} \text{ eV K}^{-1}$.

Intrinsic carrier density of the bar at six temperatures

Temperature T (K)	Intrinsic carrier density n_i (m^{-3})
300	2.03×10^{19}
340	1.40×10^{20}
380	6.89×10^{20}
420	2.41×10^{21}
460	7.32×10^{21}
500	1.78×10^{22}

(a) **EXPLAIN** [2]

Explain why a graph of $\ln(n_i/T^{3/2})$ against $1/T$ is expected to be a straight line, and what its gradient gives.

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(b) **DETERMINE** [3]

Determine E_g from the data.

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(c) **ESTIMATE** [2]

Using the first and last rows only, estimate the uncertainty in E_g that follows from the $\pm 3\%$ uncertainty in n_i , stating the assumption you make about the two uncertainties.

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(d) **COMPARE** [2]

A student instead plots $\ln n_i$ against $1/T$ and takes $E_g = -2k \times \text{gradient}$. Compare the gap this gives with your value from (b), and account for the difference.

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(e) EVALUATE

[2]

Optical absorption measured on the same bar at 400 K shows an absorption edge at 0.66 eV. Semiconductor gaps typically narrow as the temperature rises. Evaluate whether this contradicts your value from (b).

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18
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPIV-10.03 · UPIV-10.04 · UPIV-10.05 · UPIV-10.07 · UPIV-10.09

[15]

An electron is bound to a nucleus of charge Ze . With μ the reduced mass, the radial function $u(r) = rR(r)$ of a state of orbital quantum number l satisfies the radial Schrödinger equation $-(\hbar^2/2\mu)(d^2u/dr^2) + [-Ze^2/(4\pi\epsilon_0 r) + l(l+1)\hbar^2/(2\mu r^2)]u = Eu$, with $E < 0$ for a bound state. Write $\kappa = \sqrt{-2\mu E}/\hbar$ and $a_\mu = 4\pi\epsilon_0\hbar^2/(\mu e^2)$. You may use $\int_0^\infty r^k e^{-\beta r} dr = k!/\beta^{k+1}$ for integer $k \geq 0$. For the numerical parts take $a_0 = 5.292 \times 10^{-11}$ m and 13.606 eV as the Bohr radius and the ionisation energy of hydrogen for an infinitely heavy nucleus, $m_p/m_e = 1836$ and $h = 4.136 \times 10^{-15}$ eV s.

(a) SHOW THAT

[2]

Show that a normalisable solution must behave as $e^{-\kappa r}$ at large r and as r^{l+1} near $r = 0$.

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(b) DERIVE

[5]

By substituting $u = r^{l+1}e^{-\kappa r}$ into the radial Schrödinger equation, derive the conditions under which it is an exact solution, and hence derive κ and the energy E of this state. Identify its principal quantum number n .

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(c) DETERMINE

[3]

For this state, determine the most probable radius r_{mp} and the mean radius $\langle r \rangle$ in terms of n , a_μ and Z .

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(d) **CALCULATE** [3]

Circular Rydberg states, with $l = n - 1$, are used in microwave cavity experiments. For the $n = 50$ circular state of hydrogen, calculate the most probable radius and the frequency of the photon emitted in the transition to $n = 49$.

(e) **JUSTIFY** [2]

Justify, using the electric-dipole selection rules, the claim that this state can decay by only one electric-dipole transition.

Model the universe as a uniform, pressureless dust of mass density $\rho(t)$ in which every separation grows in proportion to a scale factor $a(t)$, with $a = 1$ today. A particle of mass m moving with the dust lies at distance $r = a(t)x$ from an arbitrary origin, where x is constant, and by the shell theorem it is attracted only by the mass inside the sphere of radius r . Take $G = 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$, the present expansion rate $H_0 = \dot{a}/a = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $1 \text{ Mpc} = 3.086 \times 10^{22} \text{ m}$, the mass of a hydrogen atom as $1.674 \times 10^{-27} \text{ kg}$ and $1 \text{ yr} = 3.156 \times 10^7 \text{ s}$.

(a) DERIVE

[4]

Starting from conservation of energy for the particle, derive the Friedmann equation $(\dot{a}/a)^2 = (8\pi G/3)\rho - K/a^2$, where K is a constant, and state what the sign of K means for the future of the expansion in this Newtonian picture.

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(b) CALCULATE

[3]

The critical density ρ_c is the density for which $K = 0$. Calculate ρ_c today, and express it as a number of hydrogen atoms per cubic metre.

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(c) DERIVE

[3]

For $K = 0$ and $\rho \propto a^{-3}$, derive $a(t)$ from the Friedmann equation, taking $a = 0$ at $t = 0$, and hence obtain the age t_0 of the universe in terms of H_0 and in gigayears.

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(d) EVALUATE

[2]

The oldest globular-cluster stars in the Milky Way are dated at about 12.5 Gyr. Evaluate the flat, matter-only model in the light of this, including the value of H_0 the model would need.

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(e) CALCULATE

[2]

The cosmic microwave background was released at redshift $z = 1090$. Calculate the scale factor at that time and the factor by which the matter density then exceeded its present value.

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University Physics IV · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 MULTIPLE CHOICE · DEMANDING · UPIV-01.06 · UPIV-01.08 [1]
CORE COURSE

A clean tungsten surface of work function $\phi = 4.50$ eV is illuminated in vacuum by femtosecond laser pulses of wavelength 800 nm, and electrons are emitted even though each photon carries less energy than the work function. An electron spectrometer shows that the most energetic electrons leave with 0.15 eV at every pulse intensity used, and doubling the peak intensity I multiplies the number of electrons emitted per pulse by 7.9 ± 0.4 . Take $hc = 1240$ eV nm. Which conclusion do these measurements justify?

KEY C Each emitted electron absorbs three quanta of 1.55 eV, since $3h\nu - \phi = 0.15$ eV and an emission rate proportional to I^3 grows by $2^3 = 8$ when I doubles; energy is exchanged with the light in quanta of $h\nu$, but these data do not by themselves prove that the light field is quantised.

Each photon carries $hc/\lambda = (1240 \text{ eV nm})/(800 \text{ nm}) = 1.55$ eV. Two quanta (3.10 eV) cannot free an electron, three leave a maximum kinetic energy of $3(1.55 \text{ eV}) - 4.50 \text{ eV} = 0.15$ eV, exactly as measured, and four would leave 1.70 eV; independently, a rate proportional to I^n grows by 2^n when I doubles, and 7.9 ± 0.4 corresponds to n between 2.91 and 3.05. Two separate measurements pointing to the same integer is what justifies C. B is tempting because the threshold is usually taught as absolute, but Einstein's equation describes one quantum absorbed by one electron: absorbing several at once needs an enormous photon flux, and the rate of n -quantum absorption scales as I^n , negligible for a lamp and measurable only with intense, focused laser pulses, so nothing in the quantum picture forbids it. A appeals to the heating that intense pulses really do cause, but thermal emission would give an energy spread set by the electron temperature, widening as the intensity rises, and a yield growing exponentially with that temperature, not a fixed cut-off at exactly $3h\nu - \phi$ and a clean third-power law. D draws the right multiphoton picture and then overclaims: a classical field of frequency ν acting on an electron whose states are quantised also transfers energy only in multiples of $h\nu$, because the n th-order response to an oscillating field is resonant only when the energy change is $nh\nu$. Like the ordinary photoelectric effect, this experiment establishes that energy is exchanged in quanta of $h\nu$; showing that the field itself is quantised takes measurements on the light alone, such as photon antibunching.

U4

2 MULTIPLE CHOICE · ROUTINE · UPIV-04.05 · UPIV-04.06 [1]
CORE COURSE

At a pulsed neutron source, each pulse releases neutrons with a wide spread of speeds from the moderator at the same instant. They travel a total flight path of 12.0 m, by way of a powder sample, to a detector that records neutrons scattered elastically through 90.0° by the sample. A Bragg peak reaches the detector 6.00 ms after the pulse. Treating it as a first-order reflection, what is the spacing of the lattice planes that produce it? Take $h = 6.626 \times 10^{-34}$ J s and $m_n = 1.675 \times 10^{-27}$ kg.

KEY A 0.140 nm

The neutrons in the peak left the moderator together and keep their speed through elastic scattering, so $v = (12.0 \text{ m})/(6.00 \times 10^{-3} \text{ s}) = 2.00 \times 10^3 \text{ m s}^{-1}$ and $\lambda = h/(m_n v) = 1.98 \times 10^{-10} \text{ m}$. A deflection of 90.0° is 2θ , twice the Bragg angle measured from the planes, so $\theta = 45.0^\circ$ and $d = \lambda/(2 \sin \theta) = 1.40 \times 10^{-10} \text{ m} = 0.140 \text{ nm}$. B takes the deflection of the beam as the Bragg angle, $d = \lambda/(2 \sin 90.0^\circ)$; the angle in $2d \sin \theta = \lambda$ lies between the beam and the planes, and a beam turned through 90° meets the planes at 45° . C drops the factor of 2 from $2d \sin \theta$, counting the extra path of the ray reflected from the deeper plane on the way in but not on the way out. D writes the de Broglie relation with $\hbar$ in place of h , which makes λ 2π times too short and gives a spacing of 0.0223 nm, far smaller than any distance between neighbouring atoms in a solid.

U3

3

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04

[1]

An electron is held in a one-dimensional trap whose potential energy is not symmetric about any point. It is in the trap's ground state, described by a real, normalised wavefunction $u(x)$ whose probability density is lopsided. A uniform force then acts for a time so short that $u(x)$ does not change while it acts, delivering an impulse $\hbar k_0$ in the $+x$ direction, so that immediately afterwards the wavefunction is $u(x)e^{(ik_0x)}$. With $\phi(p)$ the momentum-space wavefunction, which statement is correct?

KEY D Before the impulse $|\phi(p)|^2 = |\phi(-p)|^2$, so $\langle p \rangle = 0$ even though $|u(x)|^2$ is lopsided; immediately afterwards the momentum distribution is the same curve shifted by $\hbar k_0$, while the position density is unchanged.

Writing $\phi(p) = (2\pi\hbar)^{-1/2} \int u(x)e^{(-ipx/\hbar)} dx$, a real u gives $\phi(-p) = \phi(p)^*$, so $|\phi(p)|^2$ is even in p and $\langle p \rangle = 0$: a real wavefunction is a superposition in which the components of momentum $+p$ and $-p$ always carry amplitudes of equal size, whatever the shape of u . After the impulse, $u(x)e^{(ik_0x)}$ transforms to $\phi(p - \hbar k_0)$, the same distribution translated by $\hbar k_0$, while $|u(x)e^{(ik_0x)}|^2 = |u(x)|^2$. A reads the lopsided position density into the momentum distribution, but the ground state is stationary, so $\langle p \rangle = m d\langle x \rangle/dt = 0$, and a real wavefunction carries no probability current. B is the commonest error: a phase factor that varies with x is not a global phase, and although the position density cannot see it, the Born rule applied in the momentum representation uses $|\phi(p)|^2$, which has moved by $\hbar k_0$. C gets the symmetry condition wrong, since an even u would make ϕ itself real and even, which is more than $|\phi(p)|^2$ needs in order to be even, and it gets the shift wrong too: multiplying by a single plane wave translates $\phi(p)$ without reshaping it, adding $\hbar k_0$ to $\langle p \rangle$ and leaving the variance $\langle p^2 \rangle - \langle p \rangle^2$ unchanged.

U1

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-08.04 · UPIV-08.09

[1]

A single ion held in a radio-frequency trap oscillates along the trap axis as a quantum harmonic oscillator of frequency $f = 4.00$ MHz, with energy levels $E_n = (n + \frac{1}{2})\hbar f$. After laser cooling, its axial motion is in thermal equilibrium at $T = 0.600$ mK. What is the probability that the axial motion is found in its ground state, $n = 0$? Take $h = 6.626 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K $^{-1}$.

KEY B 0.274

With $x = \hbar f/(k_B T) = (6.626 \times 10^{-34} \text{ J s})(4.00 \times 10^6 \text{ Hz})/[(1.381 \times 10^{-23} \text{ J K}^{-1})(6.00 \times 10^{-4} \text{ K})] = 0.320$, the Boltzmann weights $e^{-(n + \frac{1}{2})x}$ of the evenly spaced ladder form a geometric series with partition function $Z = e^{-(x/2)}/(1 - e^{-x})$, so $P_0 = e^{-(x/2)}/Z = 1 - e^{-x} = 0.274$. The zero-point energy multiplies every weight by the same factor $e^{-(x/2)}$ and cancels. A is e^{-x} , the ratio of the populations of $n = 1$ and $n = 0$: it is the familiar Boltzmann factor, but it compares two levels instead of giving the share of all the levels that $n = 0$ holds. C normalises over $n = 0$ and $n = 1$ only, as if the motion were a two-level system, giving $1/(1 + e^{-x})$; with $k_B T$ about $3.1\hbar f$ many levels are occupied, and the mean occupation $1/(e^x - 1)$ is 2.65 quanta. D is $e^{-(x/2)}$, the Boltzmann factor of the zero-point energy alone, taken as a probability without dividing by the partition function.

U3

5

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-09.01 · UPIV-09.02

[1]

A particle is in the simultaneous eigenstate of J^2 and J_z with quantum numbers $j = 3/2$ and $m = 3/2$. What is the expectation value $\langle J_x^2 \rangle$?

KEY A $3\hbar^2/4$

Because $J^2 = J_x^2 + J_y^2 + J_z^2$ and the state is an eigenstate of both J^2 and J_z , $\langle J_x^2 \rangle + \langle J_y^2 \rangle = [j(j+1) - m^2]\hbar^2 = (15/4 - 9/4)\hbar^2 = 3\hbar^2/2$, and symmetry under rotation about the z axis shares this equally, giving $3\hbar^2/4$. B is the classical picture of a vector lying along z . It is ruled out because $j(j+1)$ exceeds m^2 for every $j > 0$, so the angular momentum can never be fully aligned; it would also violate $\sigma_x \sigma_y \geq (\hbar/2)|\langle J_z \rangle| = 3\hbar^2/4$, a bound this state exactly saturates. C is the sum $\langle J_x^2 \rangle + \langle J_y^2 \rangle$, not yet shared between the two components. D is $j(j+1)\hbar^2$, the whole of $\langle J^2 \rangle$ attributed to a single component.

U2

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-11.04 · UPIV-11.06

[1]

A free ion has the ground configuration $[\text{Ar}]3d^3$. Applying all three of Hund's rules, what is its ground-state term symbol?

KEY **D** $^4F_{3/2}$

The first rule maximises S : three parallel spins give $S = 3/2$, a quartet. The second then maximises L within that S : with parallel spins the three electrons must occupy different m_l , and the largest sum is $2 + 1 + 0 = 3$, an F term. The third rule looks at filling: 3 electrons in a shell that holds 10 is less than half full, so $J = |L - S| = 3/2$, giving $^4F_{3/2}$. A applies the third rule the wrong way round; $J = L + S$ belongs to a shell more than half full. B maximises L before S : $m_l = 2, 2, 1$ is possible only if two electrons pair their spins, which gives $L = 5$ but $S = 1/2$, so it inverts the order of the first two rules. C is the other quartet term of d^3 , the choice of a candidate who takes the smallest rather than the largest L once S is fixed.

U1

7

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-12.06

[1]

In a one-dimensional tight-binding model an electron band has dispersion $E(k) = E_0 - 2t \cos(ka)$, with hopping energy $t = 0.50$ eV and lattice spacing $a = 0.30$ nm. What is the effective mass of an electron near the bottom of the band, at $k = 0$? Take $\hbar = 1.055 \times 10^{-34}$ J s, $e = 1.602 \times 10^{-19}$ C and $m_e = 9.109 \times 10^{-31}$ kg.

KEY **B** $0.85m_e$

The effective mass comes from the curvature of the band, $1/m^* = (1/\hbar^2)d^2E/dk^2$. At $k = 0$, $d^2E/dk^2 = 2ta^2$, so $m^* = \hbar^2/(2ta^2) = 7.7 \times 10^{-31}$ kg $= 0.85m_e$. A halves this, the result of equating $\hbar^2k^2/(2m^*)$ to $2ta^2k^2$ instead of to the ta^2k^2 that the expansion $2t \cos(ka) \approx 2t - ta^2k^2$ actually gives. C doubles it, from dropping the factor of 2 in $2t$. D has the right magnitude but the sign that belongs at the top of the band, $k = \pm\pi/a$, where the curvature is negative; near the bottom of the band an electron accelerates along the applied force, as a particle of positive mass does.

U2

8

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-13.03

[1]

In the spin-orbit shell model the proton single-particle levels fill in the order $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, each holding $2j + 1$ protons, and neutrons fill the same sequence. What spin and parity does the model predict for the ground state of ^{39}K , with $Z = 19$ and $N = 20$?

KEY **A** $3/2^+$

$N = 20$ fills the neutron levels through $1d_{3/2}$, which contributes zero angular momentum. The protons fill $1s_{1/2}$ (2), $1p_{3/2}$ (6), $1p_{1/2}$ (8), $1d_{5/2}$ (14) and $2s_{1/2}$ (16), then place 3 of a possible 4 in $1d_{3/2}$. A single hole in a j level carries the same j as a single particle would, so $J = 3/2$, and the parity is $(-1)^l$ with $l = 2$ for a d orbital, giving $3/2^+$. B takes the last completely filled level, $2s_{1/2}$, as the one that sets the spin. C has the right j but reads the parity from j , or from a p orbital, instead of from $l = 2$. D places the odd proton in $1f_{7/2}$, a level that begins to fill only beyond 20; it is the prediction for a nucleus with one nucleon outside a closed shell of 20, such as ^{41}Ca , not for ^{39}K .

U1

9

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-14.01

[1]

A neutral meson decays in flight into two photons, which an electromagnetic calorimeter records with energies 1.50 GeV and 0.500 GeV and an angle of 36.9° between their directions of travel. Treating the photons as massless, what is the rest energy of the meson?

KEY **D 0.548 GeV**

The rest energy is the invariant mass of the photon pair, the same in every frame: $(Mc^2)^2 = (E_1 + E_2)^2 - (p_{1c} + p_{2c}) \cdot (p_{1c} + p_{2c})$, and with $|p_{1c}| = E_1$ and $|p_{2c}| = E_2$ for massless photons this reduces to $(Mc^2)^2 = 2E_1E_2(1 - \cos \theta) = 2(1.50 \text{ GeV})(0.500 \text{ GeV})(1 - \cos 36.9^\circ) = 0.300 \text{ GeV}^2$, so $Mc^2 = 0.548 \text{ GeV}$, the rest energy of the η meson. A adds the two photon energies, which gives the meson's energy in the laboratory rather than its rest energy; the two differ because the meson was moving, with momentum 1.92 GeV/c. B has the wrong sign on the cross term: $(p_{1c} + p_{2c}) \cdot (p_{1c} + p_{2c}) = E_1^2 + E_2^2 + 2E_1E_2 \cos \theta$, and because this is subtracted from the energy squared the invariant contains $1 - \cos \theta$, not $1 + \cos \theta$. C drops the factor of 2 from the cross term $2p_1 \cdot p_2$, which leaves the result too small by a factor of $\sqrt{2}$.

U3

10

CORE
COURSE

MULTIPLE CHOICE · RECALL · UPIV-15.02 · UPIV-15.09

[1]

Entangled photon pairs are measured at two stations whose analyser settings are chosen randomly and independently, with the detection events at the two stations spacelike separated and the detection efficiency high enough that no fair-sampling assumption is needed. The measured CHSH combination is $S = 2.70 \pm 0.02$. Which statement correctly describes what this result establishes?

KEY **C All models in which each outcome is fixed by variables carried locally from the source, independent of the settings, are excluded; the correlations still cannot be used to send a signal faster than light, and non-local hidden-variable models are not excluded.**

Any model whose outcomes are fixed by local variables that do not depend on the settings obeys $|S| \leq 2$, so a value 35 standard deviations above 2, with the locality and detection loopholes closed as the stem describes, excludes that whole class; quantum mechanics predicts $2\sqrt{2} \approx 2.83$ for a maximally entangled pair at the optimal settings. The violation itself does not prove that signalling is impossible; C says only that these correlations give no means of signalling, because each station's own statistics are independent of the distant setting. A is the most common over-reading: the outcome statistics at either station alone do not depend on the distant setting, so the correlation appears only when the two records are brought together through an ordinary channel. B over-claims: the violation rules out locality combined with predetermined outcomes, not hidden variables as such, and non-local hidden-variable theories of the Bohmian kind reproduce the quantum predictions for this experiment. D inverts the physics: mixing in noise lowers S towards and below 2, and it is the pure maximally entangled state that gives the largest value.

U1

11

CORE
COURSE

STRUCTURED · DEMANDING · UPIV-11.03 · UPIV-11.09

[10]

A handheld X-ray fluorescence analyser is aimed at a corroded brass fitting. Its primary X-rays eject inner-shell electrons from atoms in the sample, and the detector records two strong lines at 8.04 keV and 8.63 keV, together with weaker lines, one of which lies at 8.90 keV. Model each K line with hydrogen-like levels and a screening constant σ : a transition from shell n to the K shell ($n = 1$) emits a photon of energy $E = (13.6 \text{ eV})(Z - \sigma)^2(1 - 1/n^2)$, so that the $K\alpha$ line ($n = 2$) follows Moseley's law, $E = (3/4)(13.6 \text{ eV})(Z - \sigma)^2$.

- (a)** DESCRIBE · U1 [2]
- Describe the sequence of events, beginning with the absorption of a primary X-ray photon, that leads to the emission of a $K\alpha$ photon, and say why the primary photons must exceed a threshold energy.
- 1 mark: a primary photon with energy above the K-shell binding energy is absorbed and ejects a 1s electron, leaving a K-shell vacancy; a photon below this K edge cannot create the vacancy, hence the threshold
 - 1 mark: an electron from the $n = 2$ (L) shell falls into the vacancy, and the energy difference between the two inner-shell levels is carried away by the $K\alpha$ photon (the competing Auger process ejects an electron instead)
- Candidates who describe an outer electron being excited and falling back are describing optical fluorescence. An X-ray line needs an inner-shell vacancy, which is why its energy identifies the element almost independently of its chemical state.*
- (b)** CALCULATE · U3 [3]
- Taking $\sigma = 1$, calculate the atomic numbers of the elements that produce the 8.04 keV and 8.63 keV lines, and identify the elements.
- 1 mark: rearranges Moseley's law to $Z = 1 + \sqrt{E/((3/4)(13.6 \text{ eV}))} = 1 + \sqrt{E/10.2 \text{ eV}}$, with E in eV
 - 1 mark: 8.04 keV gives $Z = 1 + \sqrt{788} = 29.1$, so $Z = 29$, copper
 - 1 mark: 8.63 keV gives $Z = 1 + \sqrt{846} = 30.1$, so $Z = 30$, zinc, the two constituents of brass
- Dividing an energy in keV by 13.6 eV without converting gives $Z \approx 1.9$ for both lines. That both answers land within 0.1 of an integer is itself the check that the units are right.*
- (c)** EXPLAIN · U1 [2]
- Explain why $\sigma \approx 1$ is a reasonable screening constant for a $K\alpha$ line.
- 1 mark: with one 1s electron ejected, one 1s electron remains; it lies almost entirely inside the orbit of the $n = 2$ electron that makes the jump, and so screens one unit of the nuclear charge
 - 1 mark: the electrons of the $n = 2$ shell and beyond are spread at the same or larger radii, so by Gauss's law they screen little, leaving an effective charge close to $Z - 1$ in a hydrogen-like energy
- The error to catch is crediting the screening to the ejected electron. That electron has left the atom; it is the 1s electron left behind, inside the orbit of the electron that jumps, that reduces the charge it sees.*
- (d)** EVALUATE · U4 [3]
- A colleague suggests that the 8.90 keV line is the $K\alpha$ line of a third element. Evaluate this suggestion against the alternative that it is the $K\beta$ line ($n = 3 \rightarrow n = 1$) of copper, and state what your conclusion implies about σ for a $K\beta$ line.
- 1 mark: as a $K\alpha$ line with $\sigma = 1$, 8.90 keV gives $Z = 1 + \sqrt{(8900/10.2)} = 30.5$, not an integer, whereas the two strong lines gave Z within 0.1 of an integer; the neighbouring $K\alpha$ lines, zinc (observed at 8.63 keV) and gallium (predicted 9.18 keV), bracket it, so the suggestion is rejected
 - 1 mark: copper $K\beta$ with $\sigma = 1$ is predicted at $(13.6 \text{ eV})(28)^2(1 - 1/9) = 9.48 \text{ keV}$, within 7% of 8.90 keV, and copper is known to be present, so the line is copper $K\beta$
 - 1 mark: matching 8.90 keV exactly needs $\sigma \approx 1.9$, because the $n = 3$ electron is screened by the remaining 1s electron and also, in part, by the eight $n = 2$ electrons; $\sigma = 1$ does not carry over from $K\alpha$ to $K\beta$
- The discriminating step is to use the precision shown in (b) as the test. A model that places two lines within 0.1 of an integer cannot then be stretched to accept $Z = 30.5$, and the 7% misfit for $K\beta$ is a statement about screening, not a failure of the identification.*

A single lithium-7 atom, of mass $m = 1.165 \times 10^{-26}$ kg, is confined along one direction between two sheets of laser light that act as impenetrable walls at $x = 0$ and $x = L$, with $L = 10.0 \mu\text{m}$. Its stationary states are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ for $0 \leq x \leq L$, with energies $E_n = n^2\pi^2\hbar^2/(2mL^2)$, and it is prepared in the ground state ψ_1 . At $t = 0$ a pulse of light lasting far less than any timescale of the atom's motion illuminates only the half of the box $0 < x < L/2$, multiplying the wavefunction there by $e^{i\pi} = -1$ and leaving the half $L/2 < x < L$ untouched. Take $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J K $^{-1}$. You may use $\int_0^{L/2} \sin(n\pi x/L) \sin(\pi x/L) dx = (L/(2\pi))[\sin((n-1)\pi/2)/(n-1) - \sin((n+1)\pi/2)/(n+1)]$ for $n \geq 2$.

(a) CALCULATE · U3

[2]

Calculate E_1 , and the temperature T at which $k_B T$ equals $E_2 - E_1$, the temperature scale that the atom must be cooled well below for its ground state to be prepared reliably.

1 1 mark: $E_1 = \pi^2\hbar^2/(2mL^2) = \pi^2(1.055 \times 10^{-34} \text{ J s})^2/[2(1.165 \times 10^{-26} \text{ kg})(1.00 \times 10^{-5} \text{ m})^2] = 4.71 \times 10^{-32} \text{ J}$

2 1 mark: $E_2 - E_1 = 3E_1 = 1.41 \times 10^{-31} \text{ J}$, so $T = 3E_1/k_B = 1.02 \times 10^{-8} \text{ K}$, about 10 nK
Converting 10.0 μm and then forgetting to square it, or taking the gap as $E_2 = 4E_1$ instead of $E_2 - E_1 = 3E_1$, are the slips this part is set to catch. At T itself the atom would still be found in an excited state about 30% of the time, which is why a single atom in a box this size has to be cooled well below 10 nK before its ground state can be prepared reliably.

(b) DERIVE · U2

[4]

Starting from the expansion of $\Psi(x, 0)$ in the stationary states and the Born rule, derive the probability P_n that a measurement of the energy made just after the pulse returns E_n . Find every n for which $P_n = 0$, and evaluate P_2 and P_4 .

1 1 mark: $\Psi(x, 0) = \sum c_n \psi_n(x)$, and orthonormality of the ψ_n gives $c_n = \int_0^L \psi_n(x) \Psi(x, 0) dx$, with $P_n = |c_n|^2$ by the Born rule

2 1 mark: $\Psi(x, 0) = -\psi_1$ in the left half and $+\psi_1$ in the right half, so c_n is the integral of $\psi_n \psi_1$ over the right half minus its integral over the left half; since $\psi_n(L-x) = (-1)^{n+1} \psi_n(x)$, the product $\psi_n \psi_1$ is symmetric about $x = L/2$ for odd n , the two integrals are equal, and $P_n = 0$ for every odd n , the ground state included

3 1 mark: for even n the product is antisymmetric about $x = L/2$, so $c_n = -2 \int_0^{L/2} \psi_n \psi_1 dx = -(2/\pi)[\sin((n-1)\pi/2)/(n-1) - \sin((n+1)\pi/2)/(n+1)]$, which gives $P_n = 16n^2/[\pi^2(n^2 - 1)^2]$ for even n

4 1 mark: $P_2 = 64/(9\pi^2) = 0.721$ and $P_4 = 256/(225\pi^2) = 0.115$

Candidates who reason that the density has not changed, and so the atom must still be in its ground state, miss that $P_1 = 0$ exactly. The symmetry argument does the work: without it every overlap has to be integrated separately and the zeros look like accidents.

(c) EXPLAIN · U1

[3]

The pulse leaves the probability density $|\Psi(x, 0)|^2$ equal to $|\psi_1(x)|^2$. Explain why the density nevertheless changes after the pulse, why it remains symmetric about the centre of the box at all later times, and why it is zero at $x = L/2$ at every $t > 0$.

1 1 mark: the pulse changes the relative phase of the two halves, so $\Psi(x, 0)$ is no longer a stationary state but a superposition of $\psi_2, \psi_4, \psi_6, \dots$; each term acquires its own phase $e^{i(-iE_n t/\hbar)}$, so the cross terms in $|\Psi|^2$ change with time

2 1 mark: every component present is antisymmetric about $x = L/2$ and keeps that symmetry as its phase evolves, so $\Psi(L-x, t) = -\Psi(x, t)$ and $|\Psi(x, t)|^2 = |\Psi(L-x, t)|^2$ at all times

3 1 mark: antisymmetry requires $\Psi(L/2, t) = -\Psi(L/2, t)$, so $\Psi(L/2, t) = 0$, as every even ψ_n has a node there; a notch opens at the centre, where the density had its maximum before the pulse

Reasoning from the density alone leads to the answer that a pure change of phase does nothing. The relative phase between parts of a wavefunction is physical: it decides which stationary states are present, and so how the state evolves.

(d) PREDICT · U1

[2]

In the real trap the light sheets form barriers of finite height $V_0 = 400E_1$, with E_1 the value found in (a), and the same pulse, which reverses the sign of the wavefunction everywhere to the left of $x = L/2$, is applied to the ground state of this finite well. Predict, with reasons, whether a measurement of the energy just after the pulse could return the ground-state energy of the finite well, and whether the atom is certain to remain in the trap.

- 1 mark: the finite well is still symmetric about $x = L/2$, so its bound states still have definite parity and its ground state is symmetric; the pulsed wavefunction is antisymmetric about the centre, so its overlap with the ground state is zero and the ground-state energy cannot be returned
- 2 1 mark: the sign jump makes the amplitudes fall off only as $1/n$, so part of the probability lies in unbound states above the barrier top (in the infinite-well estimate the states with $n > 20$, whose energies exceed $400E_1$, carry about 4%), and the atom is not certain to remain trapped

Predicting that finite walls merely lower the levels a little misses both effects that matter here: parity, which a symmetric finite well preserves exactly, and the unbound states above V_0 , which a sharp step in the wavefunction always populates.

13
CORE
COURSE

STRUCTURED · DEMANDING · UPIV-09.04 · UPIV-09.07 · UPIV-09.09

[11]

The resonance lines of the alkali atoms come from an $np \rightarrow ns$ transition that spin-orbit coupling splits into D_1 ($^2P_{1/2} \rightarrow ^2S_{1/2}$) and D_2 ($^2P_{3/2} \rightarrow ^2S_{1/2}$). The fine-structure splitting between $^2P_{3/2}$ and $^2P_{1/2}$ is 7.16×10^{-3} eV for the 4p level of potassium and 4.16×10^{-5} eV for the 2p level of lithium. Both vapours are placed in a uniform magnetic field of 0.500 T. Ignore hyperfine structure throughout. Take $\mu_B = 5.788 \times 10^{-5}$ eV T $^{-1}$ and $g_s = 2$, so that $g_J = 1 + [j(j+1) + s(s+1) - l(l+1)]/[2j(j+1)]$.

(a) CALCULATE · U2

[2]

Calculate g_J for the $^2S_{1/2}$, $^2P_{1/2}$ and $^2P_{3/2}$ levels.

- 1 mark: $^2S_{1/2}$: $g_J = 1 + (3/4 + 3/4)/(3/2) = 2$, the moment being pure spin
- 2 1 mark: $^2P_{1/2}$: $g_J = 1 + (3/4 + 3/4 - 2)/(3/2) = 2/3$; $^2P_{3/2}$: $g_J = 1 + (15/4 + 3/4 - 2)/(15/2) = 4/3$

$g_s = 2$ is what separates the anomalous Zeeman effect from the normal one. A candidate who sets $g_s = 1$ obtains $g_J = 1$ for every level and predicts the normal triplet for an alkali line, which is exactly what is not observed.

(b) DETERMINE · U2

[3]

In the weak-field limit each sublevel shifts by $g_J m_J \mu_B B$. Using the selection rule $\Delta m_J = 0, \pm 1$, determine the number of components into which the D_1 line splits and the shift of each component from the zero-field line, in units of $\mu_B B$.

- 1 mark: the $^2P_{1/2}$ sublevels shift by $\pm(1/3)\mu_B B$ and the $^2S_{1/2}$ sublevels by $\pm\mu_B B$
- 2 1 mark: all four pairings of $m_J = \pm 1/2$ in the upper level with $m_J = \pm 1/2$ in the lower level satisfy $\Delta m_J = 0, \pm 1$
- 3 1 mark: four components, at $\pm(2/3)\mu_B B$ for $\Delta m_J = 0$ and $\pm(4/3)\mu_B B$ for $\Delta m_J = \pm 1$

The frequent error is to use one g factor for both levels, which collapses the pattern into the normal triplet. The upper and lower levels shift by different amounts, and each component's shift is their difference.

(c) COMPARE · U3

[2]

Compare the fine-structure splitting with $\mu_B B$ at 0.500 T for each atom.

- 1 mark: $\mu_B B = (5.788 \times 10^{-5} \text{ eV T}^{-1})(0.500 \text{ T}) = 2.89 \times 10^{-5} \text{ eV}$
- 2 1 mark: potassium: $7.16 \times 10^{-3} \text{ eV} / 2.89 \times 10^{-5} \text{ eV} \approx 250$; lithium: $4.16 \times 10^{-5} \text{ eV} / 2.89 \times 10^{-5} \text{ eV} \approx 1.4$, so the same field is a small perturbation for potassium but comparable with the spin-orbit energy for lithium

Whether a field is weak or strong is a comparison, not a property of the field. The same 0.500 T is 250 times too weak to disturb potassium's spin-orbit coupling and comparable with lithium's.

(d) JUSTIFY · U1 [2]

Justify, for each atom, whether the D₁ pattern you found in (b) describes its spectrum at 0.500 T.

- 1 mark: potassium: the Zeeman energy is about 250 times smaller than the spin-orbit energy, so j and m_j remain good quantum numbers, $^2P_{1/2}$ and $^2P_{3/2}$ do not mix, and the four-component pattern of (b) applies
- 1 mark: lithium: the two energies are comparable, so the field mixes $^2P_{1/2}$ and $^2P_{3/2}$ sublevels of the same m_j ; j is no longer a good quantum number, the shifts are not linear in B , and the weak-field pattern does not apply (the intermediate regime)

Candidates who classify by field strength alone give the same answer for both atoms. Which quantum numbers are good is decided by which interaction is larger, and for lithium at this field neither dominates.

(e) PREDICT · U1 [2]

Predict the appearance of the lithium resonance line at 5.00 T, naming the good quantum numbers.

- 1 mark: at 5.00 T, $\mu_B B = 2.89 \times 10^{-4}$ eV, about seven times the fine-structure splitting, so the Paschen-Back limit is approached: spin and orbit decouple, and m_l and m_s , not j and m_j , are the good quantum numbers
- 1 mark: each state shifts by $(m_l + 2m_s)\mu_B B$; with $\Delta m_s = 0$ and $\Delta m_l = 0, \pm 1$ the line becomes a triplet at 0 and $\pm \mu_B B = \pm 2.89 \times 10^{-4}$ eV, like the normal Zeeman pattern, with each component slightly split by the residual spin-orbit energy

The error is to keep the Landé g factors at high field. Once spin and orbit decouple g_j describes nothing; each state's shift is $(m_l + 2m_s)\mu_B B$, and the rule $\Delta m_s = 0$ does the rest.

14 CORE COURSE

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[12]

The muon decays by $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ with a branching fraction of essentially 1 and a mean lifetime $\tau_\mu = 2.197 \times 10^{-6}$ s. The τ^- lepton decays through the same charged weak current into $e^- + \bar{\nu}_e + \nu_\tau$ with branching fraction $B_e = 0.178$, and also into $\mu^- + \bar{\nu}_\mu + \nu_\tau$ and into hadrons. Take $m_\mu c^2 = 105.66$ MeV, $m_\tau c^2 = 1776.9$ MeV and the Fermi constant $G_F/(\hbar c)^3 = 1.166 \times 10^{-5} \text{ GeV}^{-2}$. In natural units, with $\hbar = c = 1$, a decay width $\Gamma = \hbar/\tau$ has the dimensions of energy.

(a) STATE · U1 [2]

State the values of L_e , L_μ and L_τ before and after the decay $\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$, and state which conservation law forbids $\tau^- \rightarrow e^- + \gamma$.

- 1 mark: before, $(L_e, L_\mu, L_\tau) = (0, 0, 1)$; after, $e^- (1, 0, 0) + \bar{\nu}_e (-1, 0, 0) + \nu_\tau (0, 0, 1) = (0, 0, 1)$, so each lepton-family number is conserved
- 1 mark: $\tau^- \rightarrow e^- + \gamma$ changes L_τ from 1 to 0 and L_e from 0 to 1, so it violates conservation of lepton-family number, although charge, energy and total lepton number all balance

Checking only the total lepton number $L_e + L_\mu + L_\tau$ leaves $\tau^- \rightarrow e^- \gamma$ allowed. It is the separate family numbers that forbid it, and searches for that decay are searches for exactly this violation.

(b) DESCRIBE · U1 [2]

Describe the lowest-order Feynman diagram for $\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$, and give the power of the weak coupling g_W in its amplitude and in its rate.

- 1 mark: at one vertex the τ^- becomes a ν_τ by emitting a virtual W^- , which propagates to a second vertex where it becomes e^- and $\bar{\nu}_e$
- 1 mark: two vertices give an amplitude proportional to g_W^2 and a rate proportional to its square, g_W^4 ; at energies far below M_W the propagator is about $1/M_W^2$, so the rate goes as $(g_W^2/M_W^2)^2$, that is as G_F^2

A Z^0 exchange cannot turn a charged lepton into a neutrino, since charge would not balance at the vertex. The rate going as g_W^4 rather than g_W^2 is the step most often missed.

(c) SHOW THAT · U2 [2]

Show that if the width depends only on G_F and on the mass m of the decaying lepton, the result of (b) requires $\Gamma \propto G_F^2 m^5$.

1 1 mark: in natural units G_F has dimensions of energy⁻² and Γ of energy, so writing $\Gamma \propto G_F^a m^b$ requires $-2a + b = 1$

2 1 mark: the rate goes as G_F^2 , so $a = 2$ and $b = 5$, giving $\Gamma \propto G_F^2 m^5$

The m^5 law is why a lepton 16.8 times heavier opens each such channel about 1.3×10^6 times faster. The argument holds only while the final-state masses are negligible, which is true for $\tau \rightarrow e\bar{\nu}\nu$ and $\mu \rightarrow e\bar{\nu}\nu$ alike.

(d) ESTIMATE · U3 [3]

Estimate the mean lifetime of the τ , stating the assumptions you make.

1 1 mark: assumes lepton universality, so the W couples equally to τ and μ , and neglects the electron and neutrino masses, so $\Gamma(\tau \rightarrow e\bar{\nu}\nu)/\Gamma(\mu \rightarrow e\bar{\nu}\nu) = (m_\tau/m_\mu)^5 = (16.82)^5 = 1.35 \times 10^6$

2 1 mark: the partial width is $\Gamma(\tau \rightarrow e\bar{\nu}\nu) = B_e \hbar/\tau_\tau$, so $\tau_\tau = B_e \tau_\mu (m_\mu/m_\tau)^5$

3 1 mark: $\tau_\tau = 0.178 \times 2.197 \times 10^{-6} \text{ s} \times 7.43 \times 10^{-7} = 2.91 \times 10^{-13} \text{ s}$

Omitting the branching fraction gives $1.6 \times 10^{-12} \text{ s}$, more than five times too long. The τ is heavy enough to decay into muons and hadrons as well, and every extra channel shortens its life.

(e) EVALUATE · U4 [3]

The measured τ lifetime is $2.903 \times 10^{-13} \text{ s}$. Evaluate what the agreement with your estimate tests, and use a count of the final states available to the virtual W^- to explain why B_e is close to 0.2.

1 1 mark: the estimate agrees with the measurement to about 0.2%, so the charged weak coupling of the τ equals that of the μ to within the few-tenths-of-a-percent precision of the inputs: lepton universality holds across generations

2 1 mark: the W^- from a τ can materialise as $e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$ or $d\bar{u}$, and the quark pair counts three times, once for each colour, giving five channels of nearly equal weight and $B_e \approx 1/5$

3 1 mark: the measured 0.178 lies below 1/5 because strong-interaction (QCD) corrections raise the hadronic width: $1/B_e = 5.6$ requires the quark channels to count as about 3.6 instead of 3, an enhancement of about 20%

Counting the quark channel once gives $B_e \approx 1/3$. The factor of three from colour, confirmed by the measured branching fraction, is the same factor that appears in the R ratio.

15
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[11]

^{210}Po decays by α emission to ^{206}Pb with a half-life of 138.4 days. The atomic masses are ^{210}Po 209.982874 u, ^{206}Pb 205.974465 u and ^4He 4.002603 u, with $1 \text{ u} = 931.494 \text{ MeV}/c^2$. In the Gamow model the α particle, of reduced mass μ , meets a Coulomb barrier $V(r) = Z_d Z_\alpha \alpha \hbar c / r$ that begins at the nuclear radius R , and tunnels out to the classical turning point b at which $V(b) = Q$. The WKB exponent is $2G = 2\alpha Z_d Z_\alpha \sqrt{(2\mu c^2/Q)} [\arccos\sqrt{(R/b)} - \sqrt{((R/b)(1 - R/b))}]$, and the half-life is proportional to $e^{-(2G)}$ divided by an assault frequency. Take $\alpha \hbar c = 1.440 \text{ MeV fm}$, $\alpha = 1/137.0$, $\mu c^2 = 3.66 \times 10^3 \text{ MeV}$, $R = 9.0 \text{ fm}$, $Z_d = 82$ and $Z_\alpha = 2$.

(a) CALCULATE · U3 [2]

Calculate the Q value of the decay.

1 1 mark: $Q = [m(^{210}\text{Po}) - m(^{206}\text{Pb}) - m(^4\text{He})]c^2 = (0.005806 \text{ u})(931.494 \text{ MeV/u})$, the electron masses cancelling when atomic masses are used throughout

2 1 mark: $Q = 5.41 \text{ MeV}$

Atomic masses balance the electrons, $84 = 82 + 2$. Subtracting electron masses on one side only shifts Q by about 1 MeV, which in a barrier exponent changes the half-life by many orders of magnitude.

- (b) **CALCULATE** · U3 [1]
Calculate the outer turning point b.
- 1 1 mark: $b = Z_d Z_\alpha \hbar c / Q = (82)(2)(1.440 \text{ MeV fm}) / (5.41 \text{ MeV}) = 43.7 \text{ fm}$, nearly five times R, so $R/b = 0.21$
- The barrier is thick because Q is only a fifth of the 26 MeV Coulomb energy at R. Candidates who take b to be close to R have not used $V(b) = Q$, and a thin barrier could never produce half-lives spanning twenty orders of magnitude.*
- (c) **SHOW THAT** · U2 [3]
Using $\arccos \sqrt{x} \approx \pi/2 - \sqrt{x}$ and $\sqrt{x(1-x)} \approx \sqrt{x}$ for $x = R/b$ much less than 1, show that $2G \approx \pi \alpha Z_d Z_\alpha \sqrt{2\mu c^2 / Q} - 4\sqrt{2\alpha Z_d Z_\alpha \mu c^2 R / \hbar c}$, and hence that the second term does not depend on Q.
- 1 1 mark: the bracket becomes $\pi/2 - 2\sqrt{R/b}$, so the first term is $2\alpha Z_d Z_\alpha \sqrt{2\mu c^2 / Q} (\pi/2) = \pi \alpha Z_d Z_\alpha \sqrt{2\mu c^2 / Q}$
- 2 1 mark: the second term is $4\alpha Z_d Z_\alpha \sqrt{2\mu c^2 / Q} \sqrt{R/b}$, and with $b = Z_d Z_\alpha \hbar c / Q$, $\sqrt{R/b} = \sqrt{(RQ)/(Z_d Z_\alpha \hbar c)}$
- 3 1 mark: multiplying out, the factors of Q cancel: $4\sqrt{[\alpha^2 Z_d^2 Z_\alpha^2 (2\mu c^2 / Q)(RQ)/(Z_d Z_\alpha \hbar c)]} = 4\sqrt{2\alpha Z_d Z_\alpha \mu c^2 R / \hbar c}$, independent of Q
- The algebra turns on substituting $b = Z_d Z_\alpha \hbar c / Q$ inside $\sqrt{R/b}$. Candidates who treat b as a fixed length miss that the second term is independent of Q, which is what lets the prediction in (d) proceed without R.*
- (d) **PREDICT** · U3 [3]
 ^{214}Po also decays by α emission to a lead isotope ($Z_d = 82$), with $Q = 7.833 \text{ MeV}$. Assuming the same R, μ and assault frequency for both nuclei, predict the half-life of ^{214}Po from that of ^{210}Po using the result of (c).
- 1 1 mark: with R, μ and the assault frequency unchanged only the first term differs, so $\ln[t_{1/2}(^{210}\text{Po})/t_{1/2}(^{214}\text{Po})] = \pi \alpha Z_d Z_\alpha \sqrt{2\mu c^2} [1/\sqrt{Q(^{210}\text{Po})} - 1/\sqrt{Q(^{214}\text{Po})}]$
- 2 1 mark: $= (3.761)(85.56 \text{ MeV}^{1/2})(0.4300 - 0.3573) \text{ MeV}^{-1/2} = 23.4$, the first terms being 138.4 and 115.0
- 3 1 mark: the ratio is $e^{23.4} = 1.4 \times 10^{10}$, so $t_{1/2}(^{214}\text{Po}) = (138.4 \times 86\,400 \text{ s}) / (1.4 \times 10^{10}) = 8.3 \times 10^{-4} \text{ s}$
- The exponent is so sensitive that an error of 0.1 MeV in the Q value of ^{210}Po alone changes the predicted ratio by a factor of nearly 4. Carry Q to at least three significant figures throughout.*
- (e) **EVALUATE** · U4 [2]
The measured half-life of ^{214}Po is 164 μs . Evaluate the model in the light of this comparison.
- 1 1 mark: the prediction, 0.83 ms, is about five times the measured 0.164 ms, yet the model spans a half-life ratio of order 10^{10} : the measured ln ratio is 25.0 against the predicted 23.4, so the Q dependence of the barrier accounts for nearly all of the ten orders of magnitude
- 2 1 mark: agreement to within a factor of a few is all the model can claim, because the half-life depends exponentially on R and on the approximations made; it also omits nuclear structure, and ^{210}Po , with the closed neutron shell $N = 126$, has a reduced probability of preforming an α cluster, lengthening its half-life relative to ^{214}Po ($N = 130$), in the direction of the discrepancy
- Calling a factor of five a failure ignores the ten orders of magnitude the model spans. Calling it a success without saying what the model omits is not an evaluation either.*

A sample-return probe leaves a space station, which defines an inertial frame S, and travels in a straight line at 0.600c to a relay beacon at rest in S at a distance of 3.00 light-years. At the beacon it reverses in a negligibly short time and returns at 0.800c. Work in units in which $c = 1$ light-year per year, label events by their S coordinates (t, x), and take the departure as the event (0, 0). Let S' be the inertial frame moving with the probe on the outbound leg and S'' the inertial frame moving with it on the return leg.

(a) CALCULATE · U3

[3]

Calculate the time that elapses on the station clock, and the proper time that elapses on the probe's clock, for the round trip.

- 1 1 mark: station: $3.00/0.600 = 5.00$ y out and $3.00/0.800 = 3.75$ y back, a total of 8.75 y
- 2 1 mark: probe, leg by leg with $\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$: $5.00 \text{ y} \times 0.800 = 4.00 \text{ y}$ and $3.75 \text{ y} \times 0.600 = 2.25 \text{ y}$
- 3 1 mark: total proper time 6.25 y, so the probe returns 2.50 y younger than the station clock reads

Using a single γ for the whole trip, or one built from the average speed, is the error this part catches. Proper time accumulates leg by leg along the worldline.

(b) DETERMINE · U2

[4]

Determine the reading of the station clock that is simultaneous with the turnaround event, first in S' and then in S'' .

- 1 1 mark: the turnaround is the event (5.00 y, 3.00 ly); in S' , with $v = +0.600c$ and $\gamma = 1.25$, $t' = \gamma(t - vx/c^2) = 1.25(5.00 - 1.80) = 4.00$ y, the probe's own clock reading
- 2 1 mark: a station event ($t_s, 0$) has $t' = \gamma t_s$, and setting this equal to 4.00 y gives $t_s = 3.20$ y
- 3 1 mark: in S'' , with $v = -0.800c$ and $\gamma = 5/3$, $t'' = \gamma(t + 0.800x) = (5/3)(5.00 + 2.40) = 12.3$ y
- 4 1 mark: setting $\gamma t_s = 12.3$ y gives $t_s = 7.40$ y, so the station reading simultaneous with the turnaround jumps from 3.20 y to 7.40 y

The trap is the sign of v in S'' . Using $+0.800c$ puts the station clock at 2.60 y just after the turnaround, earlier than just before it, and no account of the return leg can then be reconciled with 8.75 y.

(c) SHOW THAT · U2

[2]

Show that the probe's account, with the station clock running slow on each leg plus the jump found in (b), reproduces the station's total of 8.75 y.

- 1 1 mark: in S' the station clock runs slow by $\gamma = 1.25$, advancing $4.00 \text{ y}/1.25 = 3.20$ y; in S'' it runs slow by $\gamma = 5/3$, advancing $2.25 \text{ y} \times 0.600 = 1.35$ y
- 2 1 mark: $3.20 \text{ y} + (7.40 - 3.20) \text{ y} + 1.35 \text{ y} = 8.75 \text{ y}$, agreeing with the station's own clock

Candidates who apply time dilation to the station clock on both legs and stop there get 4.55 y and conclude that the station ages less than the probe. The missing 4.20 y is the change of simultaneity at the turnaround.

(d) EVALUATE · U1

[2]

Evaluate the claim: "The probe ages less because it accelerated, and the harder it accelerates at the turnaround, the more time it loses."

- 1 1 mark: the ageing difference is fixed by the speeds and distances of the two inertial legs and is 2.50 y however brief the turnaround; the size of the acceleration does not appear in it, so the second half of the claim is false
- 2 1 mark: the first half is right only in a limited sense: the acceleration makes the probe's worldline non-inertial and breaks the symmetry, since the probe changes inertial frame and with it its simultaneity (the 4.20 y jump), while the station stays in one frame, and of all worldlines joining departure and return the inertial one has the greatest proper time

Acceleration is needed to break the symmetry but does not set the size of the effect. A candidate who cannot say where the 2.50 y comes from without mentioning acceleration has not separated the two.

The intrinsic carrier density n_i of an undoped semiconductor bar is found from Hall-effect and conductivity measurements at six temperatures, and each value of n_i carries a random uncertainty of $\pm 3\%$. For a non-degenerate intrinsic semiconductor with a temperature-independent band gap E_g , $n_i = CT^{3/2}\exp(-E_g/2kT)$, where C is a constant. Take $k = 8.617 \times 10^{-5} \text{ eV K}^{-1}$.

(a) EXPLAIN · U1

[2]

Explain why a graph of $\ln(n_i/T^{3/2})$ against $1/T$ is expected to be a straight line, and what its gradient gives.

- 1 mark: taking logarithms, $\ln(n_i/T^{3/2}) = \ln C - (E_g/2k)(1/T)$, which is linear in $1/T$ with the constant intercept $\ln C$
- 1 mark: the gradient is $-E_g/2k$, so $E_g = -2k \times \text{gradient}$; the $T^{3/2}$ comes from the effective densities of states of the two bands and must be divided out, or it curves the line

Candidates who plot $\ln n_i$ against $1/T$ without dividing by $T^{3/2}$ are the subject of part (d). The linearisation must isolate a term that is exactly linear in $1/T$.

(b) DETERMINE · U4

[3]

Determine E_g from the data.

- 1 mark: computes $1/T = 3.333, 2.941, 2.632, 2.381, 2.174$ and $2.000 (\times 10^{-3} \text{ K}^{-1})$ and $\ln(n_i/T^{3/2}) = 35.90, 37.64, 39.07, 40.17, 41.15$ and 41.91
- 1 mark: plots or fits a straight line, the points lying on it within the ± 0.03 expected from $\pm 3\%$ uncertainties, with gradient $-4.5 \times 10^3 \text{ K}$ (least-squares $-4.52 \times 10^3 \text{ K}$; end points $-4.51 \times 10^3 \text{ K}$)
- 1 mark: $E_g = 2(8.617 \times 10^{-5} \text{ eV K}^{-1})(4.52 \times 10^3 \text{ K}) = 0.78 \text{ eV}$ (accept 0.77 eV to 0.79 eV)

Using $\log_{10}$ in place of $\ln$ with $E_g = -2k \times \text{gradient}$ understates the gap by a factor of 2.30. Checking the scatter about the line against the stated $\pm 3\%$ is what shows the model fits.

(c) ESTIMATE · U4

[2]

Using the first and last rows only, estimate the uncertainty in E_g that follows from the $\pm 3\%$ uncertainty in n_i , stating the assumption you make about the two uncertainties.

- 1 mark: assuming the two uncertainties are independent, the difference of the two logarithms is uncertain by $\sqrt{2} \times 0.03 = 0.042$, over a span in $1/T$ of $1.333 \times 10^{-3} \text{ K}^{-1}$, so the gradient is uncertain by about 32 K
- 1 mark: $\delta E_g = 2(8.617 \times 10^{-5} \text{ eV K}^{-1})(32 \text{ K}) = 0.0055 \text{ eV}$, so $E_g = 0.779 \pm 0.005 \text{ eV}$, an uncertainty of about 0.7%

Candidates who add the two 3% uncertainties linearly obtain 0.008 eV , which is acceptable if that assumption is stated. What matters is the size: a few meV, far smaller than the effects in (d) and (e).

(d) COMPARE · U4

[2]

A student instead plots $\ln n_i$ against $1/T$ and takes $E_g = -2k \times \text{gradient}$. Compare the gap this gives with your value from (b), and account for the difference.

- 1 mark: $\ln n_i = 44.46, 46.39, 47.98, 49.23, 50.34$ and 51.23 gives a gradient of about $-5.09 \times 10^3 \text{ K}$ and an apparent gap of 0.88 eV , about 0.10 eV (13%) larger than 0.78 eV
- 1 mark: $\ln n_i$ contains $(3/2)\ln T$, which rises with T and so steepens the line; its contribution to the apparent gap is about $3kT$ at the mid-range temperature of about 390 K , $3(8.6 \times 10^{-5} \text{ eV K}^{-1})(390 \text{ K}) \approx 0.10 \text{ eV}$, some twenty times the random uncertainty from (c)

The discriminating step is recognising that the 0.10 eV difference is systematic and nearly twenty times the random uncertainty. A slowly varying power of T is invisible by eye on a logarithmic plot and still shifts the gap by 13% .

(e) EVALUATE · U4

[2]

Optical absorption measured on the same bar at 400 K shows an absorption edge at 0.66 eV. Semiconductor gaps typically narrow as the temperature rises. Evaluate whether this contradicts your value from (b).

- 1 mark: if $E_g(T) = E_{g0} - \beta T$, then $\exp(-E_g/2kT) = \exp(\beta/2k)\exp(-E_{g0}/2kT)$, so the linear term changes only the constant C , and the gradient in (b) returns E_{g0} , the gap extrapolated to 0 K, not the gap at the measurement temperatures
- 1 mark: there is no contradiction: both results fit $E_{g0} = 0.78$ eV with $\beta = (0.78 - 0.66) \text{ eV}/(400 \text{ K}) = 3.0 \times 10^{-4} \text{ eV K}^{-1}$, a typical narrowing, so the transport fit and the optical edge together determine both E_{g0} and β

Candidates who put the 0.12 eV difference down to experimental error have not compared it with the ± 0.005 eV found in (c). The transport fit and the optical edge measure different things, and together they determine both E_{g0} and β .

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[15]

An electron is bound to a nucleus of charge Ze . With μ the reduced mass, the radial function $u(r) = rR(r)$ of a state of orbital quantum number l satisfies the radial Schrödinger equation $-(\hbar^2/2\mu)(d^2u/dr^2) + [-Ze^2/(4\pi\epsilon_0 r) + l(l+1)\hbar^2/(2\mu r^2)]u = Eu$, with $E < 0$ for a bound state. Write $\kappa = \sqrt{-2\mu E}/\hbar$ and $a_\mu = 4\pi\epsilon_0\hbar^2/(\mu e^2)$. You may use $\int_0^\infty r^k e^{-\beta r} dr = k!/\beta^{k+1}$ for integer $k \geq 0$. For the numerical parts take $a_0 = 5.292 \times 10^{-11}$ m and 13.606 eV as the Bohr radius and the ionisation energy of hydrogen for an infinitely heavy nucleus, $m_p/m_e = 1836$ and $\hbar = 4.136 \times 10^{-15}$ eV s.

(a) SHOW THAT · U2

[2]

Show that a normalisable solution must behave as $e^{-\kappa r}$ at large r and as r^{l+1} near $r = 0$.

- 1 mark: as $r \rightarrow \infty$ both potential terms vanish, leaving $d^2u/dr^2 = \kappa^2 u$, with solutions $e^{\pm\kappa r}$; normalisability rejects $e^{+\kappa r}$, so $u \propto e^{-\kappa r}$
- 1 mark: as $r \rightarrow 0$ the centrifugal term dominates, $d^2u/dr^2 \approx l(l+1)u/r^2$, and $u = r^s$ requires $s(s-1) = l(l+1)$, so $s = l+1$ or $s = -l$; $u \propto r^{-l}$ makes $R = u/r$ diverge, non-normalisably for $l \geq 1$ and, for $l = 0$, as $1/r$, which fails the Schrödinger equation at the origin, so $u \propto r^{l+1}$

Rejecting $u \propto r^{-l}$ by saying only that it diverges misses the $l = 0$ case, where $R = 1/r$ is still square-integrable. There the solution fails the Schrödinger equation itself, since $\nabla^2(1/r)$ is a delta function.

(b) DERIVE · U2

[5]

By substituting $u = r^{l+1}e^{-\kappa r}$ into the radial Schrödinger equation, derive the conditions under which it is an exact solution, and hence derive κ and the energy E of this state. Identify its principal quantum number n .

- 1 mark: $u' = [(l+1)/r - \kappa]u$ and $u'' = [l(l+1)/r^2 - 2\kappa(l+1)/r + \kappa^2]u$
- 1 mark: on substitution the $1/r^2$ terms, $-(\hbar^2/2\mu)l(l+1)/r^2$ and $+l(l+1)\hbar^2/(2\mu r^2)$, cancel identically
- 1 mark: the $1/r$ terms cancel only if $\hbar^2\kappa(l+1)/\mu = Ze^2/(4\pi\epsilon_0)$, so $\kappa = Z/[(l+1)a_\mu]$
- 1 mark: the constant terms give $E = -\hbar^2\kappa^2/(2\mu) = -Z^2\hbar^2/[2\mu a_\mu^2(l+1)^2] = -\mu Z^2 e^4/[2(4\pi\epsilon_0)^2\hbar^2(l+1)^2]$
- 1 mark: u has no radial nodes, so $n = n_r + l + 1 = l + 1$, and $E_n = -Z^2 e^2/(8\pi\epsilon_0 a_\mu n^2)$, the Bohr energy with the reduced mass, for the state with $l = n - 1$

The $1/r$ balance is the quantisation condition. A candidate who fixes κ from the constant terms alone and leaves the $1/r$ terms unbalanced has a trial function, not a solution, and no discrete spectrum.

(c) DETERMINE · U2 [3]

For this state, determine the most probable radius r_{mp} and the mean radius $\langle r \rangle$ in terms of n , a_μ and Z .

- 1 mark: the radial probability density is $P(r) \propto |u|^2 = r^{2n} e^{(-2Zr/(na_\mu))}$, the r^2 Jacobian being already contained in $u = rR$
- 1 mark: $dP/dr = 0$ gives $2n/r = 2Z/(na_\mu)$, so $r_{\text{mp}} = n^2 a_\mu / Z$
- 1 mark: $\langle r \rangle = \int r^{2n+1} e^{(-\beta r)} dr / \int r^{2n} e^{(-\beta r)} dr = (2n + 1)/\beta$ with $\beta = 2Z/(na_\mu)$, so $\langle r \rangle = n(n + 1/2) a_\mu / Z$
Multiplying $|u|^2$ by a further r^2 counts the Jacobian twice and moves the peak to $n(n + 1) a_\mu / Z$. The r^2 is already inside $u = rR$.

(d) CALCULATE · U3 [3]

Circular Rydberg states, with $l = n - 1$, are used in microwave cavity experiments. For the $n = 50$ circular state of hydrogen, calculate the most probable radius and the frequency of the photon emitted in the transition to $n = 49$.

- 1 mark: $a_\mu = a_0(1 + m_e/m_p) = (5.292 \times 10^{-11} \text{ m})(1.000545) = 5.295 \times 10^{-11} \text{ m}$
- 1 mark: $r_{\text{mp}} = 2500 a_\mu = 1.32 \times 10^{-7} \text{ m}$ (132 nm)
- 1 mark: $\Delta E = (13.606 \text{ eV}/1.000545)(1/49^2 - 1/50^2) = 2.24 \times 10^{-4} \text{ eV}$, so $f = \Delta E/h = 5.42 \times 10^{10} \text{ Hz}$ (54.2 GHz)

Omitting the reduced-mass correction shifts the frequency by about 30 MHz, a 0.05% error that a microwave measurement resolves easily.

(e) JUSTIFY · U1 [2]

Justify, using the electric-dipole selection rules, the claim that this state can decay by only one electric-dipole transition.

- 1 mark: electric-dipole transitions require $\Delta l = \pm 1$ (with $\Delta m = 0, \pm 1$), and spontaneous emission requires a final level of lower energy, $n' < 50$
- 1 mark: $\Delta l = +1$ needs $l' = 50$ and so $n' \geq 51$, which is higher in energy; $\Delta l = -1$ needs $l' = 48$ and so $n' \geq 49$, leaving only $n' = 49$, $l' = 48$, the next circular state

Applying a selection rule to n instead of to l lists every $n' < 50$. In hydrogen Δn is unrestricted; it is the constraint $l' \leq n' - 1$ that leaves a single channel.

19
CORE
COURSE

FREE RESPONSE · DERIVE A MODEL, THEN TEST IT AGAINST EVIDENCE · DISCRIMINATING · UPIV-15.06 · UPIV-15.07 [14]

Model the universe as a uniform, pressureless dust of mass density $\rho(t)$ in which every separation grows in proportion to a scale factor $a(t)$, with $a = 1$ today. A particle of mass m moving with the dust lies at distance $r = a(t)x$ from an arbitrary origin, where x is constant, and by the shell theorem it is attracted only by the mass inside the sphere of radius r . Take $G = 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$, the present expansion rate $H_0 = \dot{a}/a = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $1 \text{ Mpc} = 3.086 \times 10^{22} \text{ m}$, the mass of a hydrogen atom as $1.674 \times 10^{-27} \text{ kg}$ and $1 \text{ yr} = 3.156 \times 10^7 \text{ s}$.

(a) DERIVE · U2 [4]

Starting from conservation of energy for the particle, derive the Friedmann equation $(\dot{a}/a)^2 = (8\pi G/3)\rho - K/a^2$, where K is a constant, and state what the sign of K means for the future of the expansion in this Newtonian picture.

- 1 mark: the mass inside the sphere, $M = (4\pi/3)r^3\rho$, is fixed because the dust moves with the sphere, so ρa^3 is constant
- 1 mark: energy conservation for the particle: $\frac{1}{2}m\dot{r}^2 - GMm/r = E$, with E constant
- 1 mark: substituting $r = ax$, $\dot{r} = \dot{a}x$ and $M = (4\pi/3)a^3x^3\rho$ and dividing by $\frac{1}{2}ma^2x^2$ gives $(\dot{a}/a)^2 = (8\pi G/3)\rho + 2E/(mx^2a^2)$, so $K = -2E/(mx^2)$, a constant
- 1 mark: $K > 0$ ($E < 0$): the expansion halts and reverses; $K < 0$ ($E > 0$): it continues for ever; $K = 0$: the expansion rate falls to zero only as $t \rightarrow \infty$

Candidates who hold ρ constant while the sphere grows lose the a^{-3} dilution, and with it the content of the derivation. The shell theorem is what makes the choice of origin irrelevant.

- (b)** CALCULATE · U3 [3]
- The critical density ρ_c is the density for which $K = 0$. Calculate ρ_c today, and express it as a number of hydrogen atoms per cubic metre.
- 1 mark: $K = 0$ at $a = 1$ gives $\rho_c = 3H_0^2/(8\pi G)$, with $H_0 = (67.4 \times 10^3 \text{ m s}^{-1})/(3.086 \times 10^{22} \text{ m}) = 2.18 \times 10^{-18} \text{ s}^{-1}$
 - 1 mark: $\rho_c = 3(2.18 \times 10^{-18} \text{ s}^{-1})^2/(8\pi \times 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}) = 8.5 \times 10^{-27} \text{ kg m}^{-3}$
 - 1 mark: $\rho_c/m_H = (8.53 \times 10^{-27} \text{ kg m}^{-3})/(1.674 \times 10^{-27} \text{ kg}) = 5.1$ hydrogen atoms per cubic metre
- Mixing kilometres in the speed with metres in the megaparsec leaves H_0 a thousand times too small and ρ_c a million times too small.*
- (c)** DERIVE · U2 [3]
- For $K = 0$ and $\rho \propto a^{-3}$, derive $a(t)$ from the Friedmann equation, taking $a = 0$ at $t = 0$, and hence obtain the age t_0 of the universe in terms of H_0 and in gigayears.
- 1 mark: $\rho = \rho_c a^{-3} = (3H_0^2/8\pi G)a^{-3}$ turns the Friedmann equation into $\dot{a}^2 = H_0^2/a$, so $a^{1/2} da = H_0 dt$
 - 1 mark: integrating from $a = 0$ at $t = 0$ gives $(2/3)a^{3/2} = H_0 t$, so $a(t) = (3H_0 t/2)^{2/3} \propto t^{2/3}$
 - 1 mark: $a = 1$ today gives $t_0 = 2/(3H_0) = 3.05 \times 10^{17} \text{ s} = 9.7 \text{ Gyr}$
- Quoting $1/H_0 = 14.5 \text{ Gyr}$ as the age assumes the expansion rate has never changed. A decelerating universe was expanding faster in the past, and so is younger than $1/H_0$.*
- (d)** EVALUATE · U4 [2]
- The oldest globular-cluster stars in the Milky Way are dated at about 12.5 Gyr. Evaluate the flat, matter-only model in the light of this, including the value of H_0 the model would need.
- 1 mark: 9.7 Gyr is younger than stars that formed within the universe, so the model with the measured H_0 is excluded; an age of 12.5 Gyr would need $H_0 = 2/(3t_0) \approx 52 \text{ km s}^{-1} \text{ Mpc}^{-1}$, well below the measured value of about $67 \text{ km s}^{-1} \text{ Mpc}^{-1}$
 - 1 mark: the model must change, not H_0 : a component that makes the expansion accelerate, such as a cosmological constant, means the expansion was slower in the past and the universe older for the same H_0 (about 13.8 Gyr for matter and dark-energy fractions of 0.315 and 0.685 with $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$), and pressureless dust alone cannot supply it
- An age younger than the oldest stars is a contradiction, not a tension. The discriminating answer names what must change, and checks that lowering H_0 to fit would conflict with its measurement.*
- (e)** CALCULATE · U3 [2]
- The cosmic microwave background was released at redshift $z = 1090$. Calculate the scale factor at that time and the factor by which the matter density then exceeded its present value.
- 1 mark: $a = 1/(1 + z) = 1/1091 = 9.17 \times 10^{-4}$
 - 1 mark: $\rho/\rho_0 = a^{-3} = 1091^3 = 1.30 \times 10^9$
- Using $a = 1/z$ is a negligible error at $z = 1090$ but a factor of two at $z = 1$. Redshift measures the stretch of wavelengths, $\lambda_{\text{obs}}/\lambda_{\text{emit}} = 1 + z = 1/a$.*

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 27 MARKS · 23%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 39 MARKS · 34%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

30 MARKS · 26%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

20 MARKS · 17%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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