

University Physics V · Final Examination

A cumulative final on University Physics V, weighted towards its second half: the hydrogen atom, multi-electron atoms and spectroscopy, nuclear structure and decay, fission and fusion, particle classification, the Standard Model, conservation-law audits and quantum information, with the state-space, spin and oscillator methods of the first half sampled throughout.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
150 minutes	120	19	10

SECTION	UPV · Final · University Physics V — Final examination
IN THE FORMAT OF	A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations
DEMAND	Recall 2 · Routine 7 · Demanding 8 · Discriminating 2 <i>Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.</i>
COURSE UNITS	UPV-01 · UPV-02 · UPV-03 · UPV-04 · UPV-05 · UPV-06 · UPV-07 · UPV-08 · UPV-09 · UPV-10 · UPV-11 · UPV-12 · UPV-13 · UPV-14 · UPV-15 · UPV-16 · UPV-17 · UPV-18 · UPV-19 · UPV-20

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Physics V · Final Examination

Answer all questions. 120 marks in 150 minutes.

1

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-02.08 · UPV-03.07

[1]

A single photon on an integrated-optics chip is divided equally among three waveguides, leaving it in $|\psi\rangle = (|1\rangle + |2\rangle + |3\rangle)/\sqrt{3}$, where $|k\rangle$ is the state in which the photon travels in waveguide k . A heater on waveguide 3 then multiplies the amplitude in that waveguide by $e^{i\pi}$. One of two measurements is made afterwards: M_1 asks whether the photon is in waveguide 3, with projector $|3\rangle\langle 3|$, and M_2 asks whether it is in the original state $|\psi\rangle$, with projector $|\psi\rangle\langle\psi|$. How does the probability of the outcome 'yes' in each measurement compare with its value when the heater is off?

- A Neither changes: M_1 stays at $1/3$ and M_2 stays at 1 , because multiplying an amplitude by a phase factor cannot change a probability.
- B M_1 stays at $1/3$, and M_2 falls from 1 to $1/9$.
- C M_1 stays at $1/3$, and M_2 falls from 1 to $1/3$.
- D M_1 stays at $1/3$, and M_2 falls from 1 to 0 .

Answer

2

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-06.04

[1]

Conduction electrons travelling in the $+x$ direction along a single-channel nanowire meet a thin impurity layer at $x = 0$, modelled as the potential $V(x) = \alpha\delta(x)$ with $\alpha > 0$. The potential is zero on both sides of the layer, so the wavenumber k is the same on both sides. Matching ψ at $x = 0$, together with the jump in $d\psi/dx$ that the delta function requires, gives a reflected amplitude $r = -i\beta/(1 + i\beta)$ and a transmitted amplitude $t = 1/(1 + i\beta)$ for unit incident amplitude, where $\beta = m\alpha/(\hbar^2 k)$ and m is the effective mass of the conduction electrons. For electrons with $\beta = 2$, what is the reflection coefficient R , the ratio of the reflected to the incident probability current?

- A $1/5$
- B $4/9$
- C $4/5$
- D $2/\sqrt{5}$

Answer

3CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-07.05 · UPV-07.07

[1]

A mode of a superconducting microwave resonator behaves as a harmonic oscillator of angular frequency ω , with Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ and number states $|n\rangle$ that satisfy $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. A sequence of control pulses prepares the mode in $|\psi\rangle = \eta(1 + a^\dagger)^2|0\rangle$, where 1 is the identity operator and η is a positive constant that normalises the state. What is the energy uncertainty ΔE in this state?

- A $(2\sqrt{5/7})\hbar\omega$
- B $\hbar\omega/\sqrt{3}$
- C $(2\sqrt{5/7} + 1/2)\hbar\omega$
- D $\sqrt{(12/7)}\hbar\omega$

Answer **4**CORE
COURSE

MULTIPLE CHOICE · RECALL · UPV-08.07 · UPV-09.03

[1]

A particle is in a simultaneous eigenstate of L^2 and L_z with quantum numbers $l = 2$ and $m_l = 2$. The component L_x is then measured. Which list gives every result that measurement can return?

- A $+2\hbar$ only, because the state already has a definite component of angular momentum
- B 0 only, because $\langle L_x \rangle = 0$ in this state
- C $-\hbar, 0$ and $+\hbar$
- D $-2\hbar, -\hbar, 0, +\hbar$ and $+2\hbar$

Answer **5**CORE
COURSE

MULTIPLE CHOICE · RECALL · UPV-11.05 · UPV-11.06

[1]

In the non-relativistic Coulomb model of hydrogen, with the electron's spin carried as a tensor factor but no spin-dependent term in the Hamiltonian, how many orthonormal states share the energy $E_4 = -0.850$ eV?

- A 7
- B 32
- C 16
- D 8

Answer

6

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-12.07

[1]

Treat the ground configuration of nickel as $[\text{Ar}] 3d^8 4s^2$, with the $4s^2$ subshell closed. Using antisymmetry and Hund's rules, what is the ground term?

- A $^1G_4$
- B $^3F_2$
- C $^3G_5$
- D $^3F_4$

Answer

7

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-14.04 · UPV-14.09

[1]

A nucleus with $A = 236$ and binding energy per nucleon 7.59 MeV splits into two fragments: one with $A = 138$ and 8.39 MeV per nucleon, and one with $A = 98$ and 8.60 MeV per nucleon. Ignoring any emitted neutrons, what energy is released?

- A 209 MeV
- B -209 MeV
- C 0.887 MeV
- D 1.79×10^3 MeV

Answer

8

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-15.02

[1]

A nuclide decays by α emission with branching fraction 0.25 and by β^- emission with branching fraction 0.75. Its half-life, found by following the number of undecayed nuclei, is 48.0 min. A detector that records only the α particles is used instead. With what half-life does the α count rate fall?

- A 12.0 min
- B 192 min
- C 48.0 min
- D 64.0 min

Answer

9

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-17.03 · UPV-19.06 · UPV-19.09

[1]

Which of these proposed processes is forbidden by a conservation law of the Standard Model with massless neutrinos?

- A $\mu^- \rightarrow e^- + \gamma$
- B $K^- + p \rightarrow \Omega^- + K^+ + K^0$
- C $\bar{\nu}_e + p \rightarrow n + e^+$
- D $\pi^+ \rightarrow \mu^+ + \nu_\mu$

Answer

Which of these normalised two-qubit states, each written in the computational basis $|q_1q_2\rangle$, is entangled?

- A $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$
- B $(|00\rangle - |01\rangle + |10\rangle - |11\rangle)/2$
- C $(|00\rangle + i|01\rangle + i|10\rangle - |11\rangle)/2$
- D $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$

Answer

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.0670 m_e$ in a one-dimensional infinite square well of width $L = 12.0 \text{ nm}$, with eigenstates $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ for $0 \leq x \leq L$ and energies $E_n = n^2 \pi^2 \hbar^2 / (2m^* L^2)$. At $t = 0$ it is prepared in $\Psi(x, 0) = [3\psi_1(x) + 4\psi_2(x)]/5$. Use $m_e = 9.109 \times 10^{-31} \text{ kg}$, $\hbar = 1.055 \times 10^{-34} \text{ J s}$, $h = 6.626 \times 10^{-34} \text{ J s}$ and $e = 1.602 \times 10^{-19} \text{ C}$. You may use $\int_0^L \psi_1(x) \times \psi_2(x) dx = -16L/(9\pi^2)$.

(a) CALCULATE

[2]

Calculate E_1 and E_2 , in meV.

(b) SHOW THAT

[2]

Show that the expectation value of the energy is $73E_1/25$ at every time t .

(c) DETERMINE

[3]

Determine $\langle x \rangle$ as a function of time, giving the centre and the amplitude of its oscillation in nm.

(d) CALCULATE

[1]

Calculate the period of the oscillation of $\langle x \rangle$.

(e) PREDICT

[2]

The electron is instead prepared in $[3\psi_1 + 4\psi_3]/5$. Predict how $\langle x \rangle$ behaves in time, and give the reason.

X-ray spectra of the hot gas in galaxy clusters contain lines from O^{7+} , an oxygen nucleus ($Z = 8$) with a single bound electron. Treat the ion with the non-relativistic Coulomb Hamiltonian and an infinitely heavy nucleus, for which $E_n = -(13.6 \text{ eV})Z^2/n^2$. The ground-state radial probability density is $P(r) = (4Z^3/a_0^3) r^2 e^{(-2Zr/a_0)}$, with $a_0 = 52.9 \text{ pm}$. Use $hc = 1239.8 \text{ eV nm}$.

(a) CALCULATE

[3]

Calculate the ionisation energy of ground-state O^{7+} , and the photon energy and wavelength of its $2p \rightarrow 1s$ (Lyman- α) line.

(b) SHOW THAT

[3]

Show that the most probable distance of the electron from the nucleus in the ground state is a_0/Z , and calculate that distance for O^{7+} .

(c) JUSTIFY

[3]

The electron is excited into one of the $n = 3$ states. Justify which of 3s, 3p and 3d can decay to 1s by emitting a single electric-dipole photon, and state the route to 1s for each of the others.

(d) EXPLAIN

[2]

In this model the 3s, 3p and 3d levels are exactly degenerate. Explain why this degeneracy is special to the Coulomb potential, and name the effect that splits these levels in a real O^{7+} ion.

A water sample sits in the uniform static field $B_0 = 9.40$ T of an NMR magnet, directed along $+z$. Consider the spin of a single hydrogen nucleus. The proton's magnetic moment is $\mu = (g_p \mu_N / \hbar) S$, with $g_p = 5.586$ and $\mu_N = 5.051 \times 10^{-27}$ J T $^{-1}$, so the Hamiltonian is $H = -\mu \cdot B = -\omega_0 S_z$ with $\omega_0 = g_p \mu_N B_0 / \hbar$. Take $\hbar = 1.055 \times 10^{-34}$ J s. A radio-frequency pulse, treated as instantaneous, leaves the spin at $t = 0$ in the $+\hbar/2$ eigenstate of $\hat{n} \cdot S$, where $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is the unit vector with polar angle θ and azimuth ϕ . The spin operators are $S_i = (\hbar/2)\sigma_i$, and in the basis of the S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ the Pauli matrices are $\sigma_x = (0, 1; 1, 0)$, $\sigma_y = (0, -i; i, 0)$ and $\sigma_z = (1, 0; 0, -1)$, with rows separated by semicolons.

(a) DETERMINE

[2]

Determine the matrix of $\hat{n} \cdot S$ in the S_z basis, and its normalised eigenvector with eigenvalue $+\hbar/2$ written as a combination of $|\uparrow\rangle$ and $|\downarrow\rangle$.

(b) SHOW THAT

[3]

Using the propagator $U(t) = \exp(-iHt/\hbar)$, show that at time t the spin is in the $+\hbar/2$ eigenstate of $\hat{n}(t) \cdot S$, where $\hat{n}(t)$ has the same polar angle θ and the azimuth $\phi - \omega_0 t$.

(c) CALCULATE

[2]

Calculate ω_0 , and, for $0 < \theta < \pi$, the shortest time after $t = 0$ at which the spin is again in the same physical state as at $t = 0$.

(d) DESCRIBE

[2]

A receiver coil with its axis along x picks up a voltage proportional to $d\langle S_x \rangle / dt$. Describe the signal it records, and how the amplitude of that signal depends on θ .

(e) SHOW THAT

[3]

For operators A and B the Robertson relation reads $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$. Apply it to H and S_x , and show that at every time t this spin obeys $\Delta E \Delta S_x \geq (\hbar/2) |d\langle S_x \rangle / dt|$, where ΔE is the uncertainty in its energy.

The magnesium line at $\lambda = 517.3 \text{ nm}$ is the transition from the upper level $3s4s \ ^3S_1$ to the lower level $3s3p \ ^3P_1$. The source is placed in a uniform magnetic field $B = 1.20 \text{ T}$, weak enough that LS coupling holds and each level shifts by $\Delta E = g_J \mu_B B M_J$, with $g_J = 1 + [J(J+1) + S(S+1) - L(L+1)]/[2J(J+1)]$. Use $\mu_B = 5.788 \times 10^{-5} \text{ eV T}^{-1}$ and $hc = 1239.8 \text{ eV nm}$. In the lower term, the $^3P_1$ and $^3P_2$ levels are 5.05 meV apart.

(a) CALCULATE

[2]

Calculate the Landé g factors of the upper and lower levels.

(b) JUSTIFY

[2]

Justify that this line is an allowed electric-dipole transition.

(c) DETERMINE

[4]

Determine how many Zeeman components the line splits into, and the shift of each from the zero-field photon energy in units of $\mu_B B$.

(d) CALCULATE

[2]

Calculate the wavelength separation between the two outermost components.

(e) ESTIMATE

[2]

Estimate the field above which this anomalous Zeeman pattern gives way to the Paschen-Back regime, stating the criterion you use.

A π^- beam strikes a liquid-hydrogen target, and the reaction $\pi^- + p \rightarrow K^0 + \Lambda^0$ is studied. The Λ^0 then decays as $\Lambda^0 \rightarrow p + \pi^-$ with mean lifetime 2.63×10^{-10} s. Rest energies: $m_\pi c^2 = 139.57$ MeV, $m_p c^2 = 938.27$ MeV, $m_K c^2 = 497.61$ MeV and $m_\Lambda c^2 = 1115.68$ MeV. The quark contents are $K^0 = d\bar{s}$ and $\Lambda^0 = uds$, and a strange quark carries strangeness $S = -1$.

(a) STATE

[2]

State the total charge, baryon number and strangeness on each side of the production reaction, and state which interaction can mediate it.

(b) DERIVE

[3]

Derive, from four-momentum conservation, an expression for the minimum laboratory energy E_π of the pion, with the target proton at rest, for which the reaction can occur.

(c) CALCULATE

[2]

Calculate the threshold kinetic energy and momentum of the pion.

(d) COMPARE

[2]

The Σ^0 baryon, also uds , decays as $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ with mean lifetime 7.4×10^{-20} s. Compare its lifetime with that of the Λ^0 , relating the difference to the interaction responsible for each decay.

(e) CALCULATE

[3]

Calculate the momentum of the proton in the rest frame of the Λ^0 for the decay $\Lambda^0 \rightarrow p + \pi^-$.

A source sends one qubit of each pair to Alice and the other to Bob, in the state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Alice measures the observable $A(a) = \cos a Z + \sin a X$ on her qubit and Bob measures $B(b) = \cos b Z + \sin b X$ on his, each with outcomes ± 1 , where X and Z are Pauli matrices. The correlation is $E(a, b) = \langle \Phi^+ | A(a) \otimes B(b) | \Phi^+ \rangle$, and the CHSH combination is $S = E(a, b) + E(a, b') + E(a', b) - E(a', b')$. Every local hidden-variable model obeys $|S| \leq 2$.

(a) SHOW THAT

[3]

Show that $E(a, b) = \cos(a - b)$.

(b) CALCULATE

[2]

Calculate S for the settings $a = 0$, $a' = \pi/2$, $b = \pi/4$ and $b' = -\pi/4$.

(c) DETERMINE

[3]

Noise in the source replaces $|\Phi^+\rangle\langle\Phi^+|$ by the density operator $\rho = p|\Phi^+\rangle\langle\Phi^+| + (1 - p)I/4$, with $0 \leq p \leq 1$. Determine S for the same settings as a function of p , and the range of p for which these settings show a violation.

(d) EVALUATE

[2]

An experiment using these settings reports $S = 2.38 \pm 0.05$. Evaluate what this result establishes and what it does not.

A radiopharmacy confirms the identity of a batch of fluorine-18 by measuring its half-life. A sample is placed in a well counter and the counts in 100 s are recorded at intervals; t is the time at the middle of each 100 s count, measured from the middle of the first. With no sample present, the counter records 1200 counts in 1000 s. The site's quality-control rule accepts the batch if the measured half-life lies between 105 min and 115 min. The accepted half-life of fluorine-18 is 109.8 min.

Gross counts recorded in 100 s intervals

t / min	Gross counts in 100 s
0	48252
60	32822
120	22668
180	15665
240	10629
300	7284
360	5123

(a) **EXPLAIN** [2]

Explain why the background must be subtracted from each reading before logarithms are taken, and why the uncertainty in the background itself can be neglected here.

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(b) **DETERMINE** [3]

Determine the decay constant λ from the data, using either a straight-line fit of $\ln(\text{net counts})$ against t or two well-separated readings.

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(c) **CALCULATE** [2]

Calculate the half-life and the mean lifetime.

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(d) **ESTIMATE** [3]

Estimate the uncertainty in λ that arises from counting statistics, assuming each reading is Poisson-distributed and using the two readings from part (b).

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(e) EVALUATE

[2]

Evaluate whether the batch passes the quality-control rule, and whether the data show any sign of a longer-lived radionuclidic impurity.

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18
CORE
COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPV-14.04 · UPV-16.06

[15]

In a deuterium-tritium plasma the reaction ${}^2\text{H} + {}^3\text{H} \rightarrow {}^4\text{He} + \text{n}$ releases the energy that fusion devices aim to harvest. Atomic masses: $m({}^2\text{H}) = 2.014102 \text{ u}$, $m({}^3\text{H}) = 3.016049 \text{ u}$ and $m({}^4\text{He}) = 4.002603 \text{ u}$; the neutron mass is 1.008665 u ; $1 \text{ u} = 931.494 \text{ MeV}/c^2$. For a Maxwellian plasma at temperature T , the non-resonant reaction rate per pair of nuclei is $\langle \sigma v \rangle \propto \int_0^\infty S(E) \exp[-E/(kT) - \sqrt{E_G/E}] dE$, where E is the centre-of-mass energy, $S(E)$ varies slowly with E , $\exp(-E/kT)$ is the Maxwell-Boltzmann factor and $\exp(-\sqrt{E_G/E})$ is the probability of tunnelling through the Coulomb barrier. For D-T the Gamow energy is $E_G = 2\mu c^2(\pi\alpha Z_1 Z_2)^2 = 1182 \text{ keV}$.

(a) CALCULATE

[2]

Calculate the Q value of the reaction.

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(b) SKETCH

[2]

On one set of axes against E , sketch the Maxwell-Boltzmann factor, the tunnelling factor and their product, marking the energy E_0 at which the product peaks.

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(c) DERIVE

[3]

Starting from the rate integral given in the stem, derive an expression for E_0 in terms of E_G and kT by maximising the exponent of its integrand.

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(d) SHOW THAT

[3]

By expanding the exponent to second order about E_0 , show that the integrand is approximately proportional to $\exp[-(E - E_0)^2/(\Delta/2)^2]$ with $\Delta = 4\sqrt{(E_0 kT/3)}$.

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(e) **CALCULATE**

[3]

Calculate E_0 and Δ for $kT = 10.0$ keV, and comment on how sharply the Gaussian form localises the reacting energies at this temperature.

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(f) **EVALUATE**

[2]

The Coulomb barrier between a deuteron and a triton at nuclear contact is about 0.44 MeV. A student argues that since $kT = 10.0$ keV is some 44 times smaller, fusion in this plasma must be negligible. Evaluate this argument.

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19
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-18.04 · UPV-18.05

[16]

In the electroweak model the Higgs field is a complex doublet ϕ with hypercharge $\frac{1}{2}$ and potential $V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$, with $\mu^2 > 0$ and $\lambda > 0$. Its covariant derivative is $D_\mu = \partial_\mu - ig(\sigma^a/2)W^a_\mu - i(g'/2)B_\mu$, summed over $a = 1, 2, 3$, where σ^a are the Pauli matrices, and the Lagrangian contains $(D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)$. In unitary gauge the vacuum is $\phi_0 = (0, v/\sqrt{2})^T$. A real vector field V_μ of mass M contributes $\frac{1}{2}M^2V_\mu V^\mu$ to the Lagrangian, and the charged fields $W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$ of mass M_W contribute $M_W^2W^+_\mu W^-_\mu$. Take $g = 0.652$, $g' = 0.357$, $v = 246$ GeV and $\lambda = 0.129$.

(a) **DERIVE**

[3]

Derive the vacuum value v in terms of μ and λ . Then, writing $\phi = (0, (v + h)/\sqrt{2})^T$, derive the mass M_H of the physical scalar h .

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(b) **STATE**

[2]

State how many real field components the doublet has, how many generators of $SU(2)_L \times U(1)_Y$ the vacuum breaks, and what becomes of the corresponding field components.

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(c) **SHOW THAT** [4]

Show that the kinetic term evaluated on ϕ_0 gives $M_W = gv/2$, and that the neutral fields acquire the mass-squared matrix $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$ in the basis (W^3, B) , with rows separated by semicolons.

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(d) **DETERMINE** [3]

Determine the eigenvalues and normalised eigenvectors of M^2 , identify the photon and the Z boson, and express the Weinberg angle θ_W and the ratio M_W/M_Z in terms of g and g' .

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(e) **CALCULATE** [2]

Calculate M_W , M_Z , $\sin^2\theta_W$ and M_H .

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(f) **EVALUATE** [2]

The measured values are $M_W = 80.37$ GeV, $M_Z = 91.19$ GeV and $M_H = 125.1$ GeV. Evaluate which of your results are genuine predictions of the model and which only return the numbers put in.

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University Physics V · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-02.08 · UPV-03.07 **[1]**
CORE COURSE

A single photon on an integrated-optics chip is divided equally among three waveguides, leaving it in $|\psi\rangle = (|1\rangle + |2\rangle + |3\rangle)/\sqrt{3}$, where $|k\rangle$ is the state in which the photon travels in waveguide k . A heater on waveguide 3 then multiplies the amplitude in that waveguide by $e^{i\pi}$. One of two measurements is made afterwards: M_1 asks whether the photon is in waveguide 3, with projector $|3\rangle\langle 3|$, and M_2 asks whether it is in the original state $|\psi\rangle$, with projector $|\psi\rangle\langle\psi|$. How does the probability of the outcome 'yes' in each measurement compare with its value when the heater is off?

KEY B M_1 stays at 1/3, and M_2 falls from 1 to 1/9.

With the heater on, the state is $|\psi\rangle = (|1\rangle + |2\rangle - |3\rangle)/\sqrt{3}$. A is the misreading this item is built to catch: a phase factor is unobservable only when it multiplies the whole state, whereas $e^{i\pi} = -1$ on one component is a relative phase, and any measurement whose projector mixes the waveguide states can detect it. C finds the overlap correctly, $\langle\psi|\psi\rangle = (1 + 1 - 1)/3 = 1/3$, but stops at the amplitude, when the Born rule takes its squared modulus. D carries over the two-path interferometer, where a phase of π on one arm makes the two amplitudes cancel; with three equal amplitudes, reversing one leaves $1 + 1 - 1 = 1$, not 0. M_1 is unchanged in every option because $|\langle 3|\psi\rangle|^2 = |-1/\sqrt{3}|^2 = 1/3$: a phase on one basis component leaves that component's modulus alone. Only B pairs this with $P(M_2) = |1/3|^2 = 1/9$.

U1

2 MULTIPLE CHOICE · ROUTINE · UPV-06.04 **[1]**
CORE COURSE

Conduction electrons travelling in the +x direction along a single-channel nanowire meet a thin impurity layer at $x = 0$, modelled as the potential $V(x) = \alpha\delta(x)$ with $\alpha > 0$. The potential is zero on both sides of the layer, so the wavenumber k is the same on both sides. Matching ψ at $x = 0$, together with the jump in $d\psi/dx$ that the delta function requires, gives a reflected amplitude $r = -i\beta/(1 + i\beta)$ and a transmitted amplitude $t = 1/(1 + i\beta)$ for unit incident amplitude, where $\beta = m\alpha/(\hbar^2 k)$ and m is the effective mass of the conduction electrons. For electrons with $\beta = 2$, what is the reflection coefficient R , the ratio of the reflected to the incident probability current?

KEY C 4/5

The reflected wave has the same wavenumber as the incident wave, so no velocity factor enters and $R = |r|^2 = \beta^2/(1 + \beta^2) = 4/5$; $T = |t|^2 = 1/(1 + \beta^2) = 1/5$ confirms $R + T = 1$. A is that transmission coefficient T , the right calculation answering the wrong question. B comes from dropping the i in the denominator and treating $|1 + i\beta|$ as $1 + \beta = 3$, which gives $(2/3)^2 = 4/9$; the modulus is $\sqrt{1 + \beta^2} = \sqrt{5}$, and with this error $R + T$ comes to $5/9$ instead of 1. D is $|r| = 2/\sqrt{5} \approx 0.894$ left unsquared, but a probability current is proportional to the squared modulus of an amplitude, not to the amplitude itself. The check $R + T = 1$ exposes both B and D.

U3

3

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-07.05 · UPV-07.07

[1]

A mode of a superconducting microwave resonator behaves as a harmonic oscillator of angular frequency ω , with Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ and number states $|n\rangle$ that satisfy $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. A sequence of control pulses prepares the mode in $|\psi\rangle = \eta(1 + a^\dagger)^2|0\rangle$, where 1 is the identity operator and η is a positive constant that normalises the state. What is the energy uncertainty ΔE in this state?

KEY **A** $(2\sqrt{5/7})\hbar\omega$

Expanding the operator gives $(1 + a^\dagger)^2|0\rangle = |0\rangle + 2a^\dagger|0\rangle + a^\dagger a^\dagger|0\rangle = |0\rangle + 2|1\rangle + \sqrt{2}|2\rangle$, so $\eta^2 = 1/7$ and the probabilities of $n = 0, 1$ and 2 are $1/7, 4/7$ and $2/7$. Then $\langle n \rangle = 8/7$, $\langle n^2 \rangle = 12/7$ and $(\Delta n)^2 = 12/7 - 64/49 = 20/49$; since $H = \hbar\omega(a^\dagger a + \frac{1}{2})$, $\Delta E = \hbar\omega \Delta n = (2\sqrt{5/7})\hbar\omega$. B writes $a^\dagger a^\dagger|0\rangle = |2\rangle$, missing the factor $\sqrt{1} \sqrt{2} = \sqrt{2}$ from the two raising steps; the state $|0\rangle + 2|1\rangle + |2\rangle$ has $\Delta n = 1/\sqrt{3}$. C adds the zero-point energy to the spread, but $\frac{1}{2}\hbar\omega$ shifts every level by the same amount, so it moves $\langle E \rangle$ to $(23/14)\hbar\omega$ and leaves ΔE alone. D is $\sqrt{\langle n^2 \rangle} \hbar\omega$, a root-mean-square value with $\langle n \rangle^2$ never subtracted, which mistakes how far the energy sits above the zero point for how widely it is spread.

U2

4

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[1]

A particle is in a simultaneous eigenstate of L^2 and L_z with quantum numbers $l = 2$ and $m_l = 2$. The component L_x is then measured. Which list gives every result that measurement can return?

KEY **D** $-2\hbar, -\hbar, 0, +\hbar$ and $+2\hbar$

A treats a definite L_z as a definite angular-momentum vector. L_x does not commute with L_z , so an L_z eigenstate is not an L_x eigenstate and the result of measuring L_x is not fixed. B confuses a mean with an outcome: $\langle L_x \rangle = 0$ here, but a single measurement returns an eigenvalue, and the zero mean is built from several different ones. C is the spectrum for $l = 1$. Within the $l = 2$ subspace L_x has the same eigenvalues as L_z , because a rotation carries one into the other, and the stretched state $|2, 2\rangle$ has a non-zero overlap with every L_x eigenstate, with probabilities $1/16, 1/4, 3/8, 1/4$ and $1/16$.

U1

5

CORE
COURSE

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[1]

In the non-relativistic Coulomb model of hydrogen, with the electron's spin carried as a tensor factor but no spin-dependent term in the Hamiltonian, how many orthonormal states share the energy $E_4 = -0.850$ eV?

KEY **B** 32

C is $n^2 = 16$, the sum of $2l + 1$ over $l = 0$ to 3 . It is the orbital count alone, correct only before the spin factor is carried, and the stem says it is carried. A is $2l + 1$ for the largest value, $l = 3$, which counts one subshell and forgets that every l from 0 to $n - 1$ has the same energy in a pure Coulomb field. D counts four l values with two spin states each but drops the $2l + 1$ values of m_l . The full label set n, l, m_l, m_s gives $2n^2 = 32$.

U1

6

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[1]

Treat the ground configuration of nickel as $[\text{Ar}] 3d^8 4s^2$, with the $4s^2$ subshell closed. Using antisymmetry and Hund's rules, what is the ground term?

KEY **D** $^3F_4$

A d^8 subshell behaves as two holes. C maximises S and L independently: ^3G would need two holes of the same spin both in $m_l = 2$, which antisymmetry forbids, so a spin-aligned pair reaches at most $M_L = 2 + 1 = 3$, an F term. A is the ^1G term, which antisymmetry does allow, but it takes the largest L before the largest S and so reverses the order of Hund's first two rules. B has the right term, ^3F , but applies the less-than-half-filled rule $J = |L - S|$. The subshell is more than half full, so $J = L + S = 4$ and the ground term is $^3F_4$.

U1

7

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[1]

A nucleus with $A = 236$ and binding energy per nucleon 7.59 MeV splits into two fragments: one with $A = 138$ and 8.39 MeV per nucleon, and one with $A = 98$ and 8.60 MeV per nucleon. Ignoring any emitted neutrons, what energy is released?

KEY **A 209 MeV**

B has the right size and the wrong sign. The fragments are more tightly bound, so their total mass is smaller and energy is released: $Q = B(\text{fragments}) - B(\text{parent})$ is positive. C is the gain in binding energy per nucleon averaged over the fragments, 0.887 MeV , which still has to be multiplied by the 236 nucleons that share it. D is the parent's total binding energy, $236 \times 7.59 = 1791 \text{ MeV}$, the energy needed to take the nucleus apart completely rather than the energy released by rearranging it. $Q = 138 \times 8.39 + 98 \times 8.60 - 236 \times 7.59 = 1157.8 + 842.8 - 1791.2 = 209 \text{ MeV}$.

U3

8

CORE
COURSE

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[1]

A nuclide decays by α emission with branching fraction 0.25 and by β^- emission with branching fraction 0.75 . Its half-life, found by following the number of undecayed nuclei, is 48.0 min . A detector that records only the α particles is used instead. With what half-life does the α count rate fall?

KEY **C 48.0 min**

B is the partial half-life of the α channel, $48.0/0.25 = 192 \text{ min}$, and it is the trap: a partial half-life says what share of the decays go by that route, not how fast that route's rate falls. Both channels draw on the same population of undecayed nuclei, so the α activity is $\lambda_\alpha N(t)$ with $N(t) = N_0 e^{(-\lambda t)}$, and it falls with the total decay constant $\lambda = \lambda_\alpha + \lambda_\beta$. D is the partial half-life of the β^- channel, $48.0/0.75 = 64.0 \text{ min}$. A multiplies the half-life by the branching fraction instead of dividing. The α rate is one quarter of the total activity at every instant, so it falls with the same 48.0 min half-life.

U1

9

CORE
COURSE

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[1]

Which of these proposed processes is forbidden by a conservation law of the Standard Model with massless neutrinos?

KEY **A $\mu^- \rightarrow e^- + \gamma$**

B is tempting because it makes a baryon of three strange quarks, but its ledger balances: strangeness -1 on the left and $-3 + 1 + 1 = -1$ on the right, charge 0 on each side, baryon number 1 on each side, so the strong interaction can mediate it above threshold. C looks as if an antineutrino turns into a positron, but both carry $L_e = -1$, charge is $+1$ on each side and baryon number is 1 on each side: it is inverse beta decay, the reaction used to detect reactor antineutrinos. D balances muon number as $0 = -1 + 1$. A conserves charge, energy and total lepton number, but changes L_μ from 1 to 0 and L_e from 0 to 1 . With massless neutrinos each lepton-family number is conserved, so A is forbidden; the family-number violation that neutrino oscillation implies would make this decay far rarer than any search can reach.

U1

10

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[1]

Which of these normalised two-qubit states, each written in the computational basis $|q_1 q_2\rangle$, is entangled?

KEY **D $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$**

Arrange the coefficients c_{ij} of $\sum c_{ij}|ij\rangle$ as a 2×2 matrix: the state is a product, with Schmidt rank 1 , exactly when that matrix has zero determinant. A is $|+\rangle \otimes |+\rangle$, with determinant $(1 \cdot 1 - 1 \cdot 1)/4 = 0$. B carries minus signs that look like correlations, but it is $|+\rangle \otimes |-\rangle$ and its determinant is $(-1 + 1)/4 = 0$. C is the strongest distractor, with complex phases and a minus sign, yet it is $(|0\rangle + i|1\rangle) \otimes (|0\rangle + i|1\rangle)/2$ and its determinant is $(-1 - i^2)/4 = 0$. D differs from A by a single sign, and that sign makes the determinant $(-1 - 1)/4 = -1/2$, so no product state can reproduce it; it is the state a controlled-Z gate makes from $|+\rangle \otimes |+\rangle$.

U2

An electron in a GaAs quantum well is modelled as a particle of effective mass $m^* = 0.0670 m_e$ in a one-dimensional infinite square well of width $L = 12.0$ nm, with eigenstates $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ for $0 \leq x \leq L$ and energies $E_n = n^2 \pi^2 \hbar^2 / (2m^* L^2)$. At $t = 0$ it is prepared in $\Psi(x, 0) = [3\psi_1(x) + 4\psi_2(x)]/5$. Use $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.626 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C. You may use $\int_0^L \psi_1(x) \times \psi_2(x) dx = -16L/(9\pi^2)$.

(a) CALCULATE · U3

[2]

Calculate E_1 and E_2 , in meV.

$$1 \quad E_1 = \pi^2 \hbar^2 / (2m^* L^2) = \pi^2 (1.055 \times 10^{-34} \text{ J s})^2 / (2 \times 0.0670 \times 9.109 \times 10^{-31} \text{ kg} \times (12.0 \times 10^{-9} \text{ m})^2) = 6.25 \times 10^{-21} \text{ J}$$

$$2 \quad E_1 = 39.0 \text{ meV and } E_2 = 4E_1 = 156 \text{ meV}$$

Using the free-electron mass instead of the effective mass gives $E_1 = 2.61$ meV, fifteen times too small; the small effective mass is why quantum-well levels reach tens of meV.

(b) SHOW THAT · U2

[2]

Show that the expectation value of the energy is $73E_1/25$ at every time t .

$$1 \quad \langle E \rangle = |c_1|^2 E_1 + |c_2|^2 E_2 = (9/25)E_1 + (16/25)(4E_1) = 73E_1/25 (= 114 \text{ meV})$$

2 time evolution multiplies each c_n by $e^{-iE_n t/\hbar}$, which leaves every $|c_n|^2$ unchanged, so $\langle E \rangle$ is the same at all t (equivalently, the propagator commutes with H)

Writing $\langle E \rangle = (3E_1 + 4E_2)/5$ weights by amplitudes instead of probabilities; the Born rule squares the coefficients.

(c) DETERMINE · U2

[3]

Determine $\langle x \rangle$ as a function of time, giving the centre and the amplitude of its oscillation in nm.

$$1 \quad \Psi(x, t) = [3\psi_1 e^{-iE_1 t/\hbar} + 4\psi_2 e^{-iE_2 t/\hbar}]/5, \text{ with } \langle \psi_1 | x | \psi_1 \rangle = \langle \psi_2 | x | \psi_2 \rangle = L/2$$

$$2 \quad \langle x \rangle = L/2 + 2(3/5)(4/5)x_{12} \cos(\omega t), \text{ where } \omega = (E_2 - E_1)/\hbar \text{ and } x_{12} = -16L/(9\pi^2)$$

$$3 \quad \langle x \rangle = L/2 - [384/(225\pi^2)]L \cos(\omega t) = 6.00 \text{ nm} - (2.08 \text{ nm}) \cos(\omega t)$$

The oscillation lives entirely in the cross term. A candidate who adds the two diagonal contributions and stops has computed $\langle x \rangle$ for a mixture, which sits at $L/2$ for ever.

(d) CALCULATE · U3

[1]

Calculate the period of the oscillation of $\langle x \rangle$.

$$1 \quad T = 2\pi\hbar/(E_2 - E_1) = \hbar/(3E_1) = 6.626 \times 10^{-34} \text{ J s} / (3 \times 6.25 \times 10^{-21} \text{ J}) = 3.53 \times 10^{-14} \text{ s} \text{ (35.3 fs); } 2\pi\hbar/(3E_1) \text{ with the given } \hbar \text{ gives } 3.54 \times 10^{-14} \text{ s, equally acceptable}$$

The period depends on the difference $E_2 - E_1$ alone; using E_2 or E_1 by itself treats an unobservable global phase as if it set a frequency.

(e) PREDICT · U1

[2]

The electron is instead prepared in $[3\psi_1 + 4\psi_3]/5$. Predict how $\langle x \rangle$ behaves in time, and give the reason.

1 $\langle x \rangle = L/2$ at all times: it does not oscillate

2 ψ_1 and ψ_3 are both even about $x = L/2$, so $(x - L/2)\psi_1\psi_3$ is odd about the centre and integrates to zero, which with $\langle \psi_1 | \psi_3 \rangle = 0$ gives $x_{13} = \int \psi_1 \times \psi_3 dx = 0$; the cross term vanishes even though $|\Psi|^2$ still changes shape symmetrically in time

Many candidates expect every two-state superposition to slosh from side to side. The box selection rule for x , non-zero only when $m - n$ is odd, decides whether it does.

X-ray spectra of the hot gas in galaxy clusters contain lines from O^{7+} , an oxygen nucleus ($Z = 8$) with a single bound electron. Treat the ion with the non-relativistic Coulomb Hamiltonian and an infinitely heavy nucleus, for which $E_n = -(13.6 \text{ eV})Z^2/n^2$. The ground-state radial probability density is $P(r) = (4Z^3/a_0^3) r^2 e^{-(2Zr/a_0)}$, with $a_0 = 52.9 \text{ pm}$. Use $hc = 1239.8 \text{ eV nm}$.

(a) CALCULATE · U3

[3]

Calculate the ionisation energy of ground-state O^{7+} , and the photon energy and wavelength of its $2p \rightarrow 1s$ (Lyman- α) line.

- 1 ionisation energy = $(13.6 \text{ eV}) \times 8^2 = 870 \text{ eV}$
- 2 photon energy for $2p \rightarrow 1s = 870.4 \text{ eV} \times (1 - 1/4) = 653 \text{ eV}$
- 3 $\lambda = hc/E = 1239.8 \text{ eV nm}/652.8 \text{ eV} = 1.90 \text{ nm}$

Forgetting the factor Z^2 gives 10.2 eV, the hydrogen Lyman- α line in the ultraviolet, rather than a line in the X-ray band where this one is observed.

(b) SHOW THAT · U2

[3]

Show that the most probable distance of the electron from the nucleus in the ground state is a_0/Z , and calculate that distance for O^{7+} .

- 1 $dP/dr \propto (2r - 2Zr^2/a_0)e^{-(2Zr/a_0)}$
- 2 $dP/dr = 0$ for $r > 0$ gives $r = a_0/Z$, a maximum because $P = 0$ at $r = 0$ and $P \rightarrow 0$ as $r \rightarrow \infty$
- 3 $r = 52.9 \text{ pm}/8 = 6.61 \text{ pm}$

Maximising $|\psi|^2$ instead of $r^2|\psi|^2$ returns $r = 0$: the probability per unit volume peaks at the nucleus, but the probability per unit radius carries the shell area $4\pi r^2$.

(c) JUSTIFY · U1

[3]

The electron is excited into one of the $n = 3$ states. Justify which of $3s$, $3p$ and $3d$ can decay to $1s$ by emitting a single electric-dipole photon, and state the route to $1s$ for each of the others.

- 1 the dipole operator is a rank-1 vector operator of odd parity, so the Wigner-Eckart triangle rule and parity together require $\Delta l = \pm 1$ (with $\Delta m_l = 0, \pm 1$)
- 2 only $3p \rightarrow 1s$ (Lyman- β) is allowed directly; $3s \rightarrow 1s$ has $\Delta l = 0$ and $3d \rightarrow 1s$ has $\Delta l = 2$, so both are forbidden
- 3 $3s$ and $3d$ each decay first to $2p$ ($\Delta l = -1$ from $3d$, $\Delta l = +1$ from $3s$) and then $2p \rightarrow 1s$

Candidates often allow $3d \rightarrow 1s$ because Δn is unrestricted; the rule constrains l , not n . Sending $3s$ to $2s$ is the other common slip, since that step has $\Delta l = 0$.

(d) EXPLAIN · U1

[2]

In this model the $3s$, $3p$ and $3d$ levels are exactly degenerate. Explain why this degeneracy is special to the Coulomb potential, and name the effect that splits these levels in a real O^{7+} ion.

- 1 for a $1/r$ potential the Runge-Lenz vector commutes with H and, with L , closes an $so(4)$ algebra whose multiplets contain every l from 0 to $n - 1$; any other central potential keeps only the $(2l + 1)$ -fold rotational degeneracy in m_l
- 2 relativistic fine structure (the kinetic-energy correction, spin-orbit coupling and the Darwin term), which scales as $Z^4\alpha^2$ and is about $8^4 \approx 4 \times 10^3$ times larger than in hydrogen, splits the levels by j ; levels of equal n and j are then separated by the much smaller Lamb shift

Rotational symmetry alone explains only the m_l degeneracy, so attributing the l degeneracy to spherical symmetry is the commonest error. Screening is also wrong here: O^{7+} has only one electron.

A water sample sits in the uniform static field $B_0 = 9.40$ T of an NMR magnet, directed along $+z$. Consider the spin of a single hydrogen nucleus. The proton's magnetic moment is $\mu = (g_p \mu_N / \hbar) S$, with $g_p = 5.586$ and $\mu_N = 5.051 \times 10^{-27}$ J T $^{-1}$, so the Hamiltonian is $H = -\mu \cdot B = -\omega_0 S_z$ with $\omega_0 = g_p \mu_N B_0 / \hbar$. Take $\hbar = 1.055 \times 10^{-34}$ J s. A radio-frequency pulse, treated as instantaneous, leaves the spin at $t = 0$ in the $+\hbar/2$ eigenstate of $\hat{n} \cdot S$, where $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is the unit vector with polar angle θ and azimuth ϕ . The spin operators are $S_i = (\hbar/2) \sigma_i$, and in the basis of the S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ the Pauli matrices are $\sigma_x = (0, 1; 1, 0)$, $\sigma_y = (0, -i; i, 0)$ and $\sigma_z = (1, 0; 0, -1)$, with rows separated by semicolons.

(a) DETERMINE · U2

[2]

Determine the matrix of $\hat{n} \cdot S$ in the S_z basis, and its normalised eigenvector with eigenvalue $+\hbar/2$ written as a combination of $|\uparrow\rangle$ and $|\downarrow\rangle$.

- $\hat{n} \cdot S = (\hbar/2)(\cos \theta, \sin \theta e^{-i\phi}; \sin \theta e^{i\phi}, -\cos \theta)$
- $|\hat{n}+\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi} \sin(\theta/2)|\downarrow\rangle$, normalised and checked against the first row: $\cos \theta \cos(\theta/2) + \sin \theta \sin(\theta/2) = \cos(\theta/2)$; any overall phase is accepted

Writing full angles, $\cos \theta |\uparrow\rangle + e^{i\phi} \sin \theta |\downarrow\rangle$, gives a normalised ket that is not an eigenvector. Testing the answer at $\theta = \pi$, where the eigenvector must be $|\downarrow\rangle$, catches the error at once: the full-angle form returns $-|\uparrow\rangle$.

(b) SHOW THAT · U2

[3]

Using the propagator $U(t) = \exp(-iHt/\hbar)$, show that at time t the spin is in the $+\hbar/2$ eigenstate of $\hat{n}(t) \cdot S$, where $\hat{n}(t)$ has the same polar angle θ and the azimuth $\phi - \omega_0 t$.

- $H = -\omega_0 S_z$ is diagonal in the S_z basis, with energy $-\hbar\omega_0/2$ for $|\uparrow\rangle$ and $+\hbar\omega_0/2$ for $|\downarrow\rangle$, so $U(t)$ multiplies $|\uparrow\rangle$ by $e^{i\omega_0 t/2}$ and $|\downarrow\rangle$ by $e^{-i\omega_0 t/2}$
- $U(t)|\hat{n}+\rangle = e^{i\omega_0 t/2} \cos(\theta/2)|\uparrow\rangle + e^{i\phi} e^{-i\omega_0 t/2} \sin(\theta/2)|\downarrow\rangle$
- factoring out the global phase $e^{i\omega_0 t/2}$, which changes no prediction, leaves $\cos(\theta/2)|\uparrow\rangle + e^{i(\phi - \omega_0 t)} \sin(\theta/2)|\downarrow\rangle$, the eigenvector of (a) with θ unchanged and ϕ replaced by $\phi - \omega_0 t$

Losing the minus sign in $H = -\omega_0 S_z$ gives the azimuth $\phi + \omega_0 t$, a precession in the wrong sense. The result is given, so the marks are for the route, and the sense is physical: a proton, whose moment is parallel to its spin, precesses clockwise as seen from $+z$.

(c) CALCULATE · U3

[2]

Calculate ω_0 , and, for $0 < \theta < \pi$, the shortest time after $t = 0$ at which the spin is again in the same physical state as at $t = 0$.

- $\omega_0 = g_p \mu_N B_0 / \hbar = 5.586 \times 5.051 \times 10^{-27}$ J T $^{-1} \times 9.40$ T / $(1.055 \times 10^{-34}$ J s) = 2.51×10^9 rad s $^{-1}$, a Larmor frequency $f_0 = \omega_0 / 2\pi = 400$ MHz
- the azimuth first returns to ϕ when $\omega_0 t = 2\pi$, so $t = 2\pi / \omega_0 = 2.50 \times 10^{-9}$ s; the ket is then $-|\hat{n}+\rangle$, which is the same ray

A candidate who recalls that a spinor needs a rotation through 4π to return to itself gives 5.00×10^{-9} s. After 2π the ket has only changed sign, a global phase, so the physical state has already returned. Using h in place of $\hbar$ gives f_0 where ω_0 was asked for.

(d) DESCRIBE · U1

[2]

A receiver coil with its axis along x picks up a voltage proportional to $d\langle S_x \rangle / dt$. Describe the signal it records, and how the amplitude of that signal depends on θ .

- the spin stays in the $+\hbar/2$ eigenstate of $\hat{n}(t) \cdot S$, so $\langle S \rangle = (\hbar/2)\hat{n}(t)$ keeps its length and its angle θ to the field while precessing about $+z$; $\langle S_x \rangle = (\hbar/2) \sin \theta \cos(\phi - \omega_0 t)$, so the coil records a sinusoidal voltage at $f_0 = 400$ MHz
- the amplitude is proportional to $\sin \theta$: greatest for $\theta = \pi/2$, a spin tipped into the xy -plane, and zero for $\theta = 0$ or π , when the state is an eigenstate of H and nothing precesses

Candidates who expect a signal whatever the pulse has done have not asked what precesses. When $\theta = 0$ or π every expectation value is constant, and only the part of $\langle S \rangle$ transverse to the field induces a voltage.

(e) SHOW THAT · U2 [3]

For operators A and B the Robertson relation reads $\Delta A \Delta B \geq \frac{1}{2} | \langle [A, B] \rangle |$. Apply it to H and S_x , and show that at every time t this spin obeys $\Delta E \Delta S_x \geq (\hbar/2) |d\langle S_x \rangle/dt|$, where ΔE is the uncertainty in its energy.

- 1 $[H, S_x] = -\omega_0[S_z, S_x] = -i\hbar\omega_0 S_y$, so the Robertson relation gives $\Delta E \Delta S_x \geq (\hbar\omega_0/2) | \langle S_y \rangle |$
- 2 $\langle S_x \rangle = (\hbar/2) \sin \theta \cos(\varphi - \omega_0 t)$ and $\langle S_y \rangle = (\hbar/2) \sin \theta \sin(\varphi - \omega_0 t)$, so $d\langle S_x \rangle/dt = (\hbar\omega_0/2) \sin \theta \sin(\varphi - \omega_0 t) = \omega_0 \langle S_y \rangle$
- 3 substituting $\omega_0 | \langle S_y \rangle | = |d\langle S_x \rangle/dt|$ gives $\Delta E \Delta S_x \geq (\hbar/2) |d\langle S_x \rangle/dt|$: the energy spread limits how fast $\langle S_x \rangle$ can change

Candidates who try to treat time as an operator have nothing to put into the Robertson relation. The route is through $[H, S_x]$: by Ehrenfest's theorem $\langle [H, S_x] \rangle = -i\hbar d\langle S_x \rangle/dt$, so a bound set by a commutator becomes a bound on a rate.

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[12]

The magnesium line at $\lambda = 517.3$ nm is the transition from the upper level $3s4s \ ^3S_1$ to the lower level $3s3p \ ^3P_1$. The source is placed in a uniform magnetic field $B = 1.20$ T, weak enough that LS coupling holds and each level shifts by $\Delta E = g_J \mu_B B M_J$, with $g_J = 1 + [J(J+1) + S(S+1) - L(L+1)]/[2J(J+1)]$. Use $\mu_B = 5.788 \times 10^{-5}$ eV T⁻¹ and $hc = 1239.8$ eV nm. In the lower term, the $^3P_1$ and $^3P_2$ levels are 5.05 meV apart.

(a) CALCULATE · U3 [2]

Calculate the Landé g factors of the upper and lower levels.

- 1 $^3S_1$ has $L = 0, S = 1, J = 1$: $g = 1 + (2 + 2 - 0)/4 = 2$
- 2 $^3P_1$ has $L = 1, S = 1, J = 1$: $g = 1 + (2 + 2 - 2)/4 = 3/2$

Reading the superscript 3 as $S = 3$, rather than $2S + 1 = 3$, gives meaningless g factors; so does using one electron's l in place of the term's total L.

(b) JUSTIFY · U1 [2]

Justify that this line is an allowed electric-dipole transition.

- 1 a single electron jumps $4s \rightarrow 3p$ with $\Delta l = 1$, so the configuration parity changes from even to odd, as the odd-parity dipole operator requires
- 2 $\Delta S = 0, \Delta L = 1$ and $\Delta J = 0$ between two $J = 1$ levels are all allowed; the only forbidden $\Delta J = 0$ case is $J = 0 \rightarrow J = 0$, which this is not

Candidates who quote $\Delta J = \pm 1$ alone would forbid this bright line; $\Delta J = 0$ is allowed except between two $J = 0$ levels.

(c) DETERMINE · U2 [4]

Determine how many Zeeman components the line splits into, and the shift of each from the zero-field photon energy in units of $\mu_B B$.

- 1 photon energy shift = $\mu_B B(g_u M_u - g_l M_l) = \mu_B B(2M_u - 1.5M_l)$, with $\Delta M = M_u - M_l = 0$ or ± 1 , giving seven (M_u, M_l) pairs
- 2 the $M_u = 0 \rightarrow M_l = 0$ component vanishes because $\Delta J = 0$: its Clebsch-Gordan coefficient $\langle 1\ 0; 1\ 0 | 1\ 0 \rangle$ is zero (the 3j symbol with all projections zero vanishes when $1 + 1 + 1$ is odd)
- 3 six components remain
- 4 the π components ($\Delta M = 0$) are shifted by $\pm 0.5\mu_B B$, and the σ components ($\Delta M = \pm 1$) by $\pm 1.5\mu_B B$ and $\pm 2.0\mu_B B$

Most candidates count seven components. The missing central π component is the discriminating point: it follows from Wigner-Eckart geometry, not from an extra rule to memorise.

(d) **CALCULATE** · U3 [2]

Calculate the wavelength separation between the two outermost components.

1 the outermost components are $4.0\mu_B$ apart: $4 \times 5.788 \times 10^{-5} \text{ eV T}^{-1} \times 1.20 \text{ T} = 2.78 \times 10^{-4} \text{ eV}$

2 $\Delta\lambda = \lambda^2 \Delta E / (hc) = (517.3 \text{ nm})^2 \times 2.78 \times 10^{-4} \text{ eV} / (1239.8 \text{ eV nm}) = 0.0600 \text{ nm (60.0 pm)}$

Using the shift of one outer component, $2.0\mu_B$, in place of the separation of the pair halves the answer.

(e) **ESTIMATE** · U4 [2]

Estimate the field above which this anomalous Zeeman pattern gives way to the Paschen-Back regime, stating the criterion you use.

1 criterion: the magnetic energy $\mu_B B$ becomes comparable with the fine-structure interval of the ^3P term, 5.05 meV, so that the field decouples L from S

2 $B \approx 5.05 \times 10^{-3} \text{ eV} / 5.788 \times 10^{-5} \text{ eV T}^{-1} \approx 90 \text{ T}$ (of order 10^2 T), far above 1.20 T, which confirms that the weak-field treatment used here is valid

An estimate needs its criterion; a bare number earns nothing. Comparing $\mu_B B$ with the photon energy instead of the fine-structure interval gives a field about 500 times too large.

15
CORE
COURSE

STRUCTURED · DEMANDING · UPV-17.08 · UPV-19.02 · UPV-19.03 · UPV-19.09 [12]

A π^- beam strikes a liquid-hydrogen target, and the reaction $\pi^- + p \rightarrow K^0 + \Lambda^0$ is studied. The Λ^0 then decays as $\Lambda^0 \rightarrow p + \pi^-$ with mean lifetime $2.63 \times 10^{-10} \text{ s}$. Rest energies: $m_\pi c^2 = 139.57 \text{ MeV}$, $m_p c^2 = 938.27 \text{ MeV}$, $m_K c^2 = 497.61 \text{ MeV}$ and $m_\Lambda c^2 = 1115.68 \text{ MeV}$. The quark contents are $K^0 = d\bar{s}$ and $\Lambda^0 = uds$, and a strange quark carries strangeness $S = -1$.

(a) **STATE** · U1 [2]

State the total charge, baryon number and strangeness on each side of the production reaction, and state which interaction can mediate it.

1 charge: $-1 + 1 = 0$ and $0 + 0 = 0$; baryon number: $0 + 1 = 1$ and $0 + 1 = 1$; strangeness: $0 + 0 = 0$ and $(+1) + (-1) = 0$

2 all three balance, strangeness included, so the strong interaction can mediate the production

Giving K^0 strangeness -1 because it contains strangeness unbalances the ledger; its $\bar{s}$ antiquark carries $S = +1$.

(b) **DERIVE** · U2 [3]

Derive, from four-momentum conservation, an expression for the minimum laboratory energy E_π of the pion, with the target proton at rest, for which the reaction can occur.

1 the invariant $s = (P_\pi + P_p) \cdot (P_\pi + P_p)$ is the same in every frame, and in the laboratory $s = m_\pi^2 c^4 + m_p^2 c^4 + 2E_\pi m_p c^2$

2 at threshold the products are at rest in the centre-of-momentum frame, so $s = (m_K + m_\Lambda)^2 c^4$

3 equating the two: $E_\pi = [(m_K + m_\Lambda)^2 - m_\pi^2 - m_p^2]c^2 / (2m_p)$

Equating the pion's kinetic energy to the mass deficit, as if the products could be left at rest in the laboratory, ignores momentum conservation and underestimates the threshold kinetic energy by about 30 per cent.

(c) **CALCULATE** · U3 [2]

Calculate the threshold kinetic energy and momentum of the pion.

1 $E_\pi = (1613.29^2 - 139.57^2 - 938.27^2) / (2 \times 938.27) \text{ MeV} = 907.5 \text{ MeV}$, so the kinetic energy is $E_\pi - m_\pi c^2 = 767.9 \text{ MeV}$

2 $p_\pi c = \sqrt{(E_\pi^2 - m_\pi^2 c^4)} = 896.7 \text{ MeV}$, so $p_\pi = 897 \text{ MeV}/c$

Quoting E_π as the kinetic energy is the usual slip; the rest energy has to be removed.

(d) COMPARE · U1

[2]

The Σ^0 baryon, also uds, decays as $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ with mean lifetime 7.4×10^{-20} s. Compare its lifetime with that of the Λ^0 , relating the difference to the interaction responsible for each decay.

- 1 $\Lambda^0 \rightarrow p + \pi^-$ changes strangeness from -1 to 0 , which neither the strong nor the electromagnetic interaction allows, so it is a weak decay; $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ conserves strangeness and emits a photon, so it is electromagnetic
- 2 the lifetimes differ by a factor $2.63 \times 10^{-10} \text{ s} / 7.4 \times 10^{-20} \text{ s} \approx 4 \times 10^9$; by the golden rule the width $\Gamma = \hbar/\tau$ scales as the square of the transition amplitude, so the ratio reflects how much weaker the weak amplitude is (phase space differs as well, but cannot account for nine orders of magnitude)

Saying only that weak decays are slow earns nothing; the mark is for the strangeness change that closes the faster routes.

(e) CALCULATE · U3

[3]

Calculate the momentum of the proton in the rest frame of the Λ^0 for the decay $\Lambda^0 \rightarrow p + \pi^-$.

- 1 momentum conservation makes the two momenta equal and opposite, of magnitude p^* , and energy conservation gives $\sqrt{(p^*c)^2 + m_p^2c^4} + \sqrt{(p^*c)^2 + m_{\pi}^2c^4} = m_{\Lambda}c^2$
- 2 solving: $p^*c = \sqrt{([m_{\Lambda}^2 - (m_p + m_{\pi})^2][m_{\Lambda}^2 - (m_p - m_{\pi})^2])} c^2 / (2m_{\Lambda})$
- 3 $p^* = 100.6 \text{ MeV}/c$; the proton's kinetic energy is then 5.4 MeV and the pion's 32.5 MeV , sharing $Q = 37.8 \text{ MeV}$

Treating the proton non-relativistically is harmless, but the pion's kinetic energy is almost a quarter of its rest energy: an all-non-relativistic solution gives $96 \text{ MeV}/c$, wrong in the second significant figure.

A source sends one qubit of each pair to Alice and the other to Bob, in the state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Alice measures the observable $A(a) = \cos a Z + \sin a X$ on her qubit and Bob measures $B(b) = \cos b Z + \sin b X$ on his, each with outcomes ± 1 , where X and Z are Pauli matrices. The correlation is $E(a, b) = \langle \Phi^+ | A(a) \otimes B(b) | \Phi^+ \rangle$, and the CHSH combination is $S = E(a, b) + E(a, b') + E(a', b) - E(a', b')$. Every local hidden-variable model obeys $|S| \leq 2$.

(a) SHOW THAT · U2

[3]

Show that $E(a, b) = \cos(a - b)$.

- 1 expands $A \otimes B = \cos a \cos b Z \otimes Z + \cos a \sin b Z \otimes X + \sin a \cos b X \otimes Z + \sin a \sin b X \otimes X$
- 2 $Z \otimes Z$ and $X \otimes X$ leave $|\Phi^+\rangle$ unchanged, so $\langle Z \otimes Z \rangle = \langle X \otimes X \rangle = 1$, while $Z \otimes X$ and $X \otimes Z$ map $|\Phi^+\rangle$ to $\pm(|01\rangle - |10\rangle)/\sqrt{2}$, which is orthogonal to it, so those expectation values are 0
- 3 $E(a, b) = \cos a \cos b + \sin a \sin b = \cos(a - b)$

Each qubit on its own is maximally mixed, so $\langle A \rangle = \langle B \rangle = 0$. Candidates who factorise E as $\langle A \rangle \langle B \rangle$ get zero and have assumed away the correlation the question is about.

(b) CALCULATE · U3

[2]

Calculate S for the settings $a = 0$, $a' = \pi/2$, $b = \pi/4$ and $b' = -\pi/4$.

- 1 $E(0, \pi/4) = E(0, -\pi/4) = E(\pi/2, \pi/4) = \cos(\pi/4) = 0.707$ and $E(\pi/2, -\pi/4) = \cos(3\pi/4) = -0.707$
- 2 $S = 3 \times 0.707 + 0.707 = 2\sqrt{2} = 2.83$, above the local bound of 2

The minus sign on the fourth term is where the violation comes from; dropping it gives $S = 1.41$.

(c) DETERMINE · U2

[3]

Noise in the source replaces $|\Phi^+\rangle\langle\Phi^+|$ by the density operator $\rho = p|\Phi^+\rangle\langle\Phi^+| + (1 - p)I/4$, with $0 \leq p \leq 1$. Determine S for the same settings as a function of p , and the range of p for which these settings show a violation.

- 1 $E_p(a, b) = \text{Tr}[\rho A \otimes B] = p \cos(a - b) + (1 - p)\text{Tr}(A)\text{Tr}(B)/4$
- 2 A and B are traceless, so the noise term vanishes and $S = 2\sqrt{2} p$
- 3 a violation needs $2\sqrt{2} p > 2$, so $p > 1/\sqrt{2} = 0.707$

Candidates who give the white-noise part a correlation of its own, often $\pm 1/4$, have forgotten that every Pauli combination here is traceless.

(d) EVALUATE · U4

[2]

An experiment using these settings reports $S = 2.38 \pm 0.05$. Evaluate what this result establishes and what it does not.

- 1 S exceeds 2 by $(2.38 - 2)/0.05 = 7.6$ standard uncertainties, so the correlations are incompatible with every local hidden-variable model under the test's assumptions; in the noise model it implies $p = 2.38/2.83 = 0.84$, above the threshold of 0.707
- 2 it excludes local hidden variables only if the loopholes are closed (settings chosen at spacelike separation, and detection efficient enough that the recorded pairs are not a biased sample), and it neither proves that exactly $|\Phi^+\rangle$ was prepared nor allows any signal to pass between Alice and Bob

A strong answer states the size of the violation in standard uncertainties and names the assumptions it rests on; the bare claim that quantum mechanics is proved earns neither mark.

A radiopharmacy confirms the identity of a batch of fluorine-18 by measuring its half-life. A sample is placed in a well counter and the counts in 100 s are recorded at intervals; t is the time at the middle of each 100 s count, measured from the middle of the first. With no sample present, the counter records 1200 counts in 1000 s. The site's quality-control rule accepts the batch if the measured half-life lies between 105 min and 115 min. The accepted half-life of fluorine-18 is 109.8 min.

(a) EXPLAIN · U4

[2]

Explain why the background must be subtracted from each reading before logarithms are taken, and why the uncertainty in the background itself can be neglected here.

- 1 the background adds a constant 120 counts per 100 s to an exponential, so $\ln(\text{gross counts})$ is not linear in t : the late readings sit above the straight line and a fit would give too small a λ , that is, too long a half-life
- 2 the background uncertainty is $\sqrt{1200}/10 \approx 3.5$ counts per 100 s, which shifts $\ln(\text{net counts})$ by at most $3.5/5003 \approx 7 \times 10^{-4}$ at $t = 360$ min, far below that reading's own Poisson uncertainty of about 0.014

Omitting the background, or subtracting it only after fitting $\ln(\text{gross counts})$, is the commonest procedural error in decay counting; it bends the late-time data upward and mimics a long-lived impurity.

(b) DETERMINE · U4

[3]

Determine the decay constant λ from the data, using either a straight-line fit of $\ln(\text{net counts})$ against t or two well-separated readings.

- 1 subtracts 120 from each reading, for example net counts 48132 at $t = 0$ and 5003 at $t = 360$ min, giving $\ln(\text{net counts}) = 10.782$ and 8.518
- 2 gradient = $(8.518 - 10.782)/(360 \text{ min}) = -6.29 \times 10^{-3} \text{ min}^{-1}$ (a least-squares fit to all seven points gives $-6.30 \times 10^{-3} \text{ min}^{-1}$)
- 3 $\lambda = 6.3 \times 10^{-3} \text{ min}^{-1} = 1.05 \times 10^{-4} \text{ s}^{-1}$; accept values from 6.27×10^{-3} to $6.33 \times 10^{-3} \text{ min}^{-1}$

Using $\log_{10}$ and reading its gradient as λ gives a value 2.30 times too small; only the natural logarithm has gradient $-\lambda$.

(c) CALCULATE · U3

[2]

Calculate the half-life and the mean lifetime.

- 1 $T_{1/2} = \ln 2/\lambda = 0.6931/(6.29 \times 10^{-3} \text{ min}^{-1}) = 110 \text{ min}$ (110.2 min from the two end readings, 109.9 min from the least-squares fit)
- 2 $\tau = 1/\lambda = 159 \text{ min}$

τ and $T_{1/2}$ differ by the factor $\ln 2$. Quoting $1/\lambda = 159 \text{ min}$ as the half-life would reject a batch that is in fact fluorine-18.

(d) ESTIMATE · U4

[3]

Estimate the uncertainty in λ that arises from counting statistics, assuming each reading is Poisson-distributed and using the two readings from part (b).

- 1 $\sigma[\ln(\text{net counts})] = \sqrt{(\text{gross counts})/(\text{net counts})}$: $\sqrt{48252/48132} = 0.0046$ at $t = 0$ and $\sqrt{5123/5003} = 0.0143$ at $t = 360$ min
- 2 $\delta\lambda \approx \sqrt{(0.0046^2 + 0.0143^2)}/(360 \text{ min}) = 4.2 \times 10^{-5} \text{ min}^{-1}$, a fractional uncertainty of 0.7 per cent
- 3 $\delta T_{1/2} = T_{1/2} \delta\lambda/\lambda \approx 0.7 \text{ min}$, so $T_{1/2} = 110.2 \pm 0.7 \text{ min}$ (a weighted fit to all seven readings narrows this to about $\pm 0.5 \text{ min}$)

Taking the uncertainty in $\ln N$ to be $\sqrt{N}$ rather than $\sqrt{N}/N$ makes λ look unmeasurable; using $\sqrt{(\text{net})}$ in place of $\sqrt{(\text{gross})}$ is only a minor slip here.

(e) EVALUATE · U4

[2]

Evaluate whether the batch passes the quality-control rule, and whether the data show any sign of a longer-lived radionuclidic impurity.

- 1 $T_{1/2} = 110.2 \pm 0.7$ min lies inside 105 min to 115 min and agrees with the accepted 109.8 min to within about 0.6 standard uncertainties, so the batch passes
- 2 the intermediate readings lie on both sides of a least-squares line, each within about one Poisson standard uncertainty (within two of the line through the end readings), with no systematic upward curvature at late times; a longer-lived impurity would lift the late readings above the line, so none is detectable above a few per cent of the count rate at 360 min

Concluding that the data fit a straight line, without saying what an impurity would do to that line, is description rather than evaluation.

18
CORE
COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UPV-14.04 · UPV-16.06

[15]

In a deuterium-tritium plasma the reaction ${}^2\text{H} + {}^3\text{H} \rightarrow {}^4\text{He} + \text{n}$ releases the energy that fusion devices aim to harvest. Atomic masses: $m({}^2\text{H}) = 2.014102$ u, $m({}^3\text{H}) = 3.016049$ u and $m({}^4\text{He}) = 4.002603$ u; the neutron mass is 1.008665 u; $1 \text{ u} = 931.494 \text{ MeV}/c^2$. For a Maxwellian plasma at temperature T , the non-resonant reaction rate per pair of nuclei is $\langle \sigma v \rangle \propto \int_0^\infty S(E) \exp[-E/(kT) - \sqrt{E_G/E}] dE$, where E is the centre-of-mass energy, $S(E)$ varies slowly with E , $\exp(-E/kT)$ is the Maxwell-Boltzmann factor and $\exp(-\sqrt{E_G/E})$ is the probability of tunnelling through the Coulomb barrier. For D-T the Gamow energy is $E_G = 2\mu c^2(\pi\alpha Z_1 Z_2)^2 = 1182 \text{ keV}$.

(a) CALCULATE · U3

[2]

Calculate the Q value of the reaction.

- 1 $\Delta m = 2.014102 \text{ u} + 3.016049 \text{ u} - 4.002603 \text{ u} - 1.008665 \text{ u} = 0.018883 \text{ u}$, where atomic masses may be used because the two electrons on the left are matched by the two in ${}^4\text{He}$
- 2 $Q = 0.018883 \text{ u} \times 931.494 \text{ MeV/u} = 17.59 \text{ MeV}$

Mixing atomic and nuclear masses, for instance subtracting $2m_e$ from ${}^4\text{He}$ alone, leaves two electron masses unbalanced, an error of 1.022 MeV, about 6 per cent of Q and far larger than the data's precision.

(b) SKETCH · U1

[2]

On one set of axes against E , sketch the Maxwell-Boltzmann factor, the tunnelling factor and their product, marking the energy E_0 at which the product peaks.

- 1 the Maxwell-Boltzmann factor falls monotonically from 1 at $E = 0$, and the tunnelling factor rises monotonically from 0 at $E = 0$, very flat near the origin, towards 1 at large E
- 2 their product, drawn on an expanded vertical scale and labelled as such, is a single peak at E_0 , several times kT , and is smaller than both factors at every energy

Placing the peak near kT , where the Maxwellian itself is largest ($kT/2$ for the energy distribution, kT for its flux-weighted form), misses the whole construction: the tunnelling factor pulls the reacting nuclei far into the tail.

(c) DERIVE · U2

[3]

Starting from the rate integral given in the stem, derive an expression for E_0 in terms of E_G and kT by maximising the exponent of its integrand.

- 1 writes $f(E) = -E/(kT) - E_G^{1/2}E^{-1/2}$, since $S(E)$ varies slowly and the exponential fixes where the integrand peaks
- 2 $df/dE = -1/(kT) + \frac{1}{2}E_G^{1/2}E^{-3/2} = 0$
- 3 $E_0^{3/2} = \frac{1}{2}E_G^{1/2}kT$, so $E_0 = [E_G(kT)^2/4]^{1/3}$

Differentiating $E^{-1/2}$ as $-\frac{1}{2}E^{-1/2}$, raising the power instead of lowering it, gives $E_0 = 4/[E_G(kT)^2]$, which falls as T rises and is not even an energy; checking that E_0 grows with temperature and has the dimensions of energy catches it.

(d) SHOW THAT · U2 [3]

By expanding the exponent to second order about E_0 , show that the integrand is approximately proportional to $\exp[-(E - E_0)^2/(\Delta/2)^2]$ with $\Delta = 4\sqrt{(E_0 kT/3)}$.

1 $f'(E) = -\frac{3}{4}E_G^{1/2}E^{-5/2}$, and the stationary condition gives $E_G^{1/2} = 2E_0^{3/2}/(kT)$, so $f'(E_0) = -3/(2E_0 kT)$

2 $f(E) \approx f(E_0) + \frac{1}{2}f''(E_0)(E - E_0)^2 = f(E_0) - 3(E - E_0)^2/(4E_0 kT)$, the first-order term vanishing at the maximum

3 matching $3/(4E_0 kT) = 1/(\Delta/2)^2$ gives $(\Delta/2)^2 = 4E_0 kT/3$, so $\Delta = 4\sqrt{(E_0 kT/3)}$

Candidates who evaluate f' without eliminating E_G through the stationary condition usually reach a correct but unrecognisable expression and cannot finish.

(e) CALCULATE · U3 [3]

Calculate E_0 and Δ for $kT = 10.0$ keV, and comment on how sharply the Gaussian form localises the reacting energies at this temperature.

1 $E_0 = [1182 \text{ keV} \times (10.0 \text{ keV})^2/4]^{1/3} = 30.9 \text{ keV}$, about $3.1kT$

2 $\Delta = 4\sqrt{(30.9 \text{ keV} \times 10.0 \text{ keV}/3)} = 40.6 \text{ keV}$

3 $\Delta \approx 1.3E_0$, so the window is broad rather than sharp and noticeably asymmetric, extending further above E_0 than below it: the Gaussian gives the position and width of the window but not its exact shape (a numerical integration shows the Gaussian estimate of the integral is about 4 per cent low)

Laplace's method is exact only in the limit of a peak narrow compared with its position; at D-T temperatures Δ is comparable with E_0 , and candidates who never compare the two have not checked the approximation they have just made.

(f) EVALUATE · U1 [2]

The Coulomb barrier between a deuteron and a triton at nuclear contact is about 0.44 MeV. A student argues that since $kT = 10.0$ keV is some 44 times smaller, fusion in this plasma must be negligible. Evaluate this argument.

1 the barrier height is not a threshold: nuclei tunnel through it, and at E_0 the tunnelling factor is $e^{(-\sqrt{(1182/30.9)})} = e^{(-6.18)} \approx 2 \times 10^{-3}$, small but far from zero

2 the rate is set where this factor overlaps the Maxwellian tail (about 10 per cent of nuclei have $E > E_0 \approx 3kT$), so the reaction proceeds at an appreciable rate; the student's argument holds only classically, where no pair below 0.44 MeV could react

This is the argument Gamow's factor answers. A candidate who agrees with the student has treated the barrier classically straight after deriving a formula that contains the tunnelling probability.

19
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-18.04 · UPV-18.05

[16]

In the electroweak model the Higgs field is a complex doublet ϕ with hypercharge $\frac{1}{2}$ and potential $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$, with $\mu^2 > 0$ and $\lambda > 0$. Its covariant derivative is $D_\mu = \partial_\mu - ig(\sigma^a/2)W^a_\mu - i(g'/2)B_\mu$, summed over $a = 1, 2, 3$, where σ^a are the Pauli matrices, and the Lagrangian contains $(D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi)$. In unitary gauge the vacuum is $\phi_0 = (0, v/\sqrt{2})^T$. A real vector field V_μ of mass M contributes $\frac{1}{2}M^2 V_\mu V^\mu$ to the Lagrangian, and the charged fields $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$ of mass M_W contribute $M_W^2 W^+_\mu W^{-\mu}$. Take $g = 0.652$, $g' = 0.357$, $v = 246$ GeV and $\lambda = 0.129$.

(a) DERIVE · U2 [3]

Derive the vacuum value v in terms of μ and λ . Then, writing $\phi = (0, (v + h)/\sqrt{2})^T$, derive the mass M_H of the physical scalar h .

1 with $\rho = \phi^\dagger \phi$, $dV/d\rho = -\mu^2 + 2\lambda\rho = 0$ gives $\rho = \mu^2/(2\lambda) = v^2/2$, so $v = \mu/\sqrt{\lambda}$

2 substituting $\phi^\dagger \phi = (v + h)^2/2$ gives $V = -\frac{1}{2}\mu^2(v + h)^2 + \frac{1}{4}\lambda(v + h)^4$, whose h^2 coefficient is $-\frac{1}{2}\mu^2 + (3/2)\lambda v^2 = \lambda v^2$

3 matching $\lambda v^2 h^2$ to $\frac{1}{2}M_H^2 h^2$ gives $M_H = \sqrt{(2\lambda)} v = \sqrt{2} \mu$

Reading the mass from the h^2 coefficient without the $\frac{1}{2}$ in $\frac{1}{2}M_H^2 h^2$ gives $M_H = \sqrt{\lambda} v$, too small by $\sqrt{2}$.

(b) STATE · U1

[2]

State how many real field components the doublet has, how many generators of $SU(2)_L \times U(1)_Y$ the vacuum breaks, and what becomes of the corresponding field components.

- 1 four real components; the vacuum is left invariant only by the $U(1)$ generated by the charge $Q = T_3 + Y$, so $4 - 1 = 3$ generators are broken
- 2 the three would-be Goldstone modes become the longitudinal polarisations of W^+ , W^- and Z , leaving one physical scalar, the Higgs boson h

Candidates who count four Goldstone bosons miss that one generator combination annihilates the vacuum, which is exactly why the photon stays massless.

(c) SHOW THAT · U2

[4]

Show that the kinetic term evaluated on ϕ_0 gives $M_W = gv/2$, and that the neutral fields acquire the mass-squared matrix $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$ in the basis (W^3, B) , with rows separated by semicolons.

- 1 the gauge-field matrix is $\frac{1}{2}(gW^3 + g'B, g(W^1 - iW^2); g(W^1 + iW^2), -gW^3 + g'B)$, so $D_\mu \phi_0 = -(i/2)(v/\sqrt{2})(g(W^1 - iW^2), -gW^3 + g'B)^T$
- 2 $(D_\mu \phi_0)^\dagger (D^\mu \phi_0) = (v^2/8)[g^2(W^1W^1 + W^2W^2) + (gW^3 - g'B)^2]$
- 3 $W^1W^1 + W^2W^2 = 2W^+W^-$, so the charged term is $(g^2v^2/4)W^+W^-$ and $M_W = gv/2$
- 4 $(v^2/8)(gW^3 - g'B)^2 = \frac{1}{2}(W^3, B)M^2(W^3, B)^T$ with $M^2 = (v^2/4)(g^2, -gg'; -gg', g'^2)$

The factor $\frac{1}{2}$ that separates a real field's mass term from a complex field's is where most attempts go wrong; tracking it separately for $W^\pm$ and for the neutral pair is what earns the last two marks.

(d) DETERMINE · U2

[3]

Determine the eigenvalues and normalised eigenvectors of M^2 , identify the photon and the Z boson, and express the Weinberg angle θ_W and the ratio M_W/M_Z in terms of g and g' .

- 1 $\det M^2 = (v^2/4)^2(g^2g'^2 - g'^2g^2) = 0$, so one eigenvalue is zero; its normalised eigenvector is the massless photon $A = (g'W^3 + gB)/\sqrt{g^2 + g'^2}$
- 2 the other eigenvalue equals the trace, $M_Z^2 = (v^2/4)(g^2 + g'^2)$, with normalised eigenvector the Z boson, $Z = (gW^3 - g'B)/\sqrt{g^2 + g'^2}$
- 3 writing $Z = \cos \theta_W W^3 - \sin \theta_W B$ gives $\tan \theta_W = g'/g$, and $M_W/M_Z = g/\sqrt{g^2 + g'^2} = \cos \theta_W$

Noticing the zero determinant finishes the diagonalisation in two lines; brute-force diagonalisation takes longer and often loses the normalisation of the eigenvectors.

(e) CALCULATE · U3

[2]

Calculate M_W , M_Z , $\sin^2 \theta_W$ and M_H .

- 1 $M_W = 0.652 \times 246 \text{ GeV}/2 = 80.2 \text{ GeV}$ and $M_Z = (246 \text{ GeV}/2)\sqrt{(0.652^2 + 0.357^2)} = 91.4 \text{ GeV}$
- 2 $\sin^2 \theta_W = g'^2/(g^2 + g'^2) = 0.231$ and $M_H = \sqrt{(2 \times 0.129)} \times 246 \text{ GeV} = 125 \text{ GeV}$

Taking $\sin \theta_W = g'/g$ confuses the sine with the tangent and gives $\sin^2 \theta_W = 0.30$ instead of 0.231.

The measured values are $M_W = 80.37$ GeV, $M_Z = 91.19$ GeV and $M_H = 125.1$ GeV. Evaluate which of your results are genuine predictions of the model and which only return the numbers put in.

- 1 g , g' , v and λ are inputs fitted to data, so the individual values of M_W , M_Z and M_H are not independent predictions; M_H in particular only restates λ , which the model leaves free
- 2 the genuine tree-level predictions are structural: a massless photon, and $M_W = M_Z \cos \theta_W$, which follows from the doublet's quantum numbers and tests the model once θ_W is measured independently of the masses, in neutral-current couplings; the measured ratio $80.37/91.19 = 0.881$ agrees with $\cos \theta_W = 0.877$ at the half-per-cent level that loop corrections account for

Declaring that the model predicts $M_W = 80.2$ GeV is exactly the confusion this part is set to expose: with four fitted inputs, only relations among the outputs test the theory.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 29 MARKS · 24%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 47 MARKS · 39%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

30 MARKS · 25%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

14 MARKS · 12%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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