

University Physics V · Midterm Examination

Examines Units 1 to 10: states and Hermitian operators in Dirac notation, the experimental case for quanta, Born's rule and unitary evolution, the box, the finite well and tunnelling, the oscillator's ladder, commutators and the Robertson bound, and angular momentum and spin one-half.

SUGGESTED TIME	TOTAL MARKS	QUESTIONS	MULTIPLE CHOICE
90 minutes	72	14	8

SECTION	UPV · Midterm · University Physics V — Midterm examination
IN THE FORMAT OF	A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation
DEMAND	Routine 5 · Demanding 7 · Discriminating 2 <i>Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.</i>
COURSE UNITS	UPV-01 · UPV-02 · UPV-03 · UPV-04 · UPV-05 · UPV-06 · UPV-07 · UPV-08 · UPV-09 · UPV-10

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Physics V · Midterm Examination

Answer all questions. 72 marks in 90 minutes.

1 MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-03.01 [1]
CORE COURSE

$|a\rangle$ and $|b\rangle$ are orthonormal kets, and a system is prepared in $|\psi\rangle = (|a\rangle + i|b\rangle)/\sqrt{2}$. Which of the following kets describes the same physical state as $|\psi\rangle$?

- A $(|a\rangle - i|b\rangle)/\sqrt{2}$
- B $(i|a\rangle + |b\rangle)/\sqrt{2}$
- C $(i|a\rangle - |b\rangle)/\sqrt{2}$
- D $(|a\rangle + |b\rangle)/\sqrt{2}$

Answer

2 MULTIPLE CHOICE · DEMANDING · UPV-02.03 [1]
CORE COURSE

In a semiclassical model of the interaction of light with matter, the electromagnetic field is a classical wave and only the electrons in the detector are treated quantum mechanically. Which observation can this model NOT account for?

- A Light from a single trapped atom, re-excited by one short laser pulse at a time, falls on a 50:50 beam splitter with a detector at each output, and coincident clicks at the two detectors within one pulse are far rarer than any classical wave of the same mean intensity allows.
- B The maximum kinetic energy of photoelectrons increases linearly with the frequency of the light, with gradient h .
- C No photoelectrons are emitted below a threshold frequency, however intense the light.
- D Photoelectrons are emitted with no measurable delay after very weak light is switched on.

Answer

3 MULTIPLE CHOICE · ROUTINE · UPV-03.02 · UPV-05.06 [1]
CORE COURSE

A particle is in the second excited state, $n = 3$, of an infinitely deep one-dimensional well occupying $0 < x < L$. What is the probability of finding it in the first quarter of the well, $0 < x < L/4$?

- A 0.15
- B 0.20
- C 0.25
- D 0.30

Answer

4

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-06.04

[1]

A steady beam of non-relativistic particles, each with kinetic energy E , travels in the $+x$ direction and meets a potential step of height $V_0 = 3E/4$ at $x = 0$. What fraction of the incident probability current is transmitted past the step?

- A $1/9$
- B $8/9$
- C $16/9$
- D $16/25$

Answer

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-07.02 · UPV-07.05

[1]

For a one-dimensional harmonic oscillator of mass m and angular frequency ω , the position operator is $x = \sqrt{\hbar/(2m\omega)}(a + a^\dagger)$, with $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. What is the matrix element $\langle 3|x^2|1\rangle$?

- A 0
- B $\sqrt{2} \hbar/(2m\omega)$
- C $\sqrt{6} \hbar/(2m\omega)$
- D $3\hbar/(2m\omega)$

Answer

6

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-08.07

[1]

Two Hermitian operators A and B on a finite-dimensional state space satisfy $[A, B] = 0$. Which statement must be true?

- A Every eigenvector of A is also an eigenvector of B .
- B $\Delta A \Delta B = 0$ in every state.
- C A measurement of B made straight after a measurement of A leaves the state produced by the A measurement unchanged.
- D There is an orthonormal basis made entirely of simultaneous eigenvectors of A and B .

Answer

7CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.04

[1]

X-ray photons of wavelength 60.0 pm scatter from free electrons initially at rest, and photons scattered through 120° are detected. Take $h/(m_e c) = 2.426$ pm and $hc = 1240$ eV nm. What is the kinetic energy of an electron that has scattered one of these detected photons?

- A 0.41 keV
- B 1.18 keV
- C 19.5 keV
- D 20.7 keV

Answer

8CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-10.06 · UPV-10.07

[1]

Beam P consists of spin- $\frac{1}{2}$ atoms each prepared in $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z . Beam Q is an equal statistical mixture of atoms in $|\uparrow\rangle$ and atoms in $|\downarrow\rangle$. Which measurement, repeated on many atoms from each beam, distinguishes the two beams?

- A S_x : every atom from P gives $+\hbar/2$, while atoms from Q give $+\hbar/2$ and $-\hbar/2$ equally often.
- B S_z : each atom from P gives either result with probability $\frac{1}{2}$, whereas each atom in Q already has a definite value.
- C S_y : the relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ in P can only be detected by a measurement along y.
- D None: in both beams each atom is found up along z with probability $\frac{1}{2}$, so no measurement can separate them.

Answer

An electron shared between two coupled quantum dots is described in the orthonormal basis $\{|1\rangle, |2\rangle\}$ of the two dot states by a Hamiltonian with matrix elements $H_{11} = 4.0$ meV, $H_{22} = -4.0$ meV, $H_{12} = 3.0i$ meV and $H_{21} = -3.0i$ meV. At $t = 0$ the electron is in $|1\rangle$. Take $\hbar = 6.582 \times 10^{-16}$ eV s.

(a) SHOW THAT

[2]

Show that the energy eigenvalues are $E_+ = +5.0$ meV and $E_- = -5.0$ meV.

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(b) DETERMINE

[2]

Determine the normalised eigenvector $|E_+\rangle$ as a combination of $|1\rangle$ and $|2\rangle$.

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(c) CALCULATE

[1]

Calculate the probability that a measurement of the energy at $t = 0$ gives $+5.0$ meV.

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(d) DERIVE

[3]

Starting from the time-dependent Schrödinger equation, derive an expression for the probability $P_2(t)$ that the electron is found in $|2\rangle$ at time t .

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(e) CALCULATE

[2]

Calculate the earliest time after $t = 0$ at which P_2 is greatest, and state that greatest value.

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An electron of mass $m_e = 9.109 \times 10^{-31}$ kg is bound in a symmetric one-dimensional square well occupying $-a < x < a$, with $a = 0.500$ nm. Its potential energy is zero inside the well and $V_0 = 1.20$ eV outside it. Measure the energy E from the bottom of the well and write $k = \sqrt{(2m_e E)/\hbar}$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_e V_0)}$. The even bound states satisfy $z \tan z = \sqrt{(z_0^2 - z^2)}$ and the odd bound states satisfy $-z \cot z = \sqrt{(z_0^2 - z^2)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) DETERMINE

[3]

Calculate z_0 , and hence determine how many bound states the well holds.

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(b) SHOW THAT

[2]

Show that $z = 1.15$ satisfies the even-parity condition to within 1%.

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(c) COMPARE

[3]

Calculate the ground-state energy of the electron, and compare it with the ground-state energy the electron would have in an infinitely deep well of the same width $2a$.

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(d) SKETCH

[2]

On one set of axes of ψ against x , from $x = -2a$ to $x = +2a$, sketch the wavefunctions of the two bound states. Mark $x = -a$ and $x = +a$.

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Slow neutrons, which have spin $\frac{1}{2}$ and gyromagnetic ratio $\gamma_n = -1.832 \times 10^8 \text{ rad s}^{-1} \text{ T}^{-1}$, leave a polariser in the state $|+z\rangle$. The spin operator is $S = (\hbar/2)\sigma$; in the $\{|+z\rangle, |-z\rangle\}$ basis σ_y has rows $(0, -i)$ and $(i, 0)$, and $\exp(-i\alpha\sigma_y) = \cos \alpha I - i \sin \alpha \sigma_y$ for real α . An analyser set along a unit vector $\hat{n}$ transmits only neutrons found in the $+\hbar/2$ eigenstate of $\hat{n} \cdot S$, and blocks the rest.

(a) CALCULATE

[2]

The neutrons pass first through an analyser whose $\hat{n}$ lies in the xz -plane at 45° to the $+z$ axis, and then through an analyser set along $-z$, which transmits only $|-z\rangle$. Calculate the fraction of the neutrons leaving the polariser that emerge from the second analyser.

(b) EXPLAIN

[2]

With the first analyser removed, no neutrons emerge from the $-z$ analyser. Explain why inserting an analyser that removes neutrons from the beam increases the number emerging, and identify which step in the sequence is not unitary.

(c) SHOW THAT

[3]

The 45° analyser is removed. Instead, the $|+z\rangle$ neutrons cross a region of uniform magnetic field B along $+y$, where the Hamiltonian is $H = -\gamma_n B S_y$, taking a time t . Show that on leaving the field the spin state is $\cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$, where $\omega = |\gamma_n|B$.

(d) CALCULATE

[2]

The neutrons travel at 2200 m s^{-1} through a field region 5.00 cm long. Calculate the field B that turns $|+z\rangle$ into $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$.

(e) PREDICT

[1]

Predict the fraction of the neutrons that a $+z$ analyser placed after the field region would transmit if the same field pointed along $+z$ instead of $+y$, and give the reason.

A single $^{40}\text{Ca}^+$ ion of mass $m = 6.64 \times 10^{-26} \text{ kg}$ is held in an ion trap whose potential along the trap axis is harmonic with angular frequency $\omega = 2\pi \times 1.20 \text{ MHz} = 7.54 \times 10^6 \text{ rad s}^{-1}$. Along that axis the position and momentum operators are $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ and $p = i\sqrt{m\hbar\omega/2}(a^\dagger - a)$, with $[a, a^\dagger] = 1$ and number states $|n\rangle$ of energy $(n + \frac{1}{2})\hbar\omega$. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$ and $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$.

(a) ESTIMATE

[2]

After laser cooling, the axial motion is in thermal equilibrium at $T = 0.50 \text{ mK}$. Estimate the mean number of vibrational quanta $\bar{n}$, stating the approximation you make and why it is justified.

(b) SHOW THAT

[2]

Further cooling and a laser pulse prepare the state $|\psi(0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Show that $\langle x \rangle(t) = \sqrt{\hbar/2m\omega} \cos \omega t$.

(c) DETERMINE

[4]

Determine the product $\Delta x \Delta p$ at $t = 0$ and at $\omega t = \pi/4$, expressing each as a multiple of $\hbar$.

(d) EXPLAIN

[2]

Explain why $\Delta x \Delta p$ exceeds $\hbar/2$ at every time for this state, although the ground state $|0\rangle$ reaches $\hbar/2$.

A scanning tunnelling microscope tip is held above a clean metal surface at a fixed bias of 0.100 V. With the feedback loop switched off, the tip is withdrawn in steps, and the tunnelling current I is recorded against Δd , the increase in the tip-surface gap from its starting value. In the opaque-barrier limit $I = I_0 \exp(-2\kappa\Delta d)$, where $\kappa = \sqrt{2m_e\phi}/\hbar$ and ϕ is the apparent barrier height. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

Tunnelling current as the tip is withdrawn, at a bias of 0.100 V

Reading	Increase in gap Δd (pm)	Tunnelling current I (nA)
1	0	2.50
2	40	1.13
3	80	0.471
4	120	0.218
5	160	0.0924
6	200	0.0420

- (a) **STATE** [1]
State the gradient of a graph of $\ln(I/\text{nA})$ against Δd , in terms of κ .

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- (b) **DETERMINE** [3]
Determine κ from the data.

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- (c) **CALCULATE** [2]
Calculate the apparent barrier height ϕ in electronvolts.

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- (d) **ESTIMATE** [2]
In constant-current imaging the feedback loop moves the tip to hold I fixed. Estimate the change in gap that changes the current by 10%, stating the assumption you make.

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(e) EVALUATE

[2]

Photoemission gives the work function of this surface as 4.60 eV. A student concludes that either the STM data or the tunnelling model must be wrong. Evaluate this conclusion.

For Hermitian operators A and B and a normalised state $|\psi\rangle$, write $\langle A \rangle = \langle \psi | A | \psi \rangle$ and $(\Delta A)^2 = \langle \psi | (A - \langle A \rangle)^2 | \psi \rangle$. The Cauchy-Schwarz inequality states that $\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$ for any kets $|f\rangle$ and $|g\rangle$. The components of an angular momentum obey $[J_x, J_y] = i\hbar J_z$ and its cyclic permutations, $J_{\pm} = J_x \pm iJ_y$, and the simultaneous eigenstates $|j, m\rangle$ of J^2 and J_z satisfy $J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$, $J_z |j, m\rangle = m\hbar |j, m\rangle$ and $J_{\pm} |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$.

(a) DERIVE

[4]

Starting from the Cauchy-Schwarz inequality, derive the Robertson uncertainty relation, a lower bound on $\Delta A \Delta B$ in terms of the commutator $[A, B]$.

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(b) SHOW THAT

[4]

Show that in the state $|j, m\rangle$, $\langle J_x \rangle = \langle J_y \rangle = 0$ and $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{2} \hbar^2 [j(j+1) - m^2]$.

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(c) DETERMINE

[3]

Determine the states $|j, m\rangle$ in which $\Delta J_x \Delta J_y$ is exactly equal to the Robertson bound for J_x and J_y .

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(d) EVALUATE

[3]

A student argues: 'In the state $|j = 1, m = 0\rangle$ the Robertson bound for J_x and J_y is zero, so a careful enough experiment could find both J_x and J_y with no spread in that state.' Evaluate this argument.

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University Physics V · Midterm Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1	MULTIPLE CHOICE · ROUTINE · UPV-01.01 · UPV-03.01	[1]
CORE COURSE	$a\rangle$ and $b\rangle$ are orthonormal kets, and a system is prepared in $\psi\rangle = (a\rangle + i b\rangle)/\sqrt{2}$. Which of the following kets describes the same physical state as $\psi\rangle$? KEY C $(i a\rangle - b\rangle)/\sqrt{2}$ <i>C is $i \psi\rangle$: multiplying both components by the same unit-modulus factor changes no probability in any basis, so it is the same ray in Hilbert space. A reverses the sign of the relative phase, and B is simply i times A, so A and B are the same state as each other but not the state $\psi\rangle$; the pair is a reminder that a phase can only be factored out if it multiplies both components. D is tempting because it has the same moduli and so gives the same probabilities, one half each, in the $\{ a\rangle, b\rangle\}$ basis. But a measurement in the basis $(a\rangle \pm b\rangle)/\sqrt{2}$ gives the '+' outcome with certainty for D and with probability one half for $\psi\rangle$, so a relative phase is observable even though a global phase is not.</i>	

U1

2	MULTIPLE CHOICE · DEMANDING · UPV-02.03	[1]
CORE COURSE	In a semiclassical model of the interaction of light with matter, the electromagnetic field is a classical wave and only the electrons in the detector are treated quantum mechanically. Which observation can this model NOT account for? KEY A Light from a single trapped atom, re-excited by one short laser pulse at a time, falls on a 50:50 beam splitter with a detector at each output, and coincident clicks at the two detectors within one pulse are far rarer than any classical wave of the same mean intensity allows. <i>B, C and D are the three features usually quoted as proof that light is quantised, which is exactly why they tempt. Treat the light as a classical wave of frequency $f = \omega/2\pi$ perturbing a quantised electron: first-order perturbation theory drives transitions only to states lying $\hbar\omega = hf$ above the initial one, giving $K_{\text{max}} = hf - \phi$, a straight line of gradient h against f (B), and a threshold at $hf = \phi$ whatever the amplitude (C). The same theory gives a transition rate proportional to the intensity from the moment the light is switched on, so the first electrons can leave at once (D); the long delay was a prediction for a classical electron slowly soaking up energy, not for a quantum electron in a classical field. A is different in kind. For any classical field divided equally by a beam splitter, $\langle I_1 I_2 \rangle \geq \langle I_1 \rangle \langle I_2 \rangle$, so the coincidence ratio $g^{(2)}(0)$ cannot fall below 1. A single atom gives up its energy one quantum per excitation, and the observed suppression of coincidences requires a quantised field: it is the experiment that closes the gap the photoelectric effect leaves open.</i>	

U4

3CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-03.02 · UPV-05.06

[1]

A particle is in the second excited state, $n = 3$, of an infinitely deep one-dimensional well occupying $0 < x < L$. What is the probability of finding it in the first quarter of the well, $0 < x < L/4$?

KEY D 0.30

With $\psi_3(x) = \sqrt{2/L} \sin(3\pi x/L)$, the integral of $|\psi_3|^2$ from 0 to $L/4$ is $1/4 + 1/(6\pi) = 0.30$. C is the classical value, a quarter of the probability in a quarter of the well; the quantum result can differ from it by up to $1/(2n\pi)$, which is appreciable only at small n . For $n = 3$ the quantum value is larger because the interval ends three-quarters of the way through the first lobe, which spans $0 < x < L/3$: it keeps that lobe's maximum at $x = L/6$ and leaves out only the low-density region next to the node at $x = L/3$. B is $1/4 - 1/(6\pi) = 0.20$, which follows from writing the antiderivative of $\sin^2 u$ as $u/2 + (\sin 2u)/4$, the antiderivative of $\cos^2 u$; it is in fact the probability for the next quarter, $L/4 < x < L/2$, since $|\psi_3|^2$ is symmetric about $x = L/2$ and the two quarters must share one half. A is 0.30 halved, the result of dropping the normalisation factor and using $\sin(3\pi x/L)/\sqrt{L}$ as the wavefunction.

U3

4CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-06.04

[1]

A steady beam of non-relativistic particles, each with kinetic energy E , travels in the $+x$ direction and meets a potential step of height $V_0 = 3E/4$ at $x = 0$. What fraction of the incident probability current is transmitted past the step?

KEY B 8/9

With $k_1 \propto \sqrt{E}$ and $k_2 \propto \sqrt{E - V_0}$, $k_2 = k_1/2$. The transmitted amplitude is $t = 2k_1/(k_1 + k_2) = 4/3$, and the flux ratio is $T = (k_2/k_1)|t|^2 = 8/9$, so that $R + T = 1/9 + 8/9 = 1$. C is $|t|^2$ on its own: an amplitude ratio may exceed 1 because the wave slows beyond the step, and leaving out the velocity factor k_2/k_1 produces a 'probability' greater than one. A is the reflection coefficient $R = [(k_1 - k_2)/(k_1 + k_2)]^2$, the right calculation answering the wrong question. D follows from taking $k_2/k_1 = 1/4$, the ratio of kinetic energies, instead of its square root.

U3

5CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-07.02 · UPV-07.05

[1]

For a one-dimensional harmonic oscillator of mass m and angular frequency ω , the position operator is $x = \sqrt{\hbar/(2m\omega)}(a + a^\dagger)$, with $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. What is the matrix element $\langle 3|x^2|1\rangle$?

KEY C $\sqrt{6} \hbar/(2m\omega)$

Expanding $(a + a^\dagger)^2$ gives a^2 , $a^\dagger^2$, $aa^\dagger$ and $a^\dagger a$; only $a^\dagger^2$ can take $|1\rangle$ to $|3\rangle$, and $a^\dagger a^\dagger|1\rangle = \sqrt{2}\sqrt{3}|3\rangle$, so the element is $\sqrt{6} \hbar/(2m\omega)$. A carries over the selection rule for x itself, which connects only states whose n differ by one; x^2 connects states whose n differ by zero or two, and parity allows this element because $|1\rangle$ and $|3\rangle$ are both odd. B uses $a^\dagger|n\rangle = \sqrt{n}|n+1\rangle$, putting the square root on the wrong index. D is the diagonal element $\langle 1|x^2|1\rangle = (2n+1)\hbar/(2m\omega)$ with $n = 1$, which answers a different question.

U2

6

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-08.07

[1]

Two Hermitian operators A and B on a finite-dimensional state space satisfy $[A, B] = 0$. Which statement must be true?

KEY D There is an orthonormal basis made entirely of simultaneous eigenvectors of A and B .

D is the theorem: B maps each eigenspace of A into itself, and diagonalising B within each eigenspace produces a common orthonormal eigenbasis. A is guaranteed only when A is non-degenerate, because within a degenerate eigenspace of A nothing forces B to be a multiple of the identity. Take $A = L^2$ and $B = L_z$ on the $l = 0$ and $l = 1$ states, a four-dimensional space: $(|1, 1\rangle + |1, 0\rangle)/\sqrt{2}$ is an eigenvector of L^2 but not of L_z . C fails for the same reason: if the L^2 measurement leaves that superposition, a measurement of L_z projects it onto $|1, 1\rangle$ or $|1, 0\rangle$, changing the state although the value of L^2 survives, so compatibility protects the eigenvalue, not the state. B mistakes a lower bound for a value: the Robertson bound $\frac{1}{2}|[A, B]|$ is zero, but $(|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$ has $\Delta L^2 = \hbar^2$ and $\Delta L_z = \hbar/2$, so $\Delta A \Delta B = \hbar^3/2$ in that state.

U1

7

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UPV-02.04

[1]

X-ray photons of wavelength 60.0 pm scatter from free electrons initially at rest, and photons scattered through 120° are detected. Take $\hbar/(m_e c) = 2.426$ pm and $hc = 1240$ eV nm. What is the kinetic energy of an electron that has scattered one of these detected photons?

KEY B 1.18 keV

The Compton shift is $\Delta\lambda = (\hbar/m_e c)(1 - \cos 120^\circ) = 2.426 \text{ pm} \times 1.5 = 3.64 \text{ pm}$, so $\lambda' = 63.64 \text{ pm}$, and the electron carries away the difference in photon energy: $20.67 \text{ keV} - 19.48 \text{ keV} = 1.18 \text{ keV}$. A comes from writing $1 - \cos 120^\circ = 0.5$, losing the sign of $\cos 120^\circ = -0.5$; the shift is then a third of its true size, and the recoil energy falls to about a third as well. D is the energy of the incident photon and C that of the scattered photon. Each is a correct number for a different question, and both tempt because the recoil energy is small: the wavelength changes by only about 6%.

U3

8

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UPV-10.06 · UPV-10.07

[1]

Beam P consists of spin- $\frac{1}{2}$ atoms each prepared in $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z . Beam Q is an equal statistical mixture of atoms in $|\uparrow\rangle$ and atoms in $|\downarrow\rangle$. Which measurement, repeated on many atoms from each beam, distinguishes the two beams?

KEY A S_x : every atom from P gives $+\hbar/2$, while atoms from Q give $+\hbar/2$ and $-\hbar/2$ equally often.

P is the pure state $|+\rangle$, with its Bloch vector along $+x$, while Q has density matrix $I/2$ and sits at the centre of the Bloch sphere, so an S_x measurement separates them completely. B describes a genuine difference in how the beams were prepared, but the S_z statistics are one half each for both beams, so this measurement cannot see the difference. C is right that the relative phase is what distinguishes a superposition from a mixture, but a real, positive relative phase puts the Bloch vector along x , not y : S_y gives one half each for P as well as for Q , and would only be the right choice for $(|\uparrow\rangle + i|\downarrow\rangle)/\sqrt{2}$. D generalises from one basis to every basis, which is precisely the error that the interference term in P exposes.

U1

An electron shared between two coupled quantum dots is described in the orthonormal basis $\{|1\rangle, |2\rangle\}$ of the two dot states by a Hamiltonian with matrix elements $H_{11} = 4.0$ meV, $H_{22} = -4.0$ meV, $H_{12} = 3.0i$ meV and $H_{21} = -3.0i$ meV. At $t = 0$ the electron is in $|1\rangle$. Take $\hbar = 6.582 \times 10^{-16}$ eV s.

(a) SHOW THAT · U2

[2]

Show that the energy eigenvalues are $E_+ = +5.0$ meV and $E_- = -5.0$ meV.

- 1 Sets $\det(H - EI) = 0$ with the off-diagonal product written out: $(4.0 - E)(-4.0 - E) - (3.0i)(-3.0i) = 0$, with E in meV
- 2 $(3.0i)(-3.0i) = +9.0$, so $E^2 - 16 - 9 = 0$, giving $E^2 = 25$ and $E_{\pm} = \pm 5.0$ meV

(b) DETERMINE · U2

[2]

Determine the normalised eigenvector $|E_+\rangle$ as a combination of $|1\rangle$ and $|2\rangle$.

- 1 From the first row of $(H - 5.0 \text{ meV } I)c = 0$: $(4.0 - 5.0)c_1 + 3.0i c_2 = 0$, so $c_2 = -ic_1/3$
- 2 Normalising with $|c_1|^2 + |c_2|^2 = 1$ gives $|E_+\rangle = (3|1\rangle - i|2\rangle)/\sqrt{10}$, to within an overall phase

Normalising with $c_1^2 + c_2^2$ instead of $|c_1|^2 + |c_2|^2$ gives $1/\sqrt{8}$ and a vector that is not normalised. Substituting the result into the second row, $-3.0i c_1 - 9.0 c_2 = 0$, is a ten-second check that also catches a sign error in c_2 .

(c) CALCULATE · U3

[1]

Calculate the probability that a measurement of the energy at $t = 0$ gives $+5.0$ meV.

- 1 $P(+5.0 \text{ meV}) = |\langle E_+|1\rangle|^2 = |3/\sqrt{10}|^2 = 0.90$

(d) DERIVE · U2

[3]

Starting from the time-dependent Schrödinger equation, derive an expression for the probability $P_2(t)$ that the electron is found in $|2\rangle$ at time t .

- 1 Because H is time independent, $i\hbar d|\psi\rangle/dt = H|\psi\rangle$ gives $|\psi(t)\rangle = \sum e^{(-iE_{\pm}t/\hbar)} |E_{\pm}\rangle \langle E_{\pm}|1\rangle$, with $\langle E_+|1\rangle = 3/\sqrt{10}$, $\langle E_-|1\rangle = 1/\sqrt{10}$ and $|E_-\rangle = (|1\rangle + 3i|2\rangle)/\sqrt{10}$ (or, since $H^2 = (5.0 \text{ meV})^2 I$, uses $e^{(-iHt/\hbar)} = \cos(Et/\hbar) I - i \sin(Et/\hbar) H/E$)
- 2 $\langle 2|\psi(t)\rangle = (3/\sqrt{10})(-i/\sqrt{10})e^{(-iEt/\hbar)} + (1/\sqrt{10})(3i/\sqrt{10})e^{(iEt/\hbar)} = (3i/10)(e^{(iEt/\hbar)} - e^{(-iEt/\hbar)}) = -(3/5) \sin(Et/\hbar)$, where $E = 5.0$ meV
- 3 $P_2(t) = (9/25) \sin^2(Et/\hbar) = 0.36 \sin^2[(5.0 \text{ meV})t/\hbar]$

Building $|E_-\rangle$ by orthogonality to $|E_+\rangle$ is where a forgotten complex conjugate costs marks: dropping it gives $(|1\rangle - 3i|2\rangle)/\sqrt{10}$, which is not an eigenvector of H and leads to $P_2 = 0.36 \cos^2(Et/\hbar)$, non-zero at $t = 0$ when the electron is certainly in $|1\rangle$. The other common error is to attach separate phases to the dot-basis amplitudes of $|1\rangle$; only energy eigenstates evolve by a phase alone.

(e) CALCULATE · U3

[2]

Calculate the earliest time after $t = 0$ at which P_2 is greatest, and state that greatest value.

- 1 P_2 is greatest, at 0.36, when $(5.0 \text{ meV})t/\hbar = \pi/2$
- 2 $t = \pi\hbar/(2 \times 5.0 \text{ meV}) = \pi \times 6.582 \times 10^{-16} \text{ eV s} / (1.0 \times 10^{-2} \text{ eV}) = 2.1 \times 10^{-13} \text{ s}$

A candidate who expects the electron to transfer completely has overlooked the unequal diagonal elements: the detuning caps the transfer at $|H_{12}|^2/E_+^2 = 9/25$.

An electron of mass $m_e = 9.109 \times 10^{-31} \text{ kg}$ is bound in a symmetric one-dimensional square well occupying $-a < x < a$, with $a = 0.500 \text{ nm}$. Its potential energy is zero inside the well and $V_0 = 1.20 \text{ eV}$ outside it. Measure the energy E from the bottom of the well and write $k = \sqrt{(2m_e E)/\hbar}$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_e V_0)}$. The even bound states satisfy $z \tan z = \sqrt{(z_0^2 - z^2)}$ and the odd bound states satisfy $-z \cot z = \sqrt{(z_0^2 - z^2)}$. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$ and $e = 1.602 \times 10^{-19} \text{ C}$.

(a) DETERMINE · U3

[3]

Calculate z_0 , and hence determine how many bound states the well holds.

- 1 $z_0 = (0.500 \times 10^{-9} \text{ m} / 1.055 \times 10^{-34} \text{ J s}) \times \sqrt{(2 \times 9.109 \times 10^{-31} \text{ kg} \times 1.20 \times 1.602 \times 10^{-19} \text{ J})}$, with V_0 converted to joules
- 2 $z_0 = 2.80$ (dimensionless)
- 3 An even root always exists; an odd root exists because $z_0 > \pi/2 = 1.57$; a second even root would need $z_0 > \pi = 3.14$. So the well holds 2 bound states, equivalently $\text{floor}(2z_0/\pi) + 1 = \text{floor}(1.79) + 1 = 2$

Leaving V_0 in electronvolts inside the square root gives $z_0 \approx 7 \times 10^9$ and a well with billions of bound states. A dimensionless z_0 of order one is the quickest unit check in the question.

(b) SHOW THAT · U3

[2]

Show that $z = 1.15$ satisfies the even-parity condition to within 1%.

- 1 Left side: $1.15 \times \tan(1.15 \text{ rad}) = 1.15 \times 2.234 = 2.57$
- 2 Right side: $\sqrt{(2.80^2 - 1.15^2)} = 2.55$ (2.56 with $z_0 = 2.805$), which differs from 2.57 by less than 1% (0.45% to 0.66%, depending on the z_0 used)

Evaluating $\tan 1.15$ in degrees gives a left side of 0.023 and apparent failure. Every trigonometric argument in a matching condition is in radians.

(c) COMPARE · U3

[3]

Calculate the ground-state energy of the electron, and compare it with the ground-state energy the electron would have in an infinitely deep well of the same width $2a$.

- 1 $E_1 = \hbar^2 z^2 / (2m_e a^2) = V_0 (z/z_0)^2 = 1.20 \text{ eV} \times (1.15/2.80)^2 = 0.20 \text{ eV}$
- 2 Infinite well of width $2a = 1.00 \text{ nm}$: $E_{1,\infty} = \pi^2 \hbar^2 / [2m_e (2a)^2] = 6.03 \times 10^{-20} \text{ J} = 0.376 \text{ eV}$
- 3 The finite-well value is about 0.54 of the infinite-well value because the wavefunction extends into the classically forbidden region: spread over a greater effective width, it has less curvature and less kinetic energy. Both are non-zero, as confinement to a finite region requires

Using a instead of $2a$ in the infinite-well formula quadruples $E_{1,\infty}$ to 1.51 eV, which lies above V_0 and should itself signal the error.

(d) SKETCH · U1

[2]

On one set of axes of ψ against x , from $x = -2a$ to $x = +2a$, sketch the wavefunctions of the two bound states. Mark $x = -a$ and $x = +a$.

- 1 Ground state: even with no node, cosine-like inside the well, joining smoothly (ψ and $d\psi/dx$ continuous) onto exponential decay outside, and non-zero at $x = \pm a$
- 2 First excited state: odd with a single node at $x = 0$, and with exponential tails that decay more slowly than the ground state's, because it lies nearer the top of the well ($ka \approx 1.7$ against 2.6)

The mark most often lost is for drawing wavefunctions that reach zero at $x = \pm a$, as in the infinite well. The non-zero value at the walls is the feature that lowered the energy in (c).

Slow neutrons, which have spin $\frac{1}{2}$ and gyromagnetic ratio $\gamma_n = -1.832 \times 10^8 \text{ rad s}^{-1} \text{ T}^{-1}$, leave a polariser in the state $|+z\rangle$. The spin operator is $S = (\hbar/2)\sigma$; in the $\{|+z\rangle, |-z\rangle\}$ basis σ_y has rows $(0, -i)$ and $(i, 0)$, and $\exp(-i\alpha\sigma_y) = \cos \alpha I - i \sin \alpha \sigma_y$ for real α . An analyser set along a unit vector $\hat{n}$ transmits only neutrons found in the $+\hbar/2$ eigenstate of $\hat{n} \cdot S$, and blocks the rest.

- (a) **CALCULATE** · U3 [2]
- The neutrons pass first through an analyser whose $\hat{n}$ lies in the xz -plane at 45° to the $+z$ axis, and then through an analyser set along $-z$, which transmits only $|-z\rangle$. Calculate the fraction of the neutrons leaving the polariser that emerge from the second analyser.
- $|+\hat{n}\rangle = \cos 22.5^\circ |+z\rangle + \sin 22.5^\circ |-z\rangle$, so the first analyser transmits $|\langle +\hat{n}|+z\rangle|^2 = \cos^2 22.5^\circ = 0.854$
 - The second analyser transmits $|\langle -z|+\hat{n}\rangle|^2 = \sin^2 22.5^\circ = 0.146$ of these, so the fraction is $0.854 \times 0.146 = 0.125$ (exactly $1/8$)
- Using the angle between the axes instead of half of it, $\cos^2 45^\circ \times \sin^2 45^\circ$, gives 0.25. The half-angle is the signature of spin one-half, for which opposite directions in space are orthogonal states.*
- (b) **EXPLAIN** · U1 [2]
- With the first analyser removed, no neutrons emerge from the $-z$ analyser. Explain why inserting an analyser that removes neutrons from the beam increases the number emerging, and identify which step in the sequence is not unitary.
- $|+z\rangle$ and $|-z\rangle$ are orthogonal, so with no intermediate analyser the amplitude to pass the $-z$ analyser is zero; the 45° analyser replaces the state by $|+\hat{n}\rangle$, which has amplitude $\sin 22.5^\circ$ on $|-z\rangle$
 - Each analyser performs a projective measurement: projecting with $|+\hat{n}\rangle\langle +\hat{n}|$ and renormalising what survives is not unitary, since it discards part of the beam and cannot be reversed, whereas free flight between analysers is unitary
- Saying that the first analyser 'rotates the spin' earns nothing. A rotation is unitary and loses no neutrons, whereas this analyser blocks about 15% of the beam; the increase comes from a projection that destroys the orthogonality between the prepared state and the final filter.*
- (c) **SHOW THAT** · U2 [3]
- The 45° analyser is removed. Instead, the $|+z\rangle$ neutrons cross a region of uniform magnetic field B along $+y$, where the Hamiltonian is $H = -\gamma_n B S_y$, taking a time t . Show that on leaving the field the spin state is $\cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$, where $\omega = |\gamma_n|B$.
- Because $\gamma_n < 0$, $H = |\gamma_n|B S_y = (\hbar\omega/2)\sigma_y$ with $\omega = |\gamma_n|B$; H is time independent, so the propagator is $U = \exp(-iHt/\hbar) = \exp(-i(\omega t/2)\sigma_y)$
 - By the given identity, $U = \cos(\omega t/2) I - i \sin(\omega t/2) \sigma_y$
 - $\sigma_y|+z\rangle = i|-z\rangle$, so $U|+z\rangle = \cos(\omega t/2)|+z\rangle - i \sin(\omega t/2)(i|-z\rangle) = \cos(\omega t/2)|+z\rangle + \sin(\omega t/2)|-z\rangle$
- Dropping the sign of γ_n gives $\cos(\omega t/2)|+z\rangle - \sin(\omega t/2)|-z\rangle$, a spin turning towards $-x$ instead of $+x$. The z probabilities agree, so only a careful candidate notices, but the sense of precession is physical and measurable along x .*
- (d) **CALCULATE** · U3 [2]
- The neutrons travel at 2200 m s^{-1} through a field region 5.00 cm long. Calculate the field B that turns $|+z\rangle$ into $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$.
- $|+x\rangle$ requires $\omega t/2 = \pi/4$, so $\omega t = \pi/2$, with $t = L/v = 0.0500 \text{ m} / 2200 \text{ m s}^{-1} = 2.27 \times 10^{-5} \text{ s}$
 - $B = \pi/(2|\gamma_n|t) = \pi / (2 \times 1.832 \times 10^8 \text{ rad s}^{-1} \text{ T}^{-1} \times 2.27 \times 10^{-5} \text{ s}) = 3.77 \times 10^{-4} \text{ T}$
- Setting $\omega t/2 = \pi/2$ gives $7.55 \times 10^{-4} \text{ T}$, the field that turns the spin all the way to $-z$. The spin vector turns through ωt , twice the angle that appears in the spinor components.*
- (e) **PREDICT** · U1 [1]
- Predict the fraction of the neutrons that a $+z$ analyser placed after the field region would transmit if the same field pointed along $+z$ instead of $+y$, and give the reason.
- All of them, a fraction of 1: with B along z , H is proportional to S_z , so $|+z\rangle$ is a stationary state that acquires only a global phase, and the z populations do not change
- Candidates who reason that 'the spin precesses, so some neutrons are lost' have not asked about the axis of precession: a spin along the field has nothing to precess about it.*

A single $^{40}\text{Ca}^+$ ion of mass $m = 6.64 \times 10^{-26} \text{ kg}$ is held in an ion trap whose potential along the trap axis is harmonic with angular frequency $\omega = 2\pi \times 1.20 \text{ MHz} = 7.54 \times 10^6 \text{ rad s}^{-1}$. Along that axis the position and momentum operators are $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ and $p = i\sqrt{m\hbar\omega/2}(a^\dagger - a)$, with $[a, a^\dagger] = 1$ and number states $|n\rangle$ of energy $(n + \frac{1}{2})\hbar\omega$. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$ and $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$.

(a) ESTIMATE · U4

[2]

After laser cooling, the axial motion is in thermal equilibrium at $T = 0.50 \text{ mK}$. Estimate the mean number of vibrational quanta $\bar{n}$, stating the approximation you make and why it is justified.

1 $\hbar\omega/(k_B T) = (1.055 \times 10^{-34} \text{ J s} \times 7.54 \times 10^6 \text{ rad s}^{-1})/(1.381 \times 10^{-23} \text{ J K}^{-1} \times 5.0 \times 10^{-4} \text{ K}) = 0.115$, so $k_B T \gg \hbar\omega$ and the high-temperature (equipartition) limit $\bar{n} \approx k_B T/\hbar\omega$ applies

2 $\bar{n} \approx 1/0.115 \approx 9$ (8.7); the Bose-Einstein value $1/(e^{0.115} - 1) = 8.2$ is also accepted, as is any value from 8 to 9

The regime is decided by comparing $k_B T$ with $\hbar\omega$, not by how small T looks. Candidates who call 0.5 mK 'ultracold' and place the ion in its ground state miss about eight quanta of occupation.

(b) SHOW THAT · U2

[2]

Further cooling and a laser pulse prepare the state $|\psi(0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Show that $\langle x \rangle(t) = \sqrt{\hbar/2m\omega} \cos \omega t$.

1 Each number state evolves as $e^{-i(n + \frac{1}{2})\omega t}|n\rangle$, so up to a global phase $|\psi(t)\rangle = (|0\rangle + e^{-i\omega t}|1\rangle)/\sqrt{2}$

2 Only $\langle 0|a|1\rangle = 1$ and $\langle 1|a^\dagger|0\rangle = 1$ contribute, giving $\langle a \rangle = \frac{1}{2}e^{-i\omega t}$ and $\langle a^\dagger \rangle = \frac{1}{2}e^{i\omega t}$, so $\langle x \rangle = \sqrt{\hbar/2m\omega}(\langle a \rangle + \langle a^\dagger \rangle) = \sqrt{\hbar/2m\omega} \cos \omega t$

Losing the relative phase $e^{-i\omega t}$ gives a constant $\langle x \rangle = \sqrt{\hbar/2m\omega}$. A stationary expectation value from a superposition of two different energies should prompt a check.

(c) DETERMINE · U2

[4]

Determine the product $\Delta x \Delta p$ at $t = 0$ and at $\omega t = \pi/4$, expressing each as a multiple of $\hbar$.

1 $(a + a^\dagger)^2 = a^2 + a^{\dagger 2} + 2N + 1$, and a^2 and $a^{\dagger 2}$ cannot connect $|0\rangle$ with $|1\rangle$, so $\langle x^2 \rangle = (\hbar/2m\omega)(2\langle N \rangle + 1) = \hbar/(m\omega)$ with $\langle N \rangle = \frac{1}{2}$; likewise $-(a^\dagger - a)^2 = 2N + 1 - a^2 - a^{\dagger 2}$ gives $\langle p^2 \rangle = m\hbar\omega$

2 $\langle p \rangle = i\sqrt{m\hbar\omega/2}(\langle a^\dagger \rangle - \langle a \rangle) = -\sqrt{m\hbar\omega/2} \sin \omega t$, which equals $m d\langle x \rangle/dt$ as Ehrenfest's theorem requires

3 $(\Delta x)^2 = (\hbar/2m\omega)(2 - \cos^2 \omega t)$ and $(\Delta p)^2 = (m\hbar\omega/2)(2 - \sin^2 \omega t)$, so $\Delta x \Delta p = (\hbar/2)\sqrt{(2 - \cos^2 \omega t)(2 - \sin^2 \omega t)}$

4 At $t = 0$, $\Delta x \Delta p = (\hbar/2)\sqrt{2} = 0.71\hbar$; at $\omega t = \pi/4$, $\Delta x \Delta p = (\hbar/2)(3/2) = 0.75\hbar$; both exceed $\hbar/2$

The usual error is to take $(\Delta x)^2 = \langle x^2 \rangle$, which holds only when $\langle x \rangle = 0$, as in a stationary state but not here. That route gives $\Delta x \Delta p = \hbar$ at every time and misses the oscillation of the product entirely.

(d) EXPLAIN · U1

[2]

Explain why $\Delta x \Delta p$ exceeds $\hbar/2$ at every time for this state, although the ground state $|0\rangle$ reaches $\hbar/2$.

- 1 Equality in $\Delta x \Delta p \geq \hbar/2$ requires both inequalities of the derivation to be saturated, Cauchy-Schwarz and the dropping of $\text{Re}\langle f|g\rangle$, which together demand $(p - \langle p \rangle)|\psi\rangle = i\lambda(x - \langle x \rangle)|\psi\rangle$ with λ real, and normalisability needs $\lambda > 0$; the only solutions are Gaussian wavefunctions
- 2 In position space this state is proportional to $(1 + c(t)x) e^{-(m\omega x^2/2\hbar)}$ with $c(t) = e^{(-i\omega t)\sqrt{2m\omega/\hbar}}$: a Gaussian multiplied by a polynomial, which is not a Gaussian at any t , so the bound is never reached, whereas $|0\rangle$ is a Gaussian and reaches it
'Superpositions carry more uncertainty' earns nothing: a coherent state is a superposition of every $|n\rangle$ and still reaches $\hbar/2$. The deciding property is the Gaussian shape, which follows from the condition for equality.

13

CORE
COURSE

DATA ANALYSIS · DEMANDING · UPV-06.05 · UPV-06.09

[10]

A scanning tunnelling microscope tip is held above a clean metal surface at a fixed bias of 0.100 V. With the feedback loop switched off, the tip is withdrawn in steps, and the tunnelling current I is recorded against Δd , the increase in the tip-surface gap from its starting value. In the opaque-barrier limit $I = I_0 \exp(-2\kappa\Delta d)$, where $\kappa = \sqrt{2m_e\phi}/\hbar$ and ϕ is the apparent barrier height. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $e = 1.602 \times 10^{-19}$ C.

(a) STATE · U2

[1]

State the gradient of a graph of $\ln(I/nA)$ against Δd , in terms of κ .

- 1 Gradient = -2κ , since $\ln(I/nA) = \ln(I_0/nA) - 2\kappa\Delta d$
Candidates who write $+2\kappa$, or κ , carry a sign or a factor of two into every later part; the 2 is there because the current follows the probability, the squared modulus of an amplitude decaying as $e^{(-\kappa\Delta d)}$.

(b) DETERMINE · U4

[3]

Determine κ from the data.

- 1 $\ln(I/nA)$ for the six readings: 0.916, 0.122, -0.753 , -1.523 , -2.382 , -3.170
- 2 The points lie on a straight line (no residual larger than about 0.03) of gradient $-2.05 \times 10^{-2} \text{ pm}^{-1}$ (least squares -0.0205 pm^{-1} ; accept -0.0200 to -0.0210 pm^{-1})
- 3 $\kappa = 0.0205 \text{ pm}^{-1} / 2 = 1.03 \times 10^{-2} \text{ pm}^{-1} = 1.03 \times 10^{10} \text{ m}^{-1}$ (accept 1.00×10^{10} to $1.05 \times 10^{10} \text{ m}^{-1}$)

Two errors dominate: using $\log_{10}$ instead of $\ln$, which scales the gradient by 0.434 and leads to a barrier height about five times too small, and failing to convert the gradient from pm^{-1} to m^{-1} .

(c) CALCULATE · U3

[2]

Calculate the apparent barrier height ϕ in electronvolts.

- 1 $\phi = \hbar^2 \kappa^2 / (2m_e) = (1.055 \times 10^{-34} \text{ J s})^2 \times (1.03 \times 10^{10} \text{ m}^{-1})^2 / (2 \times 9.109 \times 10^{-31} \text{ kg}) = 6.4 \times 10^{-19} \text{ J}$
- 2 $\phi = 4.0 \text{ eV}$ (accept 3.8 to 4.2 eV, following from the κ found in (b))

Because ϕ goes as κ^2 , a 2% error in the gradient becomes a 4% error in ϕ . Rounding κ to $1 \times 10^{10} \text{ m}^{-1}$ before squaring gives 3.8 eV.

(d) ESTIMATE · U4

[2]

In constant-current imaging the feedback loop moves the tip to hold I fixed. Estimate the change in gap that changes the current by 10%, stating the assumption you make.

- 1 Assumes the exponential law holds over a few picometres with the fitted κ and an unchanged prefactor I_0 , so that the change in gap is $\ln(1.10)/(2\kappa)$ for a 10% rise in current, or $\ln(1/0.90)/(2\kappa)$ for a 10% fall
- 2 $\Delta d = 0.0953 / 0.0205 \text{ pm}^{-1} \approx 5 \text{ pm}$ (4.6 pm for a rise, 5.1 pm for a fall, and 4.9 pm from the small-change form $0.10/(2\kappa)$ are all accepted), so holding the current to 10% fixes the gap to a few picometres, about 2% of an atomic diameter

The estimate rests on κ being the same for a displacement of a few picometres as over the 200 pm range fitted, and an answer that does not say so loses the first mark whatever its number.

(e) EVALUATE · U4

[2]

Photoemission gives the work function of this surface as 4.60 eV. A student concludes that either the STM data or the tunnelling model must be wrong. Evaluate this conclusion.

- 1 The fitted $\phi \approx 4.0 \text{ eV}$ is about 13% below 4.60 eV, but the apparent barrier height is not the work function: the image-charge potential rounds and lowers the barrier between tip and surface, and the tip's own work function also enters, so a lower apparent height is expected
- 2 The $\ln I$ values lie on a straight line to within about 0.03 over the whole 200 pm, which supports the exponential tunnelling model; the conclusion is not justified, because the data fit the model and the gap from 4.60 eV is explained by what ϕ represents

Strong answers separate the two things the student has run together: whether the model fits, which the straightness of the line decides, and what the fitted parameter means, which is an image-lowered average barrier rather than the work function.

14
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UPV-08.03 · UPV-09.03 · UPV-09.04

[14]

For Hermitian operators A and B and a normalised state $|\psi\rangle$, write $\langle A \rangle = \langle \psi | A | \psi \rangle$ and $(\Delta A)^2 = \langle \psi | (A - \langle A \rangle)^2 | \psi \rangle$. The Cauchy-Schwarz inequality states that $\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$ for any kets $|f\rangle$ and $|g\rangle$. The components of an angular momentum obey $[J_x, J_y] = i\hbar J_z$ and its cyclic permutations, $J_{\pm} = J_x \pm iJ_y$, and the simultaneous eigenstates $|j, m\rangle$ of J^2 and J_z satisfy $J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$, $J_z |j, m\rangle = m\hbar |j, m\rangle$ and $J_{\pm} |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$.

(a) DERIVE · U2

[4]

Starting from the Cauchy-Schwarz inequality, derive the Robertson uncertainty relation, a lower bound on $\Delta A \Delta B$ in terms of the commutator $[A, B]$.

- 1 Defines $|f\rangle = (A - \langle A \rangle)|\psi\rangle$ and $|g\rangle = (B - \langle B \rangle)|\psi\rangle$; because the shifted operators are Hermitian, $\langle f | f \rangle = (\Delta A)^2$ and $\langle g | g \rangle = (\Delta B)^2$
- 2 Cauchy-Schwarz gives $(\Delta A)^2 (\Delta B)^2 \geq |\langle f | g \rangle|^2 \geq (\text{Im} \langle f | g \rangle)^2 = [(\langle f | g \rangle - \langle g | f \rangle)/2i]^2$
- 3 $\langle f | g \rangle - \langle g | f \rangle = \langle \psi | [A - \langle A \rangle, B - \langle B \rangle] | \psi \rangle = \langle [A, B] \rangle$, because the numbers $\langle A \rangle$ and $\langle B \rangle$ commute with every operator
- 4 Hence $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$, a real bound because $[A, B]$ is anti-Hermitian and its expectation value is purely imaginary

Applying Cauchy-Schwarz to $A|\psi\rangle$ and $B|\psi\rangle$ without subtracting the means bounds $\langle A^2 \rangle \langle B^2 \rangle$, not the variances. The mean shift is the step that turns an inequality about vectors into an uncertainty relation.

(b) **SHOW THAT** · U2 [4]

Show that in the state $|j, m\rangle$, $\langle J_x \rangle = \langle J_y \rangle = 0$ and $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{2}\hbar^2[j(j+1) - m^2]$.

- 1 $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/2i$; $J_{\pm}$ changes m by ± 1 and $\langle j, m | j, m \pm 1 \rangle = 0$, so $\langle J_x \rangle = \langle J_y \rangle = 0$
- 2 $J_x^2 = \frac{1}{4}(J_+^2 + J_-^2 + J_+J_- + J_-J_+)$ and $J_y^2 = -\frac{1}{4}(J_+^2 + J_-^2 - J_+J_- - J_-J_+)$; the $J_{\pm}^2$ terms change m by ± 2 and have zero expectation value, so $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{4}\langle J_+J_- + J_-J_+ \rangle$
- 3 $J_+J_- + J_-J_+ = 2(J_x^2 + J_y^2) = 2(J^2 - J_z^2)$; equivalently $\langle J_+J_- \rangle + \langle J_-J_+ \rangle = \hbar^2[j(j+1) - m(m-1)] + \hbar^2[j(j+1) - m(m+1)]$ from the given ladder elements
- 4 $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{2}\langle J^2 - J_z^2 \rangle = \frac{1}{2}\hbar^2[j(j+1) - m^2]$

The result is given, so every mark is for the ladder algebra. Asserting $\langle J_x^2 \rangle = \langle J_y^2 \rangle$ 'by symmetry' earns nothing towards the value; the candidate must still show that the $J_{\pm}^2$ cross terms drop out.

(c) **DETERMINE** · U2 [3]

Determine the states $|j, m\rangle$ in which $\Delta J_x \Delta J_y$ is exactly equal to the Robertson bound for J_x and J_y .

- 1 With both means zero, (b) gives $\Delta J_x \Delta J_y = \frac{1}{2}\hbar^2[j(j+1) - m^2]$
- 2 The Robertson bound is $\frac{1}{2}|\langle [J_x, J_y] \rangle| = \frac{1}{2}\hbar| \langle J_z \rangle | = \frac{1}{2}\hbar^2|m|$
- 3 Equality requires $j(j+1) - m^2 = |m|$, that is $(j - |m|)(j + |m| + 1) = 0$, and since $j + |m| + 1 > 0$ this means $|m| = j$: only the extreme states $|j, \pm j\rangle$ reach the bound, which for spin $\frac{1}{2}$ includes both S_z eigenstates

Solving the quadratic in $|m|$ gives the roots j and $-(j+1)$. Rejecting the second, because $|m|$ cannot be negative, is the step most often left unstated, and without it the answer is not determined.

(d) **EVALUATE** · U1 [3]

A student argues: 'In the state $|j = 1, m = 0\rangle$ the Robertson bound for J_x and J_y is zero, so a careful enough experiment could find both J_x and J_y with no spread in that state.' Evaluate this argument.

- 1 In $|1, 0\rangle$, (b) gives $\Delta J_x = \Delta J_y = \hbar$, so $\Delta J_x \Delta J_y = \hbar^2$: the spreads belong to the prepared state, no experimental care reduces them, and a lower bound of zero is not a value the state attains
- 2 Zero spread in both would need a common eigenvector of J_x and J_y ; then $[J_x, J_y]|\psi\rangle = 0$ forces $J_z|\psi\rangle = 0$, the cyclic relations then force $J_x|\psi\rangle = J_y|\psi\rangle = 0$, so $J^2|\psi\rangle = 0$ and $j = 0$
- 3 Judgement: the argument fails. A vanishing bound is necessary but not sufficient for joint sharpness, and no state with $j \geq \frac{1}{2}$, $|1, 0\rangle$ included, has both components sharp

This part separates candidates who treat the Robertson relation as the whole content of uncertainty from those who see it as one inequality with a state-dependent right-hand side. The strongest answers give the commutator argument, which settles every state at once, rather than checking $|1, 0\rangle$ alone.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 13 MARKS · 18%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 29 MARKS · 40%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

20 MARKS · 28%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

10 MARKS · 14%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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