

University Physics IV · Midterm Examination

Where classical physics stops and what replaces it: Planck's quanta and the photoelectric evidence, the Lorentz boost and four-momentum, photons and matter waves, and the Schrödinger equation solved for the square well and the harmonic oscillator.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	73	13	7

SECTION	UPIV · Midterm · University Physics IV — Midterm examination
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	Routine 6 · Demanding 6 · Discriminating 1 <i>Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.</i>
COURSE UNITS	UPIV-01 · UPIV-02 · UPIV-03 · UPIV-04 · UPIV-05 · UPIV-06 · UPIV-07 · UPIV-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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University Physics IV · Midterm Examination

Answer all questions. 73 marks in 90 minutes.

1 MULTIPLE CHOICE · ROUTINE · UPIV-01.03 · UPIV-01.04 [1]
CORE COURSE

Cavity radiation at temperature 300 K is examined at wavelength 1.00 mm. By what factor does the Rayleigh-Jeans spectral energy density exceed the Planck spectral energy density at this wavelength? Take $h = 6.63 \times 10^{-34} \text{ J s}$, $c = 3.00 \times 10^8 \text{ m s}^{-1}$ and $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$.

- A 1.00, because the two laws agree exactly at long wavelength
- B 1.02
- C 1.05
- D 20.8

Answer

2 MULTIPLE CHOICE · ROUTINE · UPIV-03.06 [1]
CORE COURSE

A proton moves at a speed for which $\gamma = 2.00$. A force of magnitude $F_{\parallel}$ applied parallel to its velocity gives it an acceleration of magnitude a . A force of magnitude $F_{\perp}$ applied perpendicular to its velocity gives it an acceleration of the same magnitude a . What is $F_{\parallel}/F_{\perp}$ at that instant?

- A 0.250
- B 1.00
- C 4.00
- D 8.00

Answer

3 MULTIPLE CHOICE · ROUTINE · UPIV-04.07 [1]
CORE COURSE

A wave packet for a free, non-relativistic electron is built from de Broglie waves with angular frequency $\omega(k) = E/\hbar$, where $E = \hbar^2 k^2/(2m) + V_0$ and V_0 is a constant potential energy whose value depends on where the zero of energy is chosen. The packet is centred on wavenumber k_0 , and $v = \hbar k_0/m$ is the electron's classical speed. Which statement is correct?

- A The phase velocity is $v/2$ and the group velocity is v , whatever value of V_0 is chosen.
- B The phase velocity is v and the group velocity is $v/2$.
- C Both the phase velocity and the group velocity change when V_0 is changed.
- D The group velocity is v for every choice of V_0 , but the phase velocity depends on V_0 .

Answer

4

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04

[1]

Two identical narrow wave packets, ψ_L centred at $x = -d$ and ψ_R centred at $x = +d$, do not overlap anywhere. Many copies of a particle are prepared either (I) in the superposition $(\psi_L + \psi_R)/\sqrt{2}$, or (II) as a statistical mixture in which half the copies are in ψ_L and half are in ψ_R . Which measurement, repeated on many copies, distinguishes preparation I from preparation II?

- A Momentum: for I the momentum distribution is modulated by fringes of spacing $h/(2d)$, and for II it is not.
- B Position: for I the position distribution shows interference fringes between the two packets, and for II it does not.
- C The mean position $\langle x \rangle$, which is zero for I but not for II.
- D None, because in both preparations the particle is found near each centre with probability $\frac{1}{2}$.

Answer

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-06.08

[1]

Ehrenfest's theorem gives $d\langle x \rangle/dt = \langle p \rangle/m$ and $d\langle p \rangle/dt = -\langle \partial V/\partial x \rangle$. For which potential does $\langle x \rangle$ obey exactly the classical equation $m d^2\langle x \rangle/dt^2 = -V'(\langle x \rangle)$ for every state of the particle?

- A $V(x) = \alpha x^4$ with $\alpha > 0$, because it is a symmetric confining well.
- B Any potential, provided the wave packet is narrow at $t = 0$.
- C $V(x) = V_0 \cos(qx)$, because the force it exerts is bounded.
- D $V(x) = \frac{1}{2}m\omega^2 x^2 + Fx$, with F a constant.

Answer

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-07.07

[1]

Electrons of kinetic energy 8.00 eV travel in the $+x$ direction towards a potential step at which the potential energy rises abruptly from 0 to 6.00 eV. What fraction of the incident probability current is reflected?

- A 0
- B 0.111
- C 0.333
- D 0.360

Answer

For a particle of mass m in a harmonic oscillator of angular frequency ω , $\hat{x} = \sqrt{\hbar/(2m\omega)}(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. What is $\langle x^2 \rangle$ in the energy eigenstate $|n\rangle$?

- A $(n + \frac{1}{2})\hbar/(m\omega)$
- B $n\hbar/(m\omega)$
- C $(2n + 1)\hbar/(m\omega)$
- D Zero

Answer

An inspection gantry at rest in frame S has two laser gates: G_1 at $x = -60.0$ m and G_2 at $x = +60.0$ m, so the gantry's rest length is 120 m. A cargo shuttle of rest length 150 m moves in the $+x$ direction at $0.800c$. Frame S' is the shuttle's rest frame, in standard configuration with S : the origins coincide at $t = t' = 0$, when the centre of the shuttle passes $x = 0$. Each gate sits at the edge of the track and can fire a brief pulse straight across it, taking negligible time to cross; a pulse that meets the shuttle leaves a mark on its hull. The gantry controller fires both gates at $t = 0$ in S . Take $c = 3.00 \times 10^8$ m s $^{-1}$.

(a) CALCULATE

[2]

Calculate the length of the shuttle measured in S , and the length of the gantry measured in S' .

(b) CALCULATE

[3]

The firing of G_1 is event E_1 and the firing of G_2 is event E_2 . Calculate the coordinates x' and t' of both events in S' , and state which gate fires first in S' and by how much.

(c) EXPLAIN

[3]

In S' the shuttle is 150 m long and the gantry is only 72.0 m long. Explain, using your results from (b), why neither pulse strikes the shuttle in S' either.

(d) DETERMINE

[2]

A drone is launched forwards from the shuttle's nose at $0.600c$ relative to the shuttle. By composing rapidities, determine the drone's speed in S .

A secondary beam of positive kaons, each with momentum $2.00 \text{ GeV}/c$, travels along an evacuated decay tunnel. A K^+ at rest has mean lifetime $\tau = 1.238 \times 10^{-8} \text{ s}$ and decays mainly by $K^+ \rightarrow \mu^+ + \nu_\mu$. Take $m_K c^2 = 493.7 \text{ MeV}$, $m_\mu c^2 = 105.7 \text{ MeV}$, $m_p c^2 = 938.3 \text{ MeV}$, $m_\Lambda c^2 = 1115.7 \text{ MeV}$ and $c = 3.00 \times 10^8 \text{ m s}^{-1}$, and treat the neutrino as massless.

(a) CALCULATE

[2]

Calculate the mean decay length of the kaons in the laboratory.

(b) DERIVE

[3]

Starting from conservation of four-momentum, derive expressions for the muon's energy E^* and momentum p^* in the kaon rest frame, and evaluate both.

(c) DETERMINE

[2]

Determine the largest energy a muon from this beam can have in the laboratory, and state the direction of emission in the kaon rest frame that gives it.

(d) CALCULATE

[3]

The kaons are produced when a proton beam strikes protons at rest, through $p + p \rightarrow p + \Lambda + K^+$. Calculate the minimum kinetic energy of a beam proton for this reaction.

Technetium-99m, used in medical gamma imaging, emits photons of energy 140.5 keV. Before reaching the camera, many of these photons Compton scatter in soft tissue from electrons that may be treated as free and at rest. The camera rejects scattered photons only by their energy: it accepts any photon whose energy is at least 90.0% of 140.5 keV. Take $m_e c^2 = 511$ keV, $h = 6.63 \times 10^{-34}$ J s, $c = 3.00 \times 10^8$ m s⁻¹ and $e = 1.60 \times 10^{-19}$ C.

(a) DERIVE

[4]

A photon of wavelength λ scatters through angle θ from a free electron at rest. By conserving relativistic energy and both components of momentum, derive $\lambda' - \lambda = (h/(m_e c))(1 - \cos \theta)$ for the scattered wavelength λ' .

(b) CALCULATE

[3]

Calculate the largest angle through which a photon can be scattered once and still be accepted by the camera.

(c) CALCULATE

[2]

Calculate the de Broglie wavelength of the recoil electron for the scattering in (b).

(d) COMPARE

[2]

A PET camera detects 511 keV annihilation photons and uses the same fractional window, accepting photons of at least 90.0% of 511 keV. Compare its largest accepted single-scattering angle with your answer to (b), and explain the difference.

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.067m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, between $x = 0$ and $x = L$. The energy eigenfunctions are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ the electron is prepared in the state $\Psi(x, 0) = A[\sqrt{3}\psi_1(x) + \psi_2(x)]$, with A real and positive. You may use $\int_0^L x \sin(\pi x/L) \sin(2\pi x/L) dx = -8L^2/(9\pi^2)$. Take $m_e = 9.11 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $h = 6.63 \times 10^{-34}$ J s.

(a) **DETERMINE** [2]

Determine A , and the probabilities that a measurement of the energy at $t = 0$ returns E_1 and returns E_2 .

(b) **CALCULATE** [2]

The mean position $\langle x \rangle$ of the electron oscillates. Calculate the period T of this oscillation.

(c) **SHOW THAT** [3]

Show that $\langle x \rangle(t) = L/2 - (8\sqrt{3}L/(9\pi^2)) \cos \omega t$, where $\omega = (E_2 - E_1)/\hbar$.

(d) **SKETCH** [2]

On one set of axes, sketch $|\Psi(x, t)|^2$ for $0 \leq x \leq L$ at $t = 0$ and at $t = T/2$.

(e) **PREDICT** [2]

At $t = T/2$ the energy of the electron is measured and the result is E_2 . State the probability of this result, and predict $\langle x \rangle$ at all later times.

A vacuum photocell has a cathode whose surface work function is quoted by its manufacturer as 2.00 eV, and an anode made of a different metal. Monochromatic light of each wavelength in the table falls on the cathode. An external supply applies a retarding potential difference between anode and cathode, which is increased until the photocurrent just falls to zero; that reading is the stopping potential V_s . Each V_s reading has an uncertainty of ± 0.02 V, and the wavelengths may be treated as exact. The frequencies were calculated with $c = 3.00 \times 10^8$ m s⁻¹. Take $e = 1.60 \times 10^{-19}$ C.

Stopping potential against wavelength and frequency for the photocell

Setting	Wavelength λ / nm	Frequency f / 10^{14} Hz	Stopping potential V_s / V
1	300	10.00	1.83
2	340	8.82	1.36
3	380	7.89	0.95
4	420	7.14	0.66
5	460	6.52	0.39
6	500	6.00	0.19

(a) STATE

[2]

State the relation between V_s and f predicted by Einstein's photon model, and what the gradient and the vertical intercept of a graph of V_s against f represent.

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(b) DETERMINE

[4]

Using a straight-line fit of V_s against f , determine the Planck constant h and the work function ϕ indicated by the data.

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(c) ESTIMATE

[2]

Assuming the ± 0.02 V reading uncertainty is the only significant source of error, estimate the uncertainty in your value of h .

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(d) JUSTIFY

[2]

A student concludes that the cathode's true work function is the value found in (b), and that the manufacturer's 2.00 eV is wrong. Justify the alternative conclusion that the intercept measures the work function of the anode, not the cathode.

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A particle of mass m moves in the potential $V(x) = \frac{1}{2}m\omega^2x^2$, so $\hat{H} = \hat{p}^2/(2m) + \frac{1}{2}m\omega^2\hat{x}^2$, with $[\hat{x}, \hat{p}] = i\hbar$. Define $\hat{a} = \sqrt{(m\omega/(2\hbar))}(\hat{x} + i\hat{p}/(m\omega))$ and $\hat{a}^\dagger = \sqrt{(m\omega/(2\hbar))}(\hat{x} - i\hat{p}/(m\omega))$. In parts (d) and (e) the stretching vibration of a nitric oxide (NO) molecule is modelled as such an oscillator, with reduced mass $\mu = 1.24 \times 10^{-26}$ kg, vibrational frequency $f = 5.64 \times 10^{13}$ Hz and equilibrium bond length 115 pm. Take $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.63 \times 10^{-34}$ J s and $k_B = 1.38 \times 10^{-23}$ J K⁻¹. You may use $\int \exp(-\alpha x^2) dx = \sqrt{(\pi/\alpha)}$ and $\int x^2 \exp(-\alpha x^2) dx = \sqrt{\pi}/(2\alpha^{3/2})$, both taken over all x .

(a) SHOW THAT

[3]

Show that $[\hat{a}, \hat{a}^\dagger] = 1$ and that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

(b) DERIVE

[3]

Using only the two results of (a), derive the energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \dots$

(c) DERIVE

[3]

Derive the normalised ground-state wavefunction $\psi_0(x)$ from the condition $\hat{a}|0\rangle = 0$.

(d) CALCULATE

[3]

For the NO ground state, calculate σ_x and σ_p , and compare their product with the lower bound set by the commutator $[\hat{x}, \hat{p}]$.

(e) EVALUATE

[2]

Evaluate whether the harmonic model can be trusted for (i) the fraction of NO molecules in the first excited vibrational level at 300 K, and (ii) the energies of vibrational levels close to dissociation.

University Physics IV · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 CORE COURSE	MULTIPLE CHOICE · ROUTINE · UPIV-01.03 · UPIV-01.04	[1]
	Cavity radiation at temperature 300 K is examined at wavelength 1.00 mm. By what factor does the Rayleigh-Jeans spectral energy density exceed the Planck spectral energy density at this wavelength? Take $h = 6.63 \times 10^{-34} \text{ J s}$, $c = 3.00 \times 10^8 \text{ m s}^{-1}$ and $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$.	
	KEY B 1.02	
	<i>The number that decides the regime is $x = hv/k_BT = hc/(\lambda k_BT) = 0.0480$. Both laws use the same mode density, $8\pi\nu^2/c^3$; they differ only in the mean energy per mode, k_BT against $hv/(e^x - 1)$, so the ratio is $(e^x - 1)/x = 1.024$. A is tempting because $x \ll 1$ is the classical regime, but the Rayleigh-Jeans law is the limit Planck's law approaches, not one it reaches: the leading correction is $x/2$, about 2.4%. C is $1 + x = 1.048$, the result of expanding $e^x - 1$ as $x + x^2$ instead of $x + x^2/2$. D is $1/x = 20.8$, which compares k_BT with a single quantum $h\nu$, as if every mode held exactly one quantum rather than a thermal average.</i>	U3
2 CORE COURSE	MULTIPLE CHOICE · ROUTINE · UPIV-03.06	[1]
	A proton moves at a speed for which $\gamma = 2.00$. A force of magnitude $F_{\parallel}$ applied parallel to its velocity gives it an acceleration of magnitude a . A force of magnitude $F_{\perp}$ applied perpendicular to its velocity gives it an acceleration of the same magnitude a . What is $F_{\parallel}/F_{\perp}$ at that instant?	
	KEY C 4.00	
	<i>Differentiating $p = \gamma mv$, a force along the motion changes the speed and therefore γ as well, giving $F_{\parallel} = \gamma^3 ma$; a force across the motion changes only the direction, so γ stays fixed and $F_{\perp} = \gamma ma$. The ratio is $\gamma^2 = 4.00$. B is the Newtonian answer, and also the answer of anyone who treats γm as a single relativistic mass with the same inertia in every direction, which is exactly why that idea was abandoned. D is γ^3, from using the parallel result correctly but forgetting that the perpendicular force also carries a factor of γ. A inverts the ratio: it is chosen by a candidate who remembers that the two directions differ by γ^2 but assumes that deflecting a fast particle is the harder job.</i>	U2
3 CORE COURSE	MULTIPLE CHOICE · ROUTINE · UPIV-04.07	[1]
	A wave packet for a free, non-relativistic electron is built from de Broglie waves with angular frequency $\omega(k) = E/\hbar$, where $E = \hbar^2 k^2/(2m) + V_0$ and V_0 is a constant potential energy whose value depends on where the zero of energy is chosen. The packet is centred on wavenumber k_0 , and $v = \hbar k_0/m$ is the electron's classical speed. Which statement is correct?	
	KEY D The group velocity is v for every choice of V_0, but the phase velocity depends on V_0.	
	<i>The group velocity is $d\omega/dk = \hbar k_0/m = v$, and the constant $V_0/\hbar$ vanishes on differentiation. The phase velocity is $\omega/k = \hbar k_0/(2m) + V_0/(\hbar k_0)$, which depends on the arbitrary energy zero, so it cannot be a physical property of the electron; only the group velocity, the speed of the packet, can be measured. A is the tempting half-truth: $v/2$ is the phase velocity only in the convention $V_0 = 0$, and the words 'whatever value of V_0' make it false. B swaps the two, usually because the candidate assumes the wave crests must travel with the particle. C is tempting because both velocities are built from the same $\omega(k)$, so changing ω seems bound to change both; but moving the zero of energy shifts ω by a constant, which changes ω/k and leaves $d\omega/dk$ untouched.</i>	U1

4

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPIV-05.01 · UPIV-05.04

[1]

Two identical narrow wave packets, ψ_L centred at $x = -d$ and ψ_R centred at $x = +d$, do not overlap anywhere. Many copies of a particle are prepared either (I) in the superposition $(\psi_L + \psi_R)/\sqrt{2}$, or (II) as a statistical mixture in which half the copies are in ψ_L and half are in ψ_R . Which measurement, repeated on many copies, distinguishes preparation I from preparation II?

KEY A Momentum: for I the momentum distribution is modulated by fringes of spacing $h/(2d)$, and for II it is not.

In momentum space the displaced packets carry phase factors $e^{\pm ipd/\hbar}$, so the superposition has amplitude $\sqrt{2} \phi_0(p) \cos(pd/\hbar)$ and a distribution proportional to $\cos^2(pd/\hbar)$, with fringes of period $\pi\hbar/d = h/(2d)$; the mixture adds the two separate distributions, which are identical and carry no fringes. B is the double-slit picture misapplied: the interference term in position is $\psi_L^* \psi_R$, which is zero everywhere when the packets do not overlap, so the two position distributions are identical. C is tempting because a mixture sounds less even-handed than a superposition, but both preparations put probability $1/2$ near each centre, so both give $\langle x \rangle = 0$. D mistakes equal position probabilities for equal states: amplitudes add before they are squared, and the relative phase that a mixture lacks shows up as soon as the measurement is in a representation where the two amplitudes overlap.

U1

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-06.08

[1]

Ehrenfest's theorem gives $d\langle x \rangle/dt = \langle p \rangle/m$ and $d\langle p \rangle/dt = -\langle \partial V/\partial x \rangle$. For which potential does $\langle x \rangle$ obey exactly the classical equation $m d^2\langle x \rangle/dt^2 = -V'(\langle x \rangle)$ for every state of the particle?

KEY D $V(x) = \frac{1}{2}m\omega^2 x^2 + Fx$, with F a constant.

The mean follows Newton's law only if $\langle V'(x) \rangle = V'(\langle x \rangle)$, and that holds for every state only when V' is linear in x , which means V is at most quadratic. D qualifies because $V' = m\omega^2 x + F$. A fails because $V' = 4ax^3$ and $\langle x^3 \rangle$ is not in general equal to $\langle x \rangle^3$: for a packet of spread σ distributed symmetrically about its mean, $\langle x^3 \rangle = \langle x \rangle^3 + 3\langle x \rangle \sigma^2$, so a displaced packet does not follow the classical equation. The symmetry of the well is what makes A tempting, and it is irrelevant. B describes an approximation, not an identity: it holds only while the packet stays narrow compared with the distance over which V' changes, and a packet that starts narrow spreads. C fails for the same reason as A, since the cosine is non-linear, and a bounded force does nothing to make $\langle V'(x) \rangle$ equal $V'(\langle x \rangle)$.

U1

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPIV-07.07

[1]

Electrons of kinetic energy 8.00 eV travel in the $+x$ direction towards a potential step at which the potential energy rises abruptly from 0 to 6.00 eV. What fraction of the incident probability current is reflected?

KEY B 0.111

Above the step the wavenumber is $k_2 = k_1 \sqrt{(E - V_0)/E} = k_1 \sqrt{2.00/8.00} = k_1/2$. Matching ψ and ψ' at the step gives a reflected amplitude $(k_1 - k_2)/(k_1 + k_2) = 1/3$, and since current goes as the modulus squared, $R = 1/9 = 0.111$ and $T = 0.889$. A is the classical expectation, because $E > V_0$ means every particle could pass; the continuity of ψ' at an abrupt step forces a reflected wave anyway. C is the reflected amplitude left unsquared. D is $((E - (E - V_0))/(E + (E - V_0)))^2 = (6/10)^2$, the result of putting kinetic energies where the wavenumbers belong, when each wavenumber goes as the square root of its kinetic energy.

U3

For a particle of mass m in a harmonic oscillator of angular frequency ω , $\hat{x} = \sqrt{\hbar/(2m\omega)} (\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. What is $\langle x^2 \rangle$ in the energy eigenstate $|n\rangle$?

KEY **A** $(n + \frac{1}{2})\hbar/(m\omega)$

Expanding, $\hat{x}^2 = (\hbar/(2m\omega))(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$. In $|n\rangle$ the first two terms have zero diagonal elements, and $\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}$ gives $(n+1) + n = 2n+1$, so $\langle x^2 \rangle = (n + \frac{1}{2})\hbar/(m\omega)$. As a check, $\frac{1}{2}m\omega^2\langle x^2 \rangle = \frac{1}{2}E_n$, the virial split. C drops the $\frac{1}{2}$ from the prefactor, which would make the potential energy alone equal to the whole of E_n . B treats $\hat{a}\hat{a}^\dagger$ as if it were $\hat{a}^\dagger\hat{a}$, forgetting $[\hat{a}, \hat{a}^\dagger] = 1$; it would give zero spread in the ground state, which the uncertainty principle forbids. D confuses $\langle x \rangle$, which is zero by parity, with $\langle x^2 \rangle$, which is the spread and is never zero.

An inspection gantry at rest in frame S has two laser gates: G_1 at $x = -60.0$ m and G_2 at $x = +60.0$ m, so the gantry's rest length is 120 m. A cargo shuttle of rest length 150 m moves in the $+x$ direction at $0.800c$. Frame S' is the shuttle's rest frame, in standard configuration with S : the origins coincide at $t = t' = 0$, when the centre of the shuttle passes $x = 0$. Each gate sits at the edge of the track and can fire a brief pulse straight across it, taking negligible time to cross; a pulse that meets the shuttle leaves a mark on its hull. The gantry controller fires both gates at $t = 0$ in S . Take $c = 3.00 \times 10^8$ m s^{-1} .

(a) CALCULATE · U3

[2]

Calculate the length of the shuttle measured in S , and the length of the gantry measured in S' .

- $\gamma = 1/\sqrt{1 - 0.800^2} = 5/3$, so the shuttle's length in S is $150 \text{ m}/\gamma = 90.0$ m; at $t = 0$ it spans $x = -45.0$ m to $+45.0$ m, clear of both gates
- The gantry's length in S' is $120 \text{ m}/\gamma = 72.0$ m

Each frame measures the other's object as the shorter one. A candidate who divides in one case and multiplies in the other has not asked which frame owns each proper length.

(b) CALCULATE · U3

[3]

The firing of G_1 is event E_1 and the firing of G_2 is event E_2 . Calculate the coordinates x' and t' of both events in S' , and state which gate fires first in S' and by how much.

- Uses $x' = \gamma(x - vt)$ and $t' = \gamma(t - vx/c^2)$ with $t = 0$ for both events
- E_1 : $x' = -100$ m, $t' = +\gamma(0.800)(60.0 \text{ m})/c = +2.67 \times 10^{-7}$ s; E_2 : $x' = +100$ m, $t' = -2.67 \times 10^{-7}$ s
- G_2 , the gate the shuttle is heading towards, fires first in S' , 5.33×10^{-7} s before G_1

The sign of vx/c^2 is the whole question. A candidate who drops it, or who applies the time-dilation formula to two events that are not at the same place in either frame, gets the order wrong.

(c) EXPLAIN · U1

[3]

In S' the shuttle is 150 m long and the gantry is only 72.0 m long. Explain, using your results from (b), why neither pulse strikes the shuttle in S' either.

- In S' the shuttle occupies $-75.0 \text{ m} \leq x' \leq +75.0 \text{ m}$ at all times; E_2 happens at $x' = +100$ m, 25.0 m ahead of the nose, and E_1 at $x' = -100$ m, 25.0 m behind the tail, so neither event is on the hull
- Between the two firings the gantry moves $(0.800c)(5.33 \times 10^{-7} \text{ s}) = 128$ m in the $-x'$ direction, so G_2 fires before it reaches the nose and G_1 fires after it has passed the tail
- Whether a pulse meets the hull is a coincidence of events at one place, on which every frame agrees; the statement that the shuttle is between the gates when they fire depends on E_1 and E_2 being simultaneous, which is true in S and false in S' , so there is no contradiction

Answers that stop at saying that length contraction is relative earn nothing here. The marks are for placing both events relative to the hull in S' , for following the gantry's motion between the two firings, and for naming simultaneity as the one statement the frames do not share.

(d) DETERMINE · U2

[2]

A drone is launched forwards from the shuttle's nose at $0.600c$ relative to the shuttle. By composing rapidities, determine the drone's speed in S .

- Rapidities $\phi = \text{artanh } \beta$ add along a line: $\text{artanh } 0.800 = \ln 3 = 1.099$ and $\text{artanh } 0.600 = \ln 2 = 0.693$, so $\phi = \ln 6 = 1.792$
- $\beta = \tanh(1.792) = 35/37 = 0.946$, so the speed is $0.946c = 2.84 \times 10^8$ m s^{-1} , in agreement with $(0.800 + 0.600)/(1 + 0.480)$ and below c

Rapidity is the parameter that adds; velocity is not. A Galilean $1.40c$, or $\tanh(1.40) = 0.885$ from treating β itself as the rapidity, both show the difference has not been understood.

A secondary beam of positive kaons, each with momentum $2.00 \text{ GeV}/c$, travels along an evacuated decay tunnel. A K^+ at rest has mean lifetime $\tau = 1.238 \times 10^{-8} \text{ s}$ and decays mainly by $K^+ \rightarrow \mu^+ + \nu_\mu$. Take $m_K c^2 = 493.7 \text{ MeV}$, $m_\mu c^2 = 105.7 \text{ MeV}$, $m_p c^2 = 938.3 \text{ MeV}$, $m_\Lambda c^2 = 1115.7 \text{ MeV}$ and $c = 3.00 \times 10^8 \text{ m s}^{-1}$, and treat the neutrino as massless.

(a) CALCULATE · U3

[2]

Calculate the mean decay length of the kaons in the laboratory.

- $\beta\gamma = pc/(m_K c^2) = 2000 \text{ MeV}/493.7 \text{ MeV} = 4.05$ ($\gamma = 4.17$, $\beta = 0.971$)
- Mean decay length $= \beta\gamma c\tau = 4.05 \times (3.00 \times 10^8 \text{ m s}^{-1})(1.238 \times 10^{-8} \text{ s}) = 15.0 \text{ m}$
 τ is the proper lifetime, owned by the kaon's rest frame. $\beta c\tau = 3.61 \text{ m}$ ignores time dilation, and $\gamma c\tau = 15.5 \text{ m}$ uses c where the kaon's speed belongs.

(b) DERIVE · U2

[3]

Starting from conservation of four-momentum, derive expressions for the muon's energy E^* and momentum p^* in the kaon rest frame, and evaluate both.

- Writes $P_K = p_\mu + p_\nu$ and isolates the neutrino: $p_\nu^2 = (P_K - p_\mu)^2 = 0$ because the neutrino is massless
- Expands in the kaon rest frame: $m_K^2 c^4 + m_\mu^2 c^4 - 2m_K c^2 E^* = 0$, so $E^* = (m_K^2 + m_\mu^2)c^2/(2m_K) = 258.2 \text{ MeV}$
- $p^*c = \sqrt{(E^{*2} - m_\mu^2 c^4)} = (m_K^2 - m_\mu^2)c^2/(2m_K) = 235.5 \text{ MeV}$, so $p^* = 235.5 \text{ MeV}/c$, equal to the neutrino's energy divided by c

Squaring the four-vector that contains the unknown you do not want removes it in one line. Candidates who conserve energy and momentum separately usually lose the algebra, or assume the muon is non-relativistic when its kinetic energy is 152.5 MeV .

(c) DETERMINE · U3

[2]

Determine the largest energy a muon from this beam can have in the laboratory, and state the direction of emission in the kaon rest frame that gives it.

- Boosting, $E = \gamma(E^* + \beta p^*c \cos \theta^*)$, which is greatest for emission along the beam direction, $\theta^* = 0$
- $E_{\text{max}} = 4.173 \times (258.2 \text{ MeV} + 0.9709 \times 235.5 \text{ MeV}) = 2031 \text{ MeV} \approx 2.03 \text{ GeV}$
 (accept 2.02 to 2.04 GeV), just below the kaon's energy of 2060 MeV

Adding the rest-frame energy to the kaon's energy, or multiplying E^* by γ alone, ignores the momentum term in the boost. A result above 2060 MeV should have been caught as impossible.

(d) CALCULATE · U3

[3]

The kaons are produced when a proton beam strikes protons at rest, through $p + p \rightarrow p + \Lambda + K^+$. Calculate the minimum kinetic energy of a beam proton for this reaction.

- At threshold the products are at rest in the centre-of-momentum frame, so the invariant $(E_{\text{beam}} + m_p c^2)^2 - (p_{\text{beam}} c)^2$ equals $(m_p + m_\Lambda + m_K)^2 c^4 = (2547.7 \text{ MeV})^2$
- The invariant expands to $2m_p^2 c^4 + 2E_{\text{beam}} m_p c^2$, so $E_{\text{beam}} = [(2547.7)^2 - 2(938.3)^2]/(2 \times 938.3) \text{ MeV} = 2520 \text{ MeV}$
- $K_{\text{th}} = E_{\text{beam}} - m_p c^2 = 1582 \text{ MeV} \approx 1.58 \text{ GeV}$, far more than the 671 MeV of rest energy created, because momentum conservation requires the products to move in the laboratory

The commonest wrong answer is 671 MeV , the change in rest energy, which would be correct only if the target were infinitely massive. The invariant reaches the threshold without first having to work out how fast the products move in the laboratory.

Technetium-99m, used in medical gamma imaging, emits photons of energy 140.5 keV. Before reaching the camera, many of these photons Compton scatter in soft tissue from electrons that may be treated as free and at rest. The camera rejects scattered photons only by their energy: it accepts any photon whose energy is at least 90.0% of 140.5 keV. Take $m_e c^2 = 511$ keV, $h = 6.63 \times 10^{-34}$ J s, $c = 3.00 \times 10^8$ m s⁻¹ and $e = 1.60 \times 10^{-19}$ C.

(a) DERIVE · U2

[4]

A photon of wavelength λ scatters through angle θ from a free electron at rest. By conserving relativistic energy and both components of momentum, derive $\lambda' - \lambda = (h/(m_e c))(1 - \cos \theta)$ for the scattered wavelength λ' .

- 1 Energy: $pc + m_e c^2 = p'c + E_e$, with $p = h/\lambda$ and $p' = h/\lambda'$ for the incident and scattered photons
- 2 Momentum as a vector: $p_e = p - p'$, so $p_e^2 = p^2 + p'^2 - 2pp' \cos \theta$, which uses both components
- 3 Eliminates the electron with $E_e^2 = p_e^2 c^2 + m_e^2 c^4$: $(pc - p'c + m_e c^2)^2 = (p^2 + p'^2 - 2pp' \cos \theta)c^2 + m_e^2 c^4$, which reduces to $m_e c(p - p') = pp'(1 - \cos \theta)$
- 4 Divides by $m_e c pp'$ to get $1/p' - 1/p = (1 - \cos \theta)/(m_e c)$, and substitutes $1/p = \lambda/h$ to reach $\lambda' - \lambda = (h/(m_e c))(1 - \cos \theta)$

Using the non-relativistic electron kinetic energy $p_e^2/(2m_e)$ gives an approximate result, not this one, and conserving only the momentum component along the beam loses the $\cos \theta$ term. The second line removes the electron's unknown recoil angle by squaring the momentum vector; the mark for the third line is for eliminating the electron's energy and momentum through $E_e^2 = p_e^2 c^2 + m_e^2 c^4$ and reaching the reduced form, not for asserting the result.

(b) CALCULATE · U3

[3]

Calculate the largest angle through which a photon can be scattered once and still be accepted by the camera.

- 1 In energy form $1/E' - 1/E = (1 - \cos \theta)/(m_e c^2)$, with $E' = 0.900 \times 140.5$ keV = 126.45 keV
- 2 $1 - \cos \theta = 511$ keV $\times (1/126.45$ keV $- 1/140.5$ keV) = 0.404
- 3 $\cos \theta = 0.596$, so $\theta_{\max} = 53.4^\circ$

Subtracting energies as if the shift were linear in energy gives the wrong angle. The relation is linear in wavelength, equivalently in $1/E$.

(c) CALCULATE · U3

[2]

Calculate the de Broglie wavelength of the recoil electron for the scattering in (b).

- 1 $K = 140.5$ keV $- 126.45$ keV = 14.05 keV, and $pc = \sqrt{(K^2 + 2K m_e c^2)} = 120.7$ keV
- 2 $\lambda_e = hc/(pc) = (6.63 \times 10^{-34}$ J s $\times 3.00 \times 10^8$ m s⁻¹)/(120.7 $\times 10^3 \times 1.60 \times 10^{-19}$ J) = 1.03×10^{-11} m = 10.3 pm

Writing $\lambda = hc/K$ treats the electron as a photon and gives 88 pm. The non-relativistic momentum $\sqrt{2m_e K}$ gives 10.4 pm, under 1% away, which is itself worth noticing at 14 keV.

(d) COMPARE · U1

[2]

A PET camera detects 511 keV annihilation photons and uses the same fractional window, accepting photons of at least 90.0% of 511 keV. Compare its largest accepted single-scattering angle with your answer to (b), and explain the difference.

- 1 For 511 keV, $1 - \cos \theta = (511$ keV/511 keV)(1/0.900 $- 1)$ = 0.111, giving $\theta_{\max} = 27.3^\circ$, about half the 53.4° accepted at 140.5 keV
- 2 In general $1 - \cos \theta_{\max} = (m_e c^2/E)(1/0.900 - 1)$: at a given angle a photon with E well below $m_e c^2$ loses a smaller fraction of its energy, so it must scatter through a larger angle before falling below 90.0%, and the same fractional window accepts a much wider cone at 140.5 keV than at 511 keV

Candidates often expect the same fractional window to accept the same cone of scattering angles at every photon energy. The comparison turns on the ratio $E/m_e c^2$, which sets the fraction of energy lost at a given angle.

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.067m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, between $x = 0$ and $x = L$. The energy eigenfunctions are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m^*L^2)$. At $t = 0$ the electron is prepared in the state $\Psi(x, 0) = A[\sqrt{3}\psi_1(x) + \psi_2(x)]$, with A real and positive. You may use $\int_0^L x \sin(\pi x/L) \sin(2\pi x/L) dx = -8L^2/(9\pi^2)$. Take $m_e = 9.11 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $h = 6.63 \times 10^{-34}$ J s.

(a) DETERMINE · U2

[2]

Determine A , and the probabilities that a measurement of the energy at $t = 0$ returns E_1 and returns E_2 .

1 Orthonormality of ψ_1 and ψ_2 gives $|A|^2(3 + 1) = 1$, so $A = 1/2$

2 $P(E_1) = |A\sqrt{3}|^2 = 3/4$ and $P(E_2) = |A|^2 = 1/4$

Normalising by integrating the square of the sum term by term, without using orthogonality, wastes time; forgetting that the cross terms vanish gives the wrong A .

(b) CALCULATE · U3

[2]

The mean position $\langle x \rangle$ of the electron oscillates. Calculate the period T of this oscillation.

1 $E_1 = \pi^2\hbar^2/(2m^*L^2) = 6.25 \times 10^{-21}$ J (39.1 meV), so $E_2 - E_1 = 3E_1 = 1.87 \times 10^{-20}$ J

2 $T = 2\pi\hbar/(E_2 - E_1) = h/(E_2 - E_1) = 3.54 \times 10^{-14}$ s

The beat is set by the energy difference, not by E_1 or E_2 alone. Using the free-electron mass instead of m^* gives a period 15 times too long.

(c) SHOW THAT · U2

[3]

Show that $\langle x \rangle(t) = L/2 - (8\sqrt{3}L/(9\pi^2)) \cos \omega t$, where $\omega = (E_2 - E_1)/\hbar$.

1 $\Psi(x, t) = \frac{1}{2}[\sqrt{3}\psi_1 e^{(-iE_1t/\hbar)} + \psi_2 e^{(-iE_2t/\hbar)}]$, with each stationary state carrying its own phase

2 $\langle x \rangle = \frac{3}{4}\langle \psi_1 | x | \psi_1 \rangle + \frac{1}{4}\langle \psi_2 | x | \psi_2 \rangle + (\sqrt{3}/2)\langle \psi_1 | x | \psi_2 \rangle \cos \omega t$, and both diagonal terms equal $L/2$ because each $|\psi_n|^2$ is symmetric about the centre of the well

3 $\langle \psi_1 | x | \psi_2 \rangle = (2/L)(-8L^2/(9\pi^2)) = -16L/(9\pi^2)$, so $\langle x \rangle = L/2 - (8\sqrt{3}L/(9\pi^2)) \cos \omega t$, an oscillation of amplitude $0.156L = 1.87$ nm

Candidates who give both components the same phase find $\langle x \rangle$ constant and then force the stated answer. Any response that does not show the two cross terms combining into a cosine earns at most the first mark.

(d) SKETCH · U1

[2]

On one set of axes, sketch $|\Psi(x, t)|^2$ for $0 \leq x \leq L$ at $t = 0$ and at $t = T/2$.

1 $t = 0$: zero at both walls, with a single main peak left of centre near $x \approx 0.32L$, falling to zero at $x = 5L/6$ and staying very small between there and $x = L$

2 $t = T/2$: the mirror image of the $t = 0$ curve about $x = L/2$, with its main peak near $x \approx 0.68L$ and a zero at $x = L/6$; the two curves enclose equal areas

The $t = 0$ density must lean towards the side where ψ_1 and ψ_2 have the same sign, which is consistent with $\langle x \rangle = 0.344L$ from (c). A symmetric sketch at either time shows that the relative phase has been ignored.

(e) PREDICT · U1

[2]

At $t = T/2$ the energy of the electron is measured and the result is E_2 . State the probability of this result, and predict $\langle x \rangle$ at all later times.

1 The probability is $1/4$, the same at every time, since the modulus of each coefficient $c_n e^{(-iE_nt/\hbar)}$ does not change

2 The state is updated to ψ_2 (up to a phase), which is stationary, so there is no cross term and $\langle x \rangle = L/2$ at all later times: the oscillation stops

Many candidates expect the probability to oscillate along with $\langle x \rangle$. Only quantities with a cross term between different energies can change in time.

A vacuum photocell has a cathode whose surface work function is quoted by its manufacturer as 2.00 eV, and an anode made of a different metal. Monochromatic light of each wavelength in the table falls on the cathode. An external supply applies a retarding potential difference between anode and cathode, which is increased until the photocurrent just falls to zero; that reading is the stopping potential V_s . Each V_s reading has an uncertainty of ± 0.02 V, and the wavelengths may be treated as exact. The frequencies were calculated with $c = 3.00 \times 10^8 \text{ m s}^{-1}$. Take $e = 1.60 \times 10^{-19} \text{ C}$.

(a) STATE · U1

[2]

State the relation between V_s and f predicted by Einstein's photon model, and what the gradient and the vertical intercept of a graph of V_s against f represent.

1 $eV_s = hf - \phi$, so $V_s = (h/e)f - \phi/e$

2 The gradient is h/e , and the intercept on the V_s axis is $-\phi/e$ (equivalently, the intercept on the f axis is the threshold frequency ϕ/h)

Writing the maximum kinetic energy as $\frac{1}{2}mv^2$ and never connecting it to eV_s leaves no straight line to fit.

(b) DETERMINE · U4

[4]

Using a straight-line fit of V_s against f , determine the Planck constant h and the work function ϕ indicated by the data.

1 Plots or fits V_s against the tabulated f ; all six points lie on one straight line within ± 0.02 V

2 Gradient = $4.12 \times 10^{-15} \text{ V s}$ (accept 4.05×10^{-15} to $4.20 \times 10^{-15} \text{ V s}$)

3 $h = e \times \text{gradient} = 6.6 \times 10^{-34} \text{ J s}$ (accept 6.5×10^{-34} to $6.7 \times 10^{-34} \text{ J s}$)

4 Intercept on the V_s axis = -2.29 V (accept -2.19 V to -2.35 V), so $\phi = 2.29 \text{ eV}$; or threshold frequency $5.55 \times 10^{14} \text{ Hz}$ with $\phi = hf_0 = 2.29 \text{ eV}$

Reading the intercept requires extrapolating well beyond the data, since the lowest frequency is $6.00 \times 10^{14} \text{ Hz}$. A candidate who plots V_s against λ has no straight line and so no gradient to convert into h ; one who fits correctly but reads the work function off the last data point instead of extrapolating loses the final mark.

(c) ESTIMATE · U4

[2]

Assuming the ± 0.02 V reading uncertainty is the only significant source of error, estimate the uncertainty in your value of h .

1 Steepest and shallowest lines through the ends of the ± 0.02 V error bars at 300 nm and 500 nm have gradients $4.20 \times 10^{-15} \text{ V s}$ and $4.00 \times 10^{-15} \text{ V s}$

2 $\Delta h \approx \frac{1}{2}(6.72 - 6.40) \times 10^{-34} \text{ J s} = 0.16 \times 10^{-34} \text{ J s}$, so $h = (6.6 \pm 0.2) \times 10^{-34} \text{ J s}$, a 2 to 3% uncertainty (a justified least-squares standard error is also accepted: about $0.10 \times 10^{-34} \text{ J s}$ from $\sigma = 0.02 \text{ V}$ on each reading, or about $0.05 \times 10^{-34} \text{ J s}$ from the scatter of the residuals)

An uncertainty quoted to more decimal places than the value, or an uncertainty in h of $0.02 \times 10^{-34} \text{ J s}$ obtained by copying the $\pm 0.02 \text{ V}$, shows that the uncertainty has not been carried through the gradient.

(d) JUSTIFY · U4

[2]

A student concludes that the cathode's true work function is the value found in (b), and that the manufacturer's 2.00 eV is wrong. Justify the alternative conclusion that the intercept measures the work function of the anode, not the cathode.

- 1 An electron leaves the cathode with at most $hf - \phi_C$, but between two different metals there is also a contact potential difference $(\phi_A - \phi_C)/e$, so the external stopping potential satisfies $eV_s = (hf - \phi_C) - (\phi_A - \phi_C) = hf - \phi_A$
- 2 The intercept range 2.19 to 2.35 eV excludes 2.00 eV, so the 0.3 eV difference is significant and is accounted for as the contact potential; this does not depend on f , so the gradient and h are unaffected and the manufacturer's value can stand

Blaming the discrepancy on random error ignores the steepest and shallowest lines of (c), which put the intercept's own uncertainty at only about ± 0.08 V, far smaller than the 0.3 V gap. The point being tested is the difference between what the voltmeter reads and what the cathode does.

13
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION, THEN TEST THE MODEL ON A MOLECULE · DISCRIMINATING · UPIV-05.08 · UPIV-08.04 · UPIV-08.05 · UPIV-08.06 · UPIV-08.07 · UPIV-08.09

[14]

A particle of mass m moves in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$, so $\hat{H} = \hat{p}^2/(2m) + \frac{1}{2}m\omega^2 \hat{x}^2$, with $[\hat{x}, \hat{p}] = i\hbar$. Define $\hat{a} = \sqrt{(m\omega/(2\hbar))}(\hat{x} + i\hat{p}/(m\omega))$ and $\hat{a}^\dagger = \sqrt{(m\omega/(2\hbar))}(\hat{x} - i\hat{p}/(m\omega))$. In parts (d) and (e) the stretching vibration of a nitric oxide (NO) molecule is modelled as such an oscillator, with reduced mass $\mu = 1.24 \times 10^{-26}$ kg, vibrational frequency $f = 5.64 \times 10^{13}$ Hz and equilibrium bond length 115 pm. Take $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.63 \times 10^{-34}$ J s and $k_B = 1.38 \times 10^{-23}$ J K⁻¹. You may use $\int \exp(-\alpha x^2) dx = \sqrt{(\pi/\alpha)}$ and $\int x^2 \exp(-\alpha x^2) dx = \sqrt{\pi}/(2\alpha^{3/2})$, both taken over all x .

(a) SHOW THAT · U2

[3]

Show that $[\hat{a}, \hat{a}^\dagger] = 1$ and that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

- 1 $\hat{a}\hat{a}^\dagger = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - (i/(2\hbar))[\hat{x}, \hat{p}] = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] + \frac{1}{2}$
- 2 Similarly $\hat{a}^\dagger\hat{a} = (m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - \frac{1}{2}$, so $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
- 3 $(m\omega/(2\hbar))[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] = \hat{H}/(\hbar\omega)$, so $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Treating $\hat{x}$ and $\hat{p}$ as numbers makes the cross terms cancel and loses the $\frac{1}{2}$, which is the zero-point energy. The commutator must appear explicitly in the working.

(b) DERIVE · U2

[3]

Using only the two results of (a), derive the energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \dots$

- 1 $[\hat{H}, \hat{a}] = \hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}] = \hbar\omega[\hat{a}^\dagger, \hat{a}]\hat{a} = -\hbar\omega\hat{a}$, so if $\hat{H}|E\rangle = E|E\rangle$ then $\hat{H}(\hat{a}|E\rangle) = (E - \hbar\omega)\hat{a}|E\rangle$: $\hat{a}$ lowers the energy by $\hbar\omega$, and likewise $\hat{a}^\dagger$ raises it by $\hbar\omega$
- 2 $E = \langle E|\hat{H}|E\rangle = \hbar\omega(\|\hat{a}|E\rangle\|^2 + \frac{1}{2}) \geq \frac{1}{2}\hbar\omega$, so repeated lowering cannot continue and must end on a state $|0\rangle$ with $\hat{a}|0\rangle = 0$
- 3 Then $\hat{H}|0\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$; every eigenvalue lies a whole number of steps $\hbar\omega$ above this, and raising $|0\rangle$ n times reaches each one, so $E_n = (n + \frac{1}{2})\hbar\omega$

The step that separates answers is the second: without the norm argument there is no reason for the ladder to stop, and asserting $\hat{a}|0\rangle = 0$ earns nothing on its own.

(c) DERIVE · U2

[3]

Derive the normalised ground-state wavefunction $\psi_0(x)$ from the condition $\hat{a}|0\rangle = 0$.

- 1 In the position representation $\hat{p} = -i\hbar d/dx$, so $\hat{a}|0\rangle = 0$ becomes $(x + (\hbar/(m\omega)) d/dx)\psi_0 = 0$
- 2 Separating, $d\psi_0/\psi_0 = -(m\omega/\hbar)x dx$, so $\psi_0 = N \exp(-m\omega x^2/(2\hbar))$
- 3 $\int |\psi_0|^2 dx = N^2 \sqrt{(\pi\hbar/(m\omega))} = 1$, so $N = (m\omega/(\pi\hbar))^{1/4}$

A sign error in $i\hat{p}/(m\omega)$ gives $\exp(+m\omega x^2/(2\hbar))$, which cannot be normalised. A candidate who checks normalisability catches it.

(d) CALCULATE · U3

[3]

For the NO ground state, calculate σ_x and σ_p , and compare their product with the lower bound set by the commutator $[\hat{x}, \hat{p}]$.

- 1 $|\psi_0|^2 \propto \exp(-\mu\omega x^2/\hbar)$ is a Gaussian of variance $\hbar/(2\mu\omega)$; with $\omega = 2\pi f = 3.54 \times 10^{14} \text{ rad s}^{-1}$, $\sigma_x = \sqrt{\hbar/(2\mu\omega)} = 3.46 \times 10^{-12} \text{ m} = 3.46 \text{ pm}$, about 3% of the bond length
- 2 $\langle p \rangle = 0$ and $\langle p^2 \rangle/(2\mu) = \frac{1}{2}\langle \hat{H} \rangle = \frac{1}{4}\hbar\omega$ (equal kinetic and potential shares), so $\sigma_p = \sqrt{\mu\hbar\omega/2} = 1.52 \times 10^{-23} \text{ kg m s}^{-1}$
- 3 $\sigma_x\sigma_p = 5.27 \times 10^{-35} \text{ J s} = \hbar/2$, equal to the Robertson bound $\frac{1}{2}|[\hat{x}, \hat{p}]| = \hbar/2$, so the Gaussian ground state saturates it

Reading σ_x off the exponent of ψ_0 as $\sqrt{\hbar/(\mu\omega)} = 4.90 \text{ pm}$ confuses the width of ψ_0 with the width of $|\psi_0|^2$ and gives $\sigma_x\sigma_p = \hbar/\sqrt{2} = 7.46 \times 10^{-35} \text{ J s}$ (or $\hbar$ if σ_p is misread in the same way), which would put the ground state above the bound it actually meets.

(e) EVALUATE · U4

[2]

Evaluate whether the harmonic model can be trusted for (i) the fraction of NO molecules in the first excited vibrational level at 300 K, and (ii) the energies of vibrational levels close to dissociation.

- 1 (i) $\hbar f/(k_{\text{BT}}) = (6.63 \times 10^{-34} \text{ J s} \times 5.64 \times 10^{13} \text{ Hz})/(1.38 \times 10^{-23} \text{ J K}^{-1} \times 300 \text{ K}) = 9.03$, so $N_1/N_0 = e^{-9.03} = 1.2 \times 10^{-4}$, which is also the fraction in $n = 1$ because almost every molecule is in $n = 0$; this depends only on the lowest spacing, and with σ_x only about 3% of the bond length the lowest levels sample just the bottom of the well, where the quadratic approximation holds, so the model can be trusted
- 2 (ii) High levels have large amplitudes that reach the anharmonic part of the potential, which flattens towards dissociation; the real levels crowd together and end at a finite dissociation energy, while the harmonic ladder stays evenly spaced for ever, so the model cannot be trusted there

A verdict of 'good' or 'bad' with no criterion earns nothing. The criterion is the amplitude a level explores compared with the distance over which the real potential departs from a parabola, and (d) has already supplied it for $n = 0$.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 14 MARKS · 19%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 25 MARKS · 34%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

24 MARKS · 33%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

10 MARKS · 14%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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