

University Quantum Field Theory I · Final Examination

The whole course, weighted towards its second half: QED at tree level by traces and crossing, C, P and T on quantised fields, the self-energy and the Ward-Takahashi identity, vacuum polarisation and the running of α , $g - 2$ with the infrared cancellation, and the Lamb shift at order $\alpha(Z\alpha)^4$, with the first half's propagators, rates and one-loop ϕ^4 sampled.

SUGGESTED TIME 150 minutes	TOTAL MARKS 117	QUESTIONS 19	MULTIPLE CHOICE 10
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SECTION **UQFTI · Final · University Quantum Field Theory I — Final examination**

IN THE FORMAT OF A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations

DEMAND **Recall 2 · Routine 7 · Demanding 7 · Discriminating 3**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS UQFTI-01 · UQFTI-02 · UQFTI-03 · UQFTI-04 · UQFTI-05 · UQFTI-06 · UQFTI-07 · UQFTI-08 · UQFTI-09 · UQFTI-10 · UQFTI-11 · UQFTI-12 · UQFTI-13 · UQFTI-14 · UQFTI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

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University Quantum Field Theory I · Final Examination

Answer all questions. 117 marks in 150 minutes.

1

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07

[1]

For a free real scalar field of mass m , in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$, the Feynman propagator is $D_F(x - y) = \int d^4p / (2\pi)^4 i e^{(-ip \cdot (x - y))} / (p^2 - m^2 + i\varepsilon)$, with $E_p = \sqrt{(|p|^2 + m^2)}$ for the three-momentum p and $\varepsilon \rightarrow 0^+$. The p^0 integral is done first, by closing the contour in whichever half of the complex p^0 plane makes the exponential decay. Which statement is correct?

- A** The $i\varepsilon$ places both poles just below the real p^0 axis, so D_F vanishes whenever $x^0 < y^0$, as the response of a causal system must.
- B** The poles sit at $p^0 = \pm(E_p - i\varepsilon)$; for $x^0 > y^0$ the contour closes in the lower half-plane around $p^0 = +E_p$ and gives $\langle 0 | \phi(x) \phi(y) | 0 \rangle$, and for $x^0 < y^0$ it closes in the upper half-plane around $p^0 = -E_p$ and gives $\langle 0 | \phi(y) \phi(x) | 0 \rangle$, so the placement of the $i\varepsilon$ is the time ordering.
- C** Because it is time-ordered, $D_F(x - y)$ vanishes when $x - y$ is spacelike, and that is how the theory keeps measurements outside each other's light cones from interfering.
- D** For $x^0 > y^0$ the contour closes in the upper half-plane around $p^0 = -E_p$, so for that time order D_F describes an antiparticle carried from y to x .

Answer

2

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-03.02 · UQFTI-03.06

[1]

Two theories are expanded to second order in their coupling, in natural units. In theory (i) a real scalar ϕ has $L_{\text{int}} = -(g/3!)\phi^3$. In theory (ii), the scalar Yukawa model, a complex scalar ψ and a real scalar ϕ have $L_{\text{int}} = -g\psi^\dagger\psi\phi$. In each, the order- g^2 correction to the ϕ two-point function $\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle$ contains a one-loop diagram in which x_1 joins one vertex, x_2 joins the other, and the two vertices are linked by two internal lines; in (ii) these form a closed ψ loop. If each diagram is written with its vertex factors and propagators and an overall $1/S$, what are the symmetry factors S in (i) and (ii)?

- A** $S = 2$ in both
- B** $S = 6$ in (i) and $S = 1$ in (ii)
- C** $S = 1$ in both
- D** $S = 2$ in (i) and $S = 1$ in (ii)

Answer

3

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-04.02 · UQFTI-04.03

[1]

In ϕ^4 theory, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - (\lambda/4!)\phi^4$ with $0 < \lambda \ll 1$, the single particle has physical mass m . The Källén-Lehmann representation writes the exact two-point function as $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty (dM^2/2\pi) \rho(M^2) D_F(x-y; M^2)$, with $\rho(M^2) = 2\pi Z\delta(M^2 - m^2)$ plus a continuum. Where does the continuum begin, and what bounds Z ?

- A** At $M^2 = 4m^2$, the threshold for two particles, with $0 \leq Z \leq 1$ because ρ is non-negative and the canonical commutator fixes $\int (dM^2/2\pi)\rho(M^2) = 1$.
- B** At $M^2 = 4m^2$, with $Z \geq 1$, because interactions add multiparticle states to the one-particle state and so increase the weight of the pole.
- C** At $M^2 = 9m^2$, because the symmetry $\phi \rightarrow -\phi$ stops ϕ from connecting the vacuum to states with an even number of particles; $0 \leq Z \leq 1$, because ρ is non-negative and the canonical commutator fixes $\int (dM^2/2\pi)\rho(M^2) = 1$.
- D** Immediately above $M^2 = m^2$, because a particle can always be accompanied by arbitrarily soft quanta, so the pole sits at the start of a branch cut.

Answer

4

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQFTI-06.01

[1]

In d spacetime dimensions a dimensionless action with a canonical kinetic term gives a scalar field the mass dimension $(d-2)/2$. Which interaction has a dimensionless coupling, so that the theory is renormalisable but not super-renormalisable?

- A** $g\phi^3$ in $d = 6$
- B** $g\phi^3$ in $d = 4$
- C** $\lambda\phi^4$ in $d = 3$
- D** $g\phi^6$ in $d = 4$

Answer

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-07.07

[1]

A spin-zero boson ϕ of mass 2.40 GeV decays to a fermion-antifermion pair, the fermion being a Dirac particle of mass 0.90 GeV. Compare a scalar coupling $g\phi\bar{\psi}\psi$ with a pseudoscalar coupling $ig\phi\bar{\psi}\gamma^5\psi$ of the same real strength g . In both cases $\Gamma = |p| \sum |M|^2 / (8\pi m_\phi^2)$, with the sum over final spins and $|p|$ the momentum of either fermion in the rest frame of ϕ . What is $\Gamma_{\text{scalar}}/\Gamma_{\text{pseudoscalar}}$?

- A** 0.66
- B** 1.00
- C** 2.29
- D** 0.44

Answer

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-08.04 · UQFTI-08.06

[1]

In a general covariant gauge the photon propagator is $-i[g_{\mu\nu} - (1 - \xi)k_{\mu}k_{\nu}/k^2]/(k^2 + i\epsilon)$. In the tree amplitude for $e^-(p_1)e^+(p_2) \rightarrow \mu^-(p_3)\mu^+(p_4)$, the virtual photon carries $k = p_1 + p_2 = p_3 + p_4$, with $k^2 = s$. Why does the amplitude not depend on ξ ?

- A Because the photon is massless, $k^2 = 0$, so the $k_{\mu}k_{\nu}$ term is multiplied by zero.
- B Because k_{μ} contracted into the electron current gives $\bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = (m_e - m_e)\bar{v}(p_2)u(p_1) = 0$ by the Dirac equation, and likewise on the muon line, so the $k_{\mu}k_{\nu}$ term vanishes for any ξ .
- C Because one may choose $\xi = 1$, and a choice of gauge cannot change a physical amplitude.
- D Because the $k_{\mu}k_{\nu}$ term is suppressed by m_e^2/s , and vanishes once the electron mass is neglected.

Answer

7

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-09.07

[1]

At $\sqrt{s} \gg m_{\mu}$ the reaction $e^+e^- \rightarrow \mu^+\mu^-$ by one-photon exchange proceeds through four helicity combinations, because the vector coupling joins a right-handed fermion only to a left-handed antifermion, and vice versa. The e^- moves along $+z$, and θ is the angle between the outgoing μ^- and the incoming e^- . How does the amplitude for $e^-_R e^+_L \rightarrow \mu^-_L \mu^+_R$ depend on θ , and why?

- A As $(1 - \cos \theta)$: the initial pair has angular momentum $+1$ along the beam and the final pair -1 along the μ^- direction, so the amplitude is proportional to the rotation matrix element $d^1_{1,-1}(\theta)$, which vanishes in the forward direction.
- B As $(1 + \cos \theta)$: helicity is conserved at each vertex, so the μ^- prefers to follow the e^- .
- C It vanishes at every angle: helicity is conserved along a fermion line, so a right-handed electron cannot produce a left-handed muon.
- D As $\sin \theta$: the virtual photon carries one unit of angular momentum, so every helicity amplitude has the angular dependence of a dipole.

Answer

8

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQFTI-11.08 · UQFTI-11.09

[1]

A positronium level with orbital angular momentum L and total spin S has charge-conjugation parity $C = (-1)^{L+S}$, and a state of n photons has $C = (-1)^n$. Landau and Yang's theorem, quoted, forbids a state of total angular momentum $J = 1$ from decaying into two photons. Which level can annihilate directly into two photons?

- A 1^3S_1
- B 2^1P_1
- C 2^3P_1
- D 2^3P_2

Answer

9

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQFTI-12.06 · UQFTI-12.07

[1]

In renormalised QED with on-shell conditions the physical charge is $e = Z_2 Z_3^{(1/2)} e_0 / Z_1$, where Z_1 renormalises the vertex, Z_2 the charged field and Z_3 the photon field. A proton's coupling to the photon is dressed by strong interactions that the electron's is not, yet the hydrogen atom is neutral to better than one part in 10^{20} . Why are the renormalised charges of the electron and the proton exactly opposite, given opposite bare charges?

- A** The Ward-Takahashi identity gives $Z_1 = Z_2$ for every charged field, whatever its other interactions, so $e = Z_3^{(1/2)} e_0$, and Z_3 comes from the photon's own self-energy, which is common to every particle the photon meets.
- B** Strong interactions conserve charge, so they leave the proton's vertex unrenormalised.
- C** The electron and the proton both have spin $1/2$, so their vertex corrections are equal.
- D** The on-shell condition is imposed separately for each particle, so each renormalised charge is set equal to the measured one by definition.

Answer

10

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-13.01 · UQFTI-13.02

[1]

At one loop the electron loop gives the photon a self-energy $i\Pi^{\mu\nu}(q)$. Why is it computed in dimensional regularisation rather than with a sharp cutoff Λ on the Euclidean loop momentum?

- A** The sharp cutoff leaves $\Pi(q^2)$ quadratically divergent, which no counterterm can absorb, whereas dimensional regularisation makes the loop finite.
- B** The electron loop diverges in the infrared at $q^2 = 0$, which a cutoff on large momenta cannot regulate.
- C** A sharp cutoff is not invariant under shifts of the loop momentum and breaks gauge invariance: $\Pi^{\mu\nu}$ acquires a term proportional to $g^{\mu\nu}\Lambda^2$ that is not of the transverse form $(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ and would give the photon a mass, whereas dimensional regularisation keeps $\Pi^{\mu\nu}$ transverse.
- D** Both regulators keep $\Pi^{\mu\nu}$ transverse; dimensional regularisation is preferred only because its integrals are easier.

Answer

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model a complex scalar ψ of mass m_ψ and a real scalar ϕ of mass m_ϕ interact through $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\phi$. Take $m_\phi = 0.80$ GeV, $m_\psi = 0.25$ GeV and $g = 0.40$ GeV, with $\hbar = 6.582 \times 10^{-25}$ GeV s and $\hbar c = 0.1973$ GeV fm. For a particle of mass m_ϕ at rest decaying to two distinguishable spinless products, $\Gamma = |p||M|^2/(8\pi m_\phi^2)$, where M is the invariant amplitude and $|p|$ the momentum of either product.

(a) STATE

[1]

State the mass dimension of g , and whether the interaction is renormalisable, super-renormalisable or non-renormalisable.

(b) SHOW THAT

[2]

Show that, at tree level, $\Gamma(\phi \rightarrow \psi\bar{\psi}) = (g^2/16\pi m_\phi)\sqrt{1 - 4m_\psi^2/m_\phi^2}$.

(c) CALCULATE

[2]

Calculate Γ in MeV and the mean lifetime of ϕ in seconds.

(d) DETERMINE

[3]

Two ψ particles scatter at low speed. From the t-channel ϕ -exchange diagram, determine the potential $V(r)$ between them, allowing for the relativistic normalisation of the states, and state whether it attracts and what its range is in femtometres.

(e) PREDICT

[1]

Predict what becomes of ϕ , and of the use of the LSZ formula with ϕ as an external particle, if m_ϕ were lowered to 0.45 GeV with m_ψ unchanged.

Work in natural units with the metric $(+, -, -, -)$. A real scalar ϕ of mass $m = 0.40$ GeV has $\mathcal{L}_{\text{int}} = -(\lambda/4!)\phi^4$ with $\lambda = 0.80$. Two ϕ particles collide head-on with centre-of-mass energy $\sqrt{s} = 2.0$ GeV. For $2 \rightarrow 2$ scattering of equal masses in the centre-of-mass frame, $d\sigma/d\Omega = |M|^2/(64\pi^2 s)$. Two-body Lorentz-invariant phase space for distinguishable particles is $\int d\Pi_2 = |\mathbf{p}_{\text{cm}}|/(4\pi\sqrt{s})$, with $|\mathbf{p}_{\text{cm}}|$ the momentum of either particle. Use $(\hbar c)^2 = 0.3894$ GeV² mb. In minimal subtraction the one-loop β function of the theory is $\beta(\lambda) = \mu d\lambda/d\mu = 3\lambda^2/(16\pi^2)$.

(a) CALCULATE

[3]

Calculate the tree-level amplitude M and the total cross-section for $\phi\phi \rightarrow \phi\phi$ in microbarns.

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(b) SHOW THAT

[2]

At order λ^2 the s -channel one-loop diagram joins p_1 and p_2 at one vertex and p_3 and p_4 at the other, the two vertices being linked by two internal lines. Show by counting Wick contractions that it carries a symmetry factor $1/2$, so that $iM_s = \frac{1}{2}(-i\lambda)^2 \int d^4l/(2\pi)^4 [i/(l^2 - m^2 + i\epsilon)][i/((k-l)^2 - m^2 + i\epsilon)]$, with $k = p_1 + p_2$.

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(c) SHOW THAT

[3]

Show that the imaginary part of the forward amplitude at order λ^2 , which comes from this diagram alone, is $\text{Im } M = \lambda^2\beta/(32\pi)$ with $\beta = \sqrt{1 - 4m^2/s}$, and that it satisfies the optical theorem $\text{Im } M(\text{forward}) = 2E_{\text{cm}}|\mathbf{p}_{\text{cm}}|\sigma_{\text{tot}}$ at $\sqrt{s} = 2.0$ GeV.

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(d) EXPLAIN

[1]

Explain why $\text{Im } M$ is finite although the loop integral itself diverges logarithmically.

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(e) CALCULATE

[2]

Taking $\lambda(\mu_0) = 0.80$ at $\mu_0 = 0.40$ GeV in minimal subtraction, calculate λ at $\mu = 100$ GeV from the one-loop β function, and the scale at which the one-loop coupling would diverge.

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Work in natural units with the metric $(+, -, -, -)$. In QED with an electron and a muon, $e^+e^- \rightarrow \mu^+\mu^-$ proceeds at lowest order through one-photon exchange in the s channel, with vertex $-ie\gamma^\mu$ and Feynman-gauge photon propagator $-ig_{\mu\nu}/(q^2 + i\epsilon)$. Label the momenta $e^-(p_1)e^+(p_2) \rightarrow \mu^-(p_3)\mu^+(p_4)$, neglect the electron mass, keep the muon mass $m_\mu = 0.10566$ GeV, and take $\alpha = e^2/4\pi = 1/137.036$ and $(\hbar c)^2 = 0.3894$ GeV² mb. The tree amplitude is $M = (e^2/s)[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)]$, with $s = (p_1 + p_2)^2$. Use $\sum u \bar{u} = \not{p} + m$ and $\sum v \bar{v} = \not{p} - m$ over spins, $\text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$ and $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$; traces of an odd number of gamma matrices vanish.

(a) SHOW THAT

[3]

Show that the spin-averaged squared amplitude is $\frac{1}{4}\sum|M|^2 = (8e^4/s^2)[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_\mu^2(p_1 \cdot p_2)]$.

(b) SHOW THAT

[3]

In the centre-of-mass frame, with θ the angle between p_3 and p_1 and $\beta = \sqrt{(1 - 4m_\mu^2/s)}$ the muon speed, show that $d\sigma/d\Omega = (\alpha^2\beta/4s)(2 - \beta^2\sin^2\theta)$, and hence that $\sigma = (4\pi\alpha^2/3s) \times \beta(3 - \beta^2)/2$.

(c) CALCULATE

[2]

Calculate σ in nanobarns at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, and, at the lower energy, the ratio of σ to the value $4\pi\alpha^2/3s$ that massless muons would give.

(d) SKETCH

[2]

Sketch on one set of axes the angular distribution at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, each normalised to its own value at $\cos\theta = \pm 1$, that is $(d\sigma/d\Omega)/(d\sigma/d\Omega \text{ at } \cos\theta = 1)$ against $\cos\theta$ from -1 to $+1$, marking the value of each curve at $\cos\theta = 0$.

(e) DETERMINE

[2]

Determine by crossing the spin-averaged $|M|^2$ for $e^-\mu^- \rightarrow e^-\mu^-$ at energies where both masses can be neglected, in the Mandelstam variables of that process, and state the sign that crossing introduces.

Work in natural units with the metric $(+, -, -, -)$. Compton scattering $\gamma(k)e^-(p) \rightarrow \gamma(k')e^-(p')$ has, at tree level, an s-channel diagram, in which the electron absorbs k before emitting k' , and a u-channel diagram, in which it emits k' first. For an electron initially at rest and a photon of energy ω , the spin- and polarisation-summed result integrates to the Klein-Nishina total cross-section $\sigma = 2\pi r_e^2 \{[(1 + \kappa)/\kappa^2][2(1 + \kappa)/(1 + 2\kappa) - \ln(1 + 2\kappa)/\kappa] + \ln(1 + 2\kappa)/(2\kappa) - (1 + 3\kappa)/(1 + 2\kappa)^2\}$, with $\kappa = \omega/m$ and $r_e = \alpha/m$. Crossing the same amplitude into $e^+e^- \rightarrow \gamma\gamma$, with the factor $\frac{1}{2}$ for identical photons, gives for a slow pair the spin-averaged $\sigma_{v_{\text{rel}}} \rightarrow \pi\alpha^2/m^2$ as the relative speed $v_{\text{rel}} \rightarrow 0$. Take $m_e = 0.51100$ MeV, $m_\mu = 105.658$ MeV, $\alpha = 1/137.036$, $\hbar c = 197.327$ MeV fm, $\hbar = 6.582 \times 10^{-22}$ MeV s and $1 \text{ b} = 100 \text{ fm}^2$.

(a) EXPLAIN

[2]

Replacing the polarisation vector of the incoming photon by its momentum k , explain why neither Compton diagram alone satisfies the Ward identity $k_\mu M^{\mu\nu} = 0$, while their sum does.

(b) CALCULATE

[1]

Calculate the Thomson cross-section $(8\pi/3)r_e^2$ in barns.

(c) CALCULATE

[3]

A 511 keV photon from positron annihilation enters the scintillator of a positron-emission tomograph. Calculate the Klein-Nishina cross-section per electron in barns, its ratio to the Thomson value, and the largest kinetic energy the photon can give to a free electron at rest.

(d) DETERMINE

[3]

True muonium is the atom formed by a μ^+ and a μ^- . Treating it as a hydrogen-like atom bound by the Coulomb potential, determine the lifetime of its para ground state, 1^1S_0 , against annihilation into two photons, which is its only significant decay.

(e) EXPLAIN

[1]

Explain why the two Compton diagrams enter with the same sign, whereas the two diagrams of Bhabha scattering, $e^+e^- \rightarrow e^+e^-$, enter with a relative minus sign.

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. Charge conjugation is a unitary operator C on the Fock space of QED. On the Dirac field it acts as $C\psi(x)C^{-1} = M_C\bar{\psi}^T(x)$ and $C\bar{\psi}(x)C^{-1} = -\psi^T(x)M_C^{-1}$, where ^{T} is the transpose in spinor indices and the 4×4 matrix M_C satisfies $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^T$; on the photon field $CA_\mu(x)C^{-1} = -A_\mu(x)$. Write $(-1)^\mu = +1$ for $\mu = 0$ and -1 for $\mu = 1, 2, 3$. Under parity the Hermitian bilinear $i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi$ is multiplied by $-(-1)^\mu(-1)^\nu$ and $F_{\mu\nu}$ by $(-1)^\mu(-1)^\nu$; under time reversal, which is antiunitary, they are multiplied by $(-1)^\mu(-1)^\nu$ and $-(-1)^\mu(-1)^\nu$ respectively; under C both are multiplied by -1 . Each is evaluated at the transformed point. An electric dipole moment d of the electron would add $L_{\text{EDM}} = -(d/2)(i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi)F_{\mu\nu}$ to the Lagrangian, with d real, whose non-relativistic limit is the energy $-d\sigma \cdot E$. An experiment on polar molecules has found $|d| < 4.1 \times 10^{-30}$ e cm. Take $m_e = 0.511$ MeV and $\hbar c = 1.973 \times 10^{-11}$ MeV cm.

(a) SHOW THAT

[3]

Show that the current $\bar{\psi}\gamma^\mu\psi$ is odd under C while $\bar{\psi}\psi$ is even, identifying the step at which the anticommutation of the fields supplies a sign, and say what would become of the electron's coupling to the photon if the Dirac field were quantised with commutators.

(b) DETERMINE

[2]

Determine whether L_{EDM} is even or odd under each of P , C , T and CP , and check that it is even under $\bar{CPT}$.

(c) EXPLAIN

[2]

Explain, both from C invariance and from the diagrams, why the sum of the one-loop electron-loop diagrams with three external photons vanishes in QED, and what this means for the counterterms QED needs.

(d) ESTIMATE

[2]

A new interaction at a mass scale Λ could generate d at one loop. Estimate the lowest Λ consistent with the experimental bound, stating your assumptions.

(e) PREDICT

[1]

Predict, from CPT invariance alone, the electric dipole moment of the positron if the electron's were found to be d .

Work in natural units with the metric $(+, -, -, -)$. An electron of mass m bound to a nucleus of charge Ze , treated as infinitely heavy, feels at lowest order the Coulomb potential $V(r) = -Z\alpha/r$, whose Fourier transform is $\tilde{V}(q) = -4\pi Z\alpha/|q|^2$. The renormalised one-loop vertex, computed with a photon mass μ_γ , has to first order in the spacelike momentum transfer $q^2 = -|q|^2$ the form factors $F_1(q^2) = 1 + (\alpha/3\pi)(q^2/m^2)[\ln(m/\mu_\gamma) - 3/8]$ and $F_2(0) = \alpha/2\pi$. For an ns state $|\psi_{ns}(0)|^2 = (Z\alpha m)^3/(\pi n^3)$, and states with $l \geq 1$ vanish at the origin. Write $U = 4\alpha(Z\alpha)^4 m/(3\pi n^3)$. In these units, at order $\alpha(Z\alpha)^4$: the F_2 (Pauli) term shifts ns levels by $+(3/8)U$ and $2P_{1/2}$ by $-(1/8)U$; the Uehling (vacuum-polarisation) term shifts ns levels by $-(1/5)U$; Bethe's non-relativistic treatment of photons of energy below a cutoff K shifts ns levels by $U \ln(K/\langle E \rangle_{ns})$ and $2P_{1/2}$ by $-U \ln(\langle E \rangle_{2p}/(Z^2 Ry))$, where $Z^2 Ry = (Z\alpha)^2 m/2$ and the Bethe logarithms are $\ln(\langle E \rangle_{2s}/(Z^2 Ry)) = 2.8118$ and $\ln(\langle E \rangle_{2p}/(Z^2 Ry)) = -0.0300$. Matching across $(Z\alpha)^2 m \ll K \ll m$ replaces $\ln(m/\mu_\gamma)$ in the high-energy part by $\ln(m/2K) + 5/6$. For hydrogen-like helium, ${}^4\text{He}^+$ ($Z = 2$), take $\alpha = 1/137.036$, $m = 0.511$ MeV and, for $n = 2$, $U = 2170$ MHz, reduced-mass corrections neglected. The same assembly for hydrogen gives 1051.9 MHz for the $2S_{1/2}$ - $2P_{1/2}$ splitting against the measured 1057.8 MHz; for He^+ the measured splitting is 14 041 MHz.

(a) SHOW THAT

[3]

Show that the q^2 term of F_1 adds to the Coulomb potential a contact term $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, and that this raises an ns level by $U[\ln(m/\mu_\gamma) - 3/8]$ while leaving P levels unshifted.

(b) EXPLAIN

[2]

Explain why the result of (a) cannot be used on its own, and why a cutoff K with $(Z\alpha)^2 m \ll K \ll m$ can be found for He^+ , giving the two ends of the window in electronvolts.

(c) DETERMINE

[3]

Using the matching relation, determine the total order- $\alpha(Z\alpha)^4$ shift of the He^+ $2S_{1/2}$ level in units of U and in megahertz, showing that K cancels.

(d) CALCULATE

[2]

Calculate the shift of $2P_{1/2}$ and hence the $2S_{1/2}$ - $2P_{1/2}$ splitting of He^+ in megahertz.

(e) EVALUATE

[2]

Evaluate the agreement with the measured splittings of He^+ and of hydrogen, and say what the comparison suggests about the largest term omitted.

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17
CORE
COURSE

DATA ANALYSIS · DEMANDING · UQFTI-09.06 · UQFTI-13.04 · UQFTI-13.05 · UQFTI-13.06

[12]

An electron-positron collider measures the total cross-section for $e^+e^- \rightarrow \mu^+\mu^-$ at six centre-of-mass energies well below the Z boson. Z exchange and initial- and final-state radiation have been corrected for, so that what remains is one-photon exchange through the vacuum-polarisation-corrected photon propagator, for which $\sigma = 4\pi\alpha(s)^2/(3s)$, with $\alpha(s)$ the effective coupling at momentum transfer squared s; the muon mass is negligible at these energies. Work in natural units with $\alpha(0) = 1/137.036$ and $(\hbar c)^2 = 0.3894 \text{ GeV}^2 \text{ mb}$, so that $4\pi\alpha(0)^2/3 = 86.86 \text{ nb GeV}^2$. At one loop a fermion of charge $Q_f e$ with N_c colours makes α run as $\mu d\alpha/d\mu = 2N_c Q_f^2 \alpha^2/(3\pi)$ while μ is well above its mass, with $\mu^2 = s$. Every quoted uncertainty is 0.2% of σ , statistical and systematic combined. Lepton loops alone, run from $q^2 = 0$ with the measured lepton masses, give $1/\alpha = 134.02$ at $\sqrt{s} = 12.0 \text{ GeV}$ and 132.73 at $\sqrt{s} = 91.19 \text{ GeV}$.

Total cross-section for $e^+e^- \rightarrow \mu^+\mu^-$ after radiative and Z-exchange corrections

Run	$\sqrt{s} / \text{GeV}$	σ / pb	Uncertainty in σ / pb
1	12.0	652.5	1.3
2	18.0	291.8	0.6
3	25.0	152.7	0.3
4	35.0	78.51	0.16
5	45.0	47.66	0.10
6	58.0	28.86	0.06

(a) SHOW THAT

[2]

Show that if every fermion lighter than $\sqrt{s}$ contributes at one loop, $1/\alpha(s)$ is a straight line in $\ln s$ with gradient $-\sum_f N_c Q_f^2/(3\pi)$, and that $s\sigma/(86.86 \text{ nb GeV}^2) = [\alpha(s)/\alpha(0)]^2$.

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(b) DESCRIBE

[2]

Describe how you would process the readings to test this prediction and extract the gradient, including how the uncertainty in each σ becomes an uncertainty in $1/\alpha$.

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(c) DETERMINE

[3]

Determine $1/\alpha$ at each energy, the gradient and intercept of the best straight line with an estimate of the uncertainty in the gradient, and hence $\sum_f N_{cQ_f^2}$.

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(d) EVALUATE

[3]

Evaluate which of these the data support: the three charged leptons alone; the leptons with the five quarks u, d, s, c and b each counted once; or the leptons with those quarks in three colours.

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(e) PREDICT

[2]

Predict $1/\alpha$ at $\sqrt{s} = 91.19$ GeV by extrapolating your line, and say what its difference from the lepton-only running requires of hadrons, already at 12 GeV and at the Z.

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18
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COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQFTI-12.01 · UQFTI-12.03 · UQFTI-12.04 · UQFTI-12.06 · UQFTI-12.07

[15]

Work in natural units with the metric $(+, -, -, -)$, in QED with an electron of physical mass m and bare mass m_0 , in Feynman gauge with a small photon mass μ_γ . At one loop the electron self-energy is $-i\Sigma_2(p) = (-ie)^2 \int d^d k / (2\pi)^d \gamma^\mu [i(\not{k} + m)/(k^2 - m^2 + i\epsilon)] \gamma_\mu [-i/((p - k)^2 - \mu_\gamma^2 + i\epsilon)]$, continued to $d = 4 - \epsilon$ dimensions with $e^2 \rightarrow e^2 \mu^\epsilon$ to keep e dimensionless. You may use: in d dimensions $\gamma^\mu \gamma_\mu = d$ and $\gamma^\mu \not{k} \gamma_\mu = -(d - 2)\not{k}$; Feynman's parametrisation $1/(AB) = \int_0^1 dx \, 1/[xA + (1 - x)B]^2$; the Minkowski master integral $\int d^d l / (2\pi)^d \, 1/(l^2 - \Delta + i\epsilon)^2 = i\Gamma(2 - d/2) \Delta^{(d/2 - 2)} / (4\pi)^{(d/2)}$, integrals odd in l vanishing; and, as $\epsilon \rightarrow 0$, $\Gamma(\epsilon/2)(4\pi\mu^2/\Delta)^{(\epsilon/2)} = 2/\epsilon - \gamma_E + \ln(4\pi\mu^2/\Delta) + O(\epsilon)$. The physical mass is the pole of the full propagator, $m = m_0 + \Sigma_2(\not{p} = m)$ at this order. With the full vertex Γ^μ (its factor $-ie$ removed) and the full inverse propagator $\tilde{S}^{-1}(p) = \not{p} - m_0 - \Sigma(p)$, the Ward-Takahashi identity reads $q_\mu \Gamma^\mu(p + q, p) = \tilde{S}^{-1}(p + q) - \tilde{S}^{-1}(p)$. Take $\alpha = 1/137.036$, $m = 0.511$ MeV and the Planck mass 1.22×10^{19} GeV.

(a) DERIVE

[2]

Derive, by counting powers of loop momentum, the superficial degree of divergence D of a QED diagram in terms of the number B of its external photons and the number F of its external electron lines, and hence the value of D for the electron self-energy.

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- (b) **EXPLAIN** [2]
 Explain why, although $D = 1$, the mass shift can diverge only logarithmically and must be proportional to m , and contrast this with what happens to a scalar's mass.
- (c) **SHOW THAT** [4]
 Show that $\Sigma_2(p) = [e^2 \mu^\epsilon / (4\pi)^{d/2}] \int_0^1 dx [(4 - \epsilon)m - (2 - \epsilon)x \not{p}] \Gamma(2 - d/2) \Delta^{d/2 - 2}$, with $\Delta = (1 - x)m^2 + x\mu_\gamma^2 - x(1 - x)p^2$.
- (d) **DERIVE** [3]
 Derive the self-mass $\delta m = m - m_0$ at one loop, showing that it is proportional to m , logarithmic and finite as $\mu_\gamma \rightarrow 0$, and find its finite part.
- (e) **COMPARE** [2]
 Interpreting the pole as $\ln(\Lambda^2/m^2)$ for a cutoff Λ , so that $\delta m \approx (3\alpha/4\pi)m \ln(\Lambda^2/m^2)$, compare $\delta m/m$ for Λ equal to the Planck mass with the shift of order $\alpha\Lambda$ that a linearly divergent electromagnetic mass would give.
- (f) **SHOW THAT** [2]
 From the Ward-Takahashi identity show that $Z_1 = Z_2$, where near the mass shell $\tilde{S}^{-1}(p) \rightarrow Z_2^{-1}(\not{p} - m)$ and, between on-shell spinors as $q \rightarrow 0$, $\Gamma^\mu(p, p) \rightarrow Z_1^{-1}\gamma^\mu$.

Work in natural units with the metric $(+, -, -, -)$. Between on-shell spinors the full electron-photon vertex is $\bar{u}(p')\Gamma^\mu u(p)$, with $q = p' - p$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$; the Gordon identity reads $\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')[(p' + p)^\mu/2m + i\sigma^{\mu\nu}q_\nu/2m]u(p)$. At one loop in Feynman gauge, after Feynman parameters, the shift $\ell = k + yq - zp$ and use of the Dirac equation, the correction is $\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2\int d^4\ell/(2\pi)^4\int_0^1 dx dy dz \delta(x + y + z - 1)(2/D^3)\bar{u}(p')N^\mu u(p)$, with $D = \ell^2 - \Delta + i\epsilon$, $\Delta = -xyq^2 + (1 - z)^2m^2$ and $N^\mu = \gamma^\mu[-\ell^2/2 + (1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2] + (i\sigma^{\mu\nu}q_\nu/2m)[2m^2z(1 - z)]$. Use $\int d^4\ell/(2\pi)^4 1/(\ell^2 - \Delta)^3 = -i/(2(4\pi)^2\Delta)$. The measured electron anomaly is $a_e = 1.159\,652\,180\,59(13) \times 10^{-3}$, and the QED series is $a_e = \alpha/2\pi + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots$, with $C_2 = -0.328\,478\,965$, $C_3 = 1.181\,241\,456$ and $C_4 = -1.912\,245\,764$; in this question take $\alpha^{-1} = 137.035\,999$, from atom-recoil measurements. For elastic scattering of an electron of energy E through angle θ by a recoil-free nucleus, $-q^2 = 4E^2\sin^2(\theta/2)$; the order- α virtual correction multiplies the Born cross-section by $1 - (\alpha/\pi)f_{\text{IR}}(q^2)\ln(-q^2/\mu_\gamma^2)$, μ_γ being a photon mass, and the probability of emitting one soft photon with energy between μ_γ and E_γ is $(\alpha/\pi)f_{\text{IR}}(q^2)\ln(E_\gamma^2/\mu_\gamma^2)$, where for $-q^2 \gg m^2$, $f_{\text{IR}}(q^2) = \ln(-q^2/m^2) - 1$. Take $m = 0.511$ MeV.

(a) JUSTIFY [2]

Justify writing the on-shell vertex as $\Gamma^\mu = \gamma^\mu F_1(q^2) + (i\sigma^{\mu\nu}q_\nu/2m)F_2(q^2)$.

(b) SHOW THAT [2]

Show that in a slowly varying magnetic field the vertex gives the electron a magnetic moment with $g = 2[F_1(0) + F_2(0)]$.

(c) DERIVE [3]

Derive $F_2(q^2)$ as an integral over Feynman parameters, evaluate $F_2(0)$, and explain why F_2 needs neither an ultraviolet nor an infrared regulator.

(d) CALCULATE [3]

Calculate the residual, the measured a_e minus the QED series truncated after one, two and three terms, and compare each residual with the size of the next term.

- (e) **ESTIMATE** [1]
 Estimate the fractional precision with which this measurement of a_e would determine α if the QED series were known exactly, stating your assumption.
- (f) **SHOW THAT** [2]
 Show that in the cross-section measured by a detector that cannot register photons below E_I the photon mass cancels at order α , leaving the factor $1 - (\alpha/\pi)f_{\text{IR}}(q^2)\ln(-q^2/E_I^2)$.
- (g) **CALCULATE** [2]
 Calculate this correction factor for 4.0 GeV electrons scattered through 20° with $E_I = 20$ MeV, and compare it with its exponentiated form $\exp[-(\alpha/\pi)f_{\text{IR}}\ln(-q^2/E_I^2)]$.
- (h) **PREDICT** [1]
 Predict what happens, as $E_I \rightarrow 0$, to the rate of scattering with no photon above E_I and to the inclusive rate.

University Quantum Field Theory I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQFTI-01.05 · UQFTI-01.06 · UQFTI-01.07

[1]

For a free real scalar field of mass m , in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$, the Feynman propagator is $D_F(x - y) = \int d^4p/(2\pi)^4 i e^{i(-ip \cdot (x - y))}/(p^2 - m^2 + i\epsilon)$, with $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$ for the three-momentum $\mathbf{p}$ and $\epsilon \rightarrow 0^+$. The p^0 integral is done first, by closing the contour in whichever half of the complex p^0 plane makes the exponential decay. Which statement is correct?

KEY B The poles sit at $p^0 = \pm(E_p - i\epsilon')$; for $x^0 > y^0$ the contour closes in the lower half-plane around $p^0 = +E_p$ and gives $\langle 0|\phi(x)\phi(y)|0\rangle$, and for $x^0 < y^0$ it closes in the upper half-plane around $p^0 = -E_p$ and gives $\langle 0|\phi(y)\phi(x)|0\rangle$, so the placement of the $i\epsilon$ is the time ordering.

B is correct. Writing $p^2 - m^2 + i\epsilon = (p^0)^2 - E_p^2 + i\epsilon$, the zeros are at $p^0 = \pm\sqrt{E_p^2 - i\epsilon} = \pm(E_p - i\epsilon')$ with $\epsilon' = \epsilon/2E_p$: the positive-energy pole lies just below the real axis and the negative-energy pole just above it. For $x^0 > y^0$ the factor $e^{i(-ip^0(x^0 - y^0))}$ dies away where $\text{Im } p^0 < 0$, so the contour closes below, encloses $p^0 = E_p - i\epsilon'$, and the residue gives $\int d^3p/((2\pi)^3 2E_p) e^{i(-ip \cdot (x - y))} = \langle 0|\phi(x)\phi(y)|0\rangle$; for $x^0 < y^0$ it closes above and gives the other order. A describes the retarded propagator, whose poles are both pushed below the axis by $p^0 \rightarrow p^0 + i\epsilon$; closing above for $x^0 < y^0$ then encloses nothing, and it is D_R , not D_F , that vanishes before the source acts. C is the commonest misconception about D_F : outside the light cone it falls as $e^{i(-mr)}$ but does not vanish. Causality lives in the commutator $[\phi(x), \phi(y)] = D(x - y) - D(y - x)$, which is zero at spacelike separation because the two orderings there are related by a Lorentz transformation. D closes the contour on the wrong side: for $x^0 > y^0$ the exponential grows in the upper half-plane, and the pole that contributes for that time order is the positive-energy one, a particle created at y and absorbed at x ; the negative-energy pole contributes only for $x^0 < y^0$.

U1

2CORE
COURSE

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[1]

Two theories are expanded to second order in their coupling, in natural units. In theory (i) a real scalar ϕ has $\mathcal{L}_{\text{int}} = -(g/3!)\phi^3$. In theory (ii), the scalar Yukawa model, a complex scalar ψ and a real scalar ϕ have $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\phi$. In each, the order- g^2 correction to the ϕ two-point function $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ contains a one-loop diagram in which x_1 joins one vertex, x_2 joins the other, and the two vertices are linked by two internal lines; in (ii) these form a closed ψ loop. If each diagram is written with its vertex factors and propagators and an overall $1/S$, what are the symmetry factors S in (i) and (ii)?

KEY D $S = 2$ in (i) and $S = 1$ in (ii)

In (i) the second-order term carries $(1/2!)(1/3!)^2$. The external points can be attached to the two vertices in 2 ways, x_1 can join any of the 3 fields at its vertex and x_2 any of the 3 at the other, and the two remaining fields at one vertex pair with the two at the other in 2 ways: $2 \times 3 \times 3 \times 2 = 36$ contractions against $2! \times 3! \times 3! = 72$, leaving $1/2$, so $S = 2$, the automorphism that swaps the two internal lines. In (ii) the $1/2!$ is cancelled by the 2 ways of assigning x_1 and x_2 to the vertices, there is no $1/3!$ to cancel, and ψ at each vertex must contract with $\psi^\dagger$ at the other, which can be done in only one way, so $S = 1$ (D). A treats the two ψ propagators like the two ϕ lines of (i), but exchanging them reverses the arrows of ψ number, so it is not a symmetry of the diagram: what were two pairings of identical fields in (i) is here a single $\psi\psi^\dagger$ and $\psi^\dagger\psi$ contraction. B forgets that x_2 , like x_1 , can join any of the 3 fields at its vertex: $2 \times 3 \times 2 = 12$ contractions against 72 leaves $1/6$, as though one vertex kept its whole $1/3!$; counting the 3 choices for x_2 as well gives 36 and leaves only $1/2$. C assumes that the $1/n!$ of a ϕ^n coupling always cancels completely, which it does for tree diagrams; a loop whose two lines can be exchanged keeps a $1/2$.

U2

3

CORE
COURSE

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[1]

In ϕ^4 theory, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - (\lambda/4!)\phi^4$ with $0 < \lambda \ll 1$, the single particle has physical mass m . The Källén-Lehmann representation writes the exact two-point function as $\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty (dM^2/2\pi) \rho(M^2) D_F(x-y; M^2)$, with $\rho(M^2) = 2\pi Z\delta(M^2 - m^2)$ plus a continuum. Where does the continuum begin, and what bounds Z ?

KEY C At $M^2 = 9m^2$, because the symmetry $\phi \rightarrow -\phi$ stops ϕ from connecting the vacuum to states with an even number of particles; $0 \leq Z \leq 1$, because ρ is non-negative and the canonical commutator fixes $\int (dM^2/2\pi)\rho(M^2) = 1$.

C is correct. $\rho(M^2)$ collects $|\langle\lambda_0|\phi(0)|\Omega\rangle|^2$ over the states of invariant mass M . The Lagrangian is invariant under $\phi \rightarrow -\phi$, and so is the vacuum, so ϕ connects the vacuum only to states of odd particle number: the one-particle pole, then three particles from $M^2 = 9m^2$. The canonical equal-time commutator, applied to the spectral form, gives $\int (dM^2/2\pi)\rho(M^2) = 1$, that is Z plus the weight of the continuum equals 1, and since every contribution to ρ is a squared modulus, $0 \leq Z \leq 1$. A takes the generic two-particle threshold, which is right for a single scalar without the symmetry, such as ϕ^3 theory, where ϕ can create two of its own quanta; in ϕ^4 the two-particle matrix element $\langle p_1 p_2 | \phi(0) | \Omega \rangle$ vanishes identically. B inverts the sum rule: the continuum takes weight away from the pole, so Z falls below 1, and a Z above 1 would need states of negative norm. D describes QED, where massless photons put the electron pole at the end of a cut and the residue needs an infrared regulator; in ϕ^4 every quantum has mass m , so a gap separates the pole from the continuum.

U1

4

CORE
COURSE

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[1]

In d spacetime dimensions a dimensionless action with a canonical kinetic term gives a scalar field the mass dimension $(d-2)/2$. Which interaction has a dimensionless coupling, so that the theory is renormalisable but not super-renormalisable?

KEY A $g\phi^3$ in $d = 6$

The coupling of ϕ^n in d dimensions has mass dimension $d - n(d-2)/2$. For ϕ^3 in $d = 6$ this is $6 - 3 \times 2 = 0$, so the superficial degree of divergence of a diagram depends only on its external legs and a finite set of counterterms serves at every order: renormalisable (A). B gives $4 - 3 = 1$ and C gives $3 - 4 \times \frac{1}{2} = 1$, positive mass dimensions, so the degree of divergence falls with every extra vertex and only finitely many diagrams diverge at all: super-renormalisable, a stronger property than the one asked for. D gives $4 - 6 = -2$, so each vertex raises the degree of divergence by 2, functions with ever more external legs diverge at higher orders, and ever more counterterms are needed: non-renormalisable. ϕ^6 is renormalisable in $d = 3$, where its coupling is dimensionless, as ϕ^4 is in $d = 4$.

U1

5

CORE
COURSE

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[1]

A spin-zero boson ϕ of mass 2.40 GeV decays to a fermion-antifermion pair, the fermion being a Dirac particle of mass 0.90 GeV. Compare a scalar coupling $g\phi\bar{\psi}\psi$ with a pseudoscalar coupling $ig\phi\bar{\psi}\gamma^5\psi$ of the same real strength g . In both cases $\Gamma = |\mathbf{p}| \Sigma|M|^2/(8\pi m_\phi^2)$, with the sum over final spins and $|\mathbf{p}|$ the momentum of either fermion in the rest frame of ϕ . What is $\Gamma_{\text{scalar}}/\Gamma_{\text{pseudoscalar}}$?

KEY D 0.44

With $\beta = \sqrt{1 - 4m^2/m_\phi^2} = \sqrt{1 - 3.24/5.76} = 0.661$, the traces give $\Sigma|M|^2 = g^2\text{Tr}[(\not{p}_1 + m)(\not{p}_2 - m)] = 4g^2(p_1 \cdot p_2 - m^2) = 2g^2m_\phi^2\beta^2$ for the scalar coupling and $g^2\text{Tr}[(\not{p}_1 + m)(\not{p}_2 + m)] = 4g^2(p_1 \cdot p_2 + m^2) = 2g^2m_\phi^2$ for the pseudoscalar, the γ^5 on each side of $\not{p}_2 - m$ turning it into $-\not{p}_2 - m$. The phase space is the same, so the ratio is $\beta^2 = 0.4375 \approx 0.44$ (D): the scalar rate goes as β^3 and the pseudoscalar rate as β . A fermion pair has parity $(-1)^{L+1}$, so a 0^+ boson must produce it in a p wave, suppressed near threshold, and a 0^- boson in an s wave. A stops at one power of β , as though the p -wave suppression entered the rate linearly rather than through the square of an amplitude proportional to β . B supposes that γ^5 changes only a phase of the amplitude; it changes the relative sign of $\not{p}_2$ and m inside the trace, which is the difference between $p_1 \cdot p_2 - m^2$ and $p_1 \cdot p_2 + m^2$. C gives the pseudoscalar the p wave and the scalar the s wave, the parity assignment reversed.

U3

In a general covariant gauge the photon propagator is $-i[g_{\mu\nu} - (1 - \xi)k_\mu k_\nu/k^2]/(k^2 + i\epsilon)$. In the tree amplitude for $e^-(p_1)e^+(p_2) \rightarrow \mu^-(p_3)\mu^+(p_4)$, the virtual photon carries $k = p_1 + p_2 = p_3 + p_4$, with $k^2 = s$. Why does the amplitude not depend on ξ ?

KEY B Because k_μ contracted into the electron current gives $\bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) = (m_e - m_e)\bar{v}(p_2)u(p_1) = 0$ by the Dirac equation, and likewise on the muon line, so the $k_\mu k_\nu$ term vanishes for any ξ .

B is correct. Between on-shell spinors $\not{p}_1 u(p_1) = m_e u(p_1)$ and $\bar{v}(p_2)\not{p}_2 = -m_e \bar{v}(p_2)$, so $k_\mu \bar{v}(p_2)\gamma^\mu u(p_1) = 0$ exactly, with the masses kept; on the muon line $\bar{u}(p_3)(\not{p}_3 + \not{p}_4)v(p_4) = (m_\mu - m_\mu)\bar{u}(p_3)v(p_4) = 0$ likewise. Each current is conserved, and any term proportional to k_μ or k_ν drops out whatever ξ multiplies it. A confuses the internal photon with an external one: here $k^2 = s$, far from zero, and it is the conservation of the currents, not the mass shell, that removes the term. C argues in a circle: that no physical amplitude depends on ξ is what has to be shown, and it is current conservation that shows it, here at tree level and, through the Ward-Takahashi identity, to all orders. D gets the mechanism wrong: the term is not suppressed by m_e^2/s but vanishes exactly for any masses, because each current is conserved on its own, so nothing has to be neglected.

U1

At $\sqrt{s} \gg m_\mu$ the reaction $e^+e^- \rightarrow \mu^+\mu^-$ by one-photon exchange proceeds through four helicity combinations, because the vector coupling joins a right-handed fermion only to a left-handed antifermion, and vice versa. The e^- moves along $+z$, and θ is the angle between the outgoing μ^- and the incoming e^- . How does the amplitude for $e^-_R e^+_L \rightarrow \mu^-_L \mu^+_R$ depend on θ , and why?

KEY A As $(1 - \cos \theta)$: the initial pair has angular momentum $+1$ along the beam and the final pair -1 along the μ^- direction, so the amplitude is proportional to the rotation matrix element $d^1_{1,-1}(\theta)$, which vanishes in the forward direction.

A is correct. With the e^- right-handed along $+z$ and the e^+ left-handed along $-z$, both spins point along $+z$, so $J_z = +1$. The μ^- is left-handed along its direction $\hat{n}$ and the μ^+ right-handed along $-\hat{n}$, so both spins point along $-\hat{n}$, a projection -1 on $\hat{n}$. The amplitude is therefore proportional to $d^1_{1,-1}(\theta) = (1 - \cos \theta)/2$, which vanishes at $\theta = 0$, where a projection $+1$ cannot become -1 along the same axis, and is largest backwards. Squaring and summing the four combinations gives $2[(1 + \cos \theta)^2 + (1 - \cos \theta)^2] \propto 1 + \cos^2 \theta$. B gives the dependence of $e^-_R e^+_L \rightarrow \mu^-_R \mu^+_L$, in which the final spins also point along $+\hat{n}$; it credits the vertices with a correlation that comes from angular momentum. C misplaces the chirality rule: it holds along each fermion line, so it fixes the e^+ helicity once the e^- 's is chosen, but the muons are on a different line, and the photon couples to $\mu^-_L \mu^+_R$ as readily as to $\mu^-_R \mu^+_L$. D forgets that the initial and final pairs have definite projections ± 1 along their own axes, so the amplitudes are d^1 functions with $m = \pm 1$, $(1 \pm \cos \theta)/2$; a $\sin \theta$ dependence would need a projection 0, which helicity conservation at high energy forbids.

U1

8

CORE
COURSE

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[1]

A positronium level with orbital angular momentum L and total spin S has charge-conjugation parity $C = (-1)^{(L+S)}$, and a state of n photons has $C = (-1)^n$. Landau and Yang's theorem, quoted, forbids a state of total angular momentum $J = 1$ from decaying into two photons. Which level can annihilate directly into two photons?

KEY **D** 2^3P_2

Two photons have $C = +1$, so the level needs $L + S$ even and, by Landau and Yang, $J \neq 1$. 2^3P_2 has $L = 1$, $S = 1$, $C = +1$ and $J = 2$, so $J^{PC} = 2^{++}$ and two-photon annihilation is allowed (D), though in practice this level decays radiatively to lower levels far faster than it annihilates, because annihilation needs the pair at contact, where a P-wave function vanishes. A is ortho-positronium, $C = (-1)^{(0+1)} = -1$, which needs an odd number of photons and decays into three; that is why its lifetime, 142 ns, is about a thousand times longer than para-positronium's 125 ps. B has $C = (-1)^{(1+0)} = -1$, like ortho-positronium, although $S = 0$: the orbital angular momentum contributes its own sign. C has $C = +1$ and passes the charge-conjugation test, but $J = 1$, and Landau and Yang's theorem, which follows from Bose symmetry and rotational invariance for two real photons, forbids it.

U1

9

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQFTI-12.06 · UQFTI-12.07

[1]

In renormalised QED with on-shell conditions the physical charge is $e = Z_2 Z_3^{(1/2)} e_0 / Z_1$, where Z_1 renormalises the vertex, Z_2 the charged field and Z_3 the photon field. A proton's coupling to the photon is dressed by strong interactions that the electron's is not, yet the hydrogen atom is neutral to better than one part in 10^{20} . Why are the renormalised charges of the electron and the proton exactly opposite, given opposite bare charges?

KEY **A** **The Ward-Takahashi identity gives $Z_1 = Z_2$ for every charged field, whatever its other interactions, so $e = Z_3^{(1/2)} e_0$, and Z_3 comes from the photon's own self-energy, which is common to every particle the photon meets.**

A is correct. The identity follows from current conservation alone and forces the vertex and field renormalisations of any charged particle to cancel, so charge renormalisation is carried entirely by Z_3 , the vacuum polarisation of the photon, which acts in the same way on every charge. B is wrong in fact: strong interactions dress the proton's vertex heavily, giving it a charge radius near 0.84 fm and an anomalous moment of 1.79 nuclear magnetons; what they cannot change is $F_1(0) = 1$, and that is the content of $Z_1 = Z_2$. C confuses the form of the coupling with its size: the electron's anomalous moment is 0.00116 and the proton's 1.79, so the vertex corrections are wildly different, and still the charges agree. D puts the conclusion into the definition: QED has one charge parameter, fixed by Thomson scattering, and that every other particle's charge then comes out as an exact multiple of it is a prediction, not a convention.

U1

At one loop the electron loop gives the photon a self-energy $i\Pi^{\mu\nu}(q)$. Why is it computed in dimensional regularisation rather than with a sharp cutoff Λ on the Euclidean loop momentum?

KEY C A sharp cutoff is not invariant under shifts of the loop momentum and breaks gauge invariance: $\Pi^{\mu\nu}$ acquires a term proportional to $g^{\mu\nu}\Lambda^2$ that is not of the transverse form $(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ and would give the photon a mass, whereas dimensional regularisation keeps $\Pi^{\mu\nu}$ transverse.

C is correct. With a sharp cutoff the shift of loop momentum used to complete the square changes the integral by a surface term, and the result contains a piece of order $(\alpha/\pi)\Lambda^2 g^{\mu\nu}$. Contracting it with q_μ does not give zero, so the Ward identity fails, and in the resummed propagator that piece acts as a photon mass squared of order $\alpha\Lambda^2$. Dimensional regularisation has no surface terms, respects the shift and the gauge symmetry, and gives $\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ exactly, so the photon stays massless. A has the regulators backwards: dimensional regularisation does not make the loop finite but turns its logarithmic divergence into a $1/\epsilon$ pole, which Z_3 absorbs, and the transverse part of the cutoff result diverges only logarithmically too; the trouble with the cutoff is the extra term that is not transverse. B forgets that the electron in the loop is massive: $\Pi(q^2)$ is finite as $q^2 \rightarrow 0$, and QED's infrared problems come from massless photons on external or soft lines, not from this loop. D is exactly what fails for the sharp cutoff.

U1

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. In the scalar Yukawa model a complex scalar ψ of mass m_ψ and a real scalar ϕ of mass m_ϕ interact through $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\phi$. Take $m_\phi = 0.80$ GeV, $m_\psi = 0.25$ GeV and $g = 0.40$ GeV, with $\hbar = 6.582 \times 10^{-25}$ GeV s and $\hbar c = 0.1973$ GeV fm. For a particle of mass m_ϕ at rest decaying to two distinguishable spinless products, $\Gamma = |p||M|^2/(8\pi m_\phi^2)$, where M is the invariant amplitude and $|p|$ the momentum of either product.

(a) STATE · U1

[1]

State the mass dimension of g , and whether the interaction is renormalisable, super-renormalisable or non-renormalisable.

- 1 In four dimensions ψ and ϕ each have mass dimension 1, so $[g] = 4 - 3 = 1$; a coupling of positive mass dimension makes the interaction super-renormalisable

(b) SHOW THAT · U2

[2]

Show that, at tree level, $\Gamma(\phi \rightarrow \psi\bar{\psi}) = (g^2/16\pi m_\phi)\sqrt{(1 - 4m_\psi^2/m_\phi^2)}$.

- 1 One vertex and no propagator: contracting ϕ with the initial particle and $\psi, \psi^\dagger$ with the outgoing particle and antiparticle gives $iM = -ig$, so $|M|^2 = g^2$, with no factor $1/2$ because ψ and $\bar{\psi}$ are distinguishable
- 2 $|p| = \frac{1}{2}\sqrt{(m_\phi^2 - 4m_\psi^2)} = (m_\phi/2)\sqrt{(1 - 4m_\psi^2/m_\phi^2)}$, so $\Gamma = |p|g^2/(8\pi m_\phi^2) = (g^2/16\pi m_\phi)\sqrt{(1 - 4m_\psi^2/m_\phi^2)}$

(c) CALCULATE · U3

[2]

Calculate Γ in MeV and the mean lifetime of ϕ in seconds.

- 1 $\sqrt{(1 - 4m_\psi^2/m_\phi^2)} = \sqrt{(1 - 0.25/0.64)} = 0.781$, so $\Gamma = (0.16 \text{ GeV}^2 \times 0.781)/(16\pi \times 0.80 \text{ GeV}) = 3.11 \times 10^{-3} \text{ GeV} = 3.11 \text{ MeV}$
- 2 $\tau = \hbar/\Gamma = (6.582 \times 10^{-25} \text{ GeV s})/(3.11 \times 10^{-3} \text{ GeV}) = 2.12 \times 10^{-22} \text{ s}$

Candidates who divide $\hbar$ by Γ in MeV while keeping $\hbar$ in GeV s are out by a factor of 1000; the check is that $\Gamma/m_\phi = 0.004$, a narrow resonance, so τ must be a few hundred times $1/m_\phi = 8.2 \times 10^{-25} \text{ s}$.

(d) DETERMINE · U2

[3]

Two ψ particles scatter at low speed. From the t-channel ϕ -exchange diagram, determine the potential $V(r)$ between them, allowing for the relativistic normalisation of the states, and state whether it attracts and what its range is in femtometres.

- 1 The t-channel diagram gives $iM = (-ig)^2 i/(q^2 - m_\phi^2)$; for slow particles $q^0 \approx 0$ and $q^2 \approx -|q|^2$, so $M = g^2/(|q|^2 + m_\phi^2)$, the u-channel diagram supplying the exchange term that identical bosons require
- 2 Relativistic normalisation puts a factor $2m_\psi$ on each of the two particles, so the Born amplitude $M/(2m_\psi)^2$ equals $-\tilde{V}(q)$, giving $\tilde{V}(q) = -(g^2/4m_\psi^2)/(|q|^2 + m_\phi^2)$ and $V(r) = -(g^2/4m_\psi^2)e^{(-m_\phi r)}/(4\pi r)$
- 3 V is negative, so the force attracts, with dimensionless strength $g^2/(16\pi m_\psi^2) = 0.16/(16\pi \times 0.0625) = 0.051$ in the place of α , and range $1/m_\phi = 0.1973 \text{ GeV fm}/0.80 \text{ GeV} = 0.25 \text{ fm}$

The factor $1/(2m_\psi)^2$ is the part most often lost: without it $\tilde{V}(q)$ is dimensionless, $V(r)$ comes out with the dimension of an energy cubed, too large by the factor $(2m_\psi)^2$, and cannot be compared with the Born amplitude of quantum mechanics.

(e) PREDICT · U1

[1]

Predict what becomes of ϕ , and of the use of the LSZ formula with ϕ as an external particle, if m_ϕ were lowered to 0.45 GeV with m_ψ unchanged.

- 1 With m_ϕ below $2m_\psi = 0.50$ GeV the decay is kinematically forbidden, so ϕ becomes stable and its pole is an isolated one-particle pole below the $\psi\bar{\psi}$ continuum at $4m_\psi^2$, as LSZ requires; at $m_\phi = 0.80$ GeV the pole lies on the continuum and moves off the real axis by $m_\phi\Gamma$, so ϕ is not an asymptotic state

Work in natural units with the metric $(+, -, -, -)$. A real scalar ϕ of mass $m = 0.40$ GeV has $\mathcal{L}_{\text{int}} = -(\lambda/4!)\phi^4$ with $\lambda = 0.80$. Two ϕ particles collide head-on with centre-of-mass energy $\sqrt{s} = 2.0$ GeV. For $2 \rightarrow 2$ scattering of equal masses in the centre-of-mass frame, $d\sigma/d\Omega = |M|^2/(64\pi^2 s)$. Two-body Lorentz-invariant phase space for distinguishable particles is $\int d\Omega = |p_{\text{cm}}|/(4\pi\sqrt{s})$, with $|p_{\text{cm}}|$ the momentum of either particle. Use $(\hbar c)^2 = 0.3894$ GeV² mb. In minimal subtraction the one-loop β function of the theory is $\beta(\lambda) = \mu d\lambda/d\mu = 3\lambda^2/(16\pi^2)$.

(a) CALCULATE · U3

[3]

Calculate the tree-level amplitude M and the total cross-section for $\phi\phi \rightarrow \phi\phi$ in microbarns.

- 1 The $4!$ ways of contracting the four external particles with the four fields at the vertex cancel the $1/4!$, so $iM = -i\lambda$ and $M = -\lambda = -0.80$
- 2 $d\sigma/d\Omega = \lambda^2/(64\pi^2 s)$ is isotropic, and the two final particles are identical, so $\sigma = \frac{1}{2} \times 4\pi \times \lambda^2/(64\pi^2 s) = \lambda^2/(32\pi s)$
- 3 $\sigma = 0.64/(32\pi \times 4.0 \text{ GeV}^2) = 1.59 \times 10^{-3} \text{ GeV}^{-2}$, and multiplying by $0.3894 \text{ GeV}^2 \text{ mb}$ gives $6.20 \times 10^{-4} \text{ mb} = 0.620 \text{ } \mu\text{b}$

Candidates who integrate over the full 4π without the factor $\frac{1}{2}$ count every final configuration twice and report $1.24 \text{ } \mu\text{b}$.

(b) SHOW THAT · U2

[2]

At order λ^2 the s-channel one-loop diagram joins p_1 and p_2 at one vertex and p_3 and p_4 at the other, the two vertices being linked by two internal lines. Show by counting Wick contractions that it carries a symmetry factor $\frac{1}{2}$, so that $iM_{\text{s}} = \frac{1}{2}(-i\lambda)^2 \int d^4l/(2\pi)^4 [i/(l^2 - m^2 + i\epsilon)][i/((k-l)^2 - m^2 + i\epsilon)]$, with $k = p_1 + p_2$.

- 1 The second-order term carries $(1/2!)(1/4!)^2$. The incoming pair can sit at either vertex (2 ways), p_1 and p_2 can take $4 \times 3 = 12$ choices of field at theirs, p_3 and p_4 12 at the other, and the two remaining fields at one vertex pair with the two at the other in 2 ways: $2 \times 12 \times 12 \times 2 = 576$ contractions
- 2 $576 \times (1/2!)(1/4!)^2 = 576/1152 = \frac{1}{2}$, so $S = 2$, the automorphism that exchanges the two internal lines, and each vertex carries $-i\lambda$

Candidates who stop at $2 \times 12 \times 12$ forget that the two internal lines can be formed in two ways and find $\frac{1}{4}$, which the optical theorem in (c) would then show to be wrong.

(c) SHOW THAT · U2

[3]

Show that the imaginary part of the forward amplitude at order λ^2 , which comes from this diagram alone, is $\text{Im } M = \lambda^2\beta/(32\pi)$ with $\beta = \sqrt{1 - 4m^2/s}$, and that it satisfies the optical theorem $\text{Im } M(\text{forward}) = 2E_{\text{cm}}|p_{\text{cm}}|\sigma_{\text{tot}}$ at $\sqrt{s} = 2.0$ GeV.

- 1 For $s > 4m^2$ the two internal lines can go on shell together, and the $i\epsilon$ prescription gives the imaginary part by cutting the diagram, replacing each $1/(l^2 - m^2 + i\epsilon)$ by $-2\pi i\delta(l^2 - m^2)$ with positive energy; in the forward direction the t- and u-channel loops sit at $t = 0$ and $u = 4m^2 - s = -3.36 \text{ GeV}^2$, below their thresholds, and stay real
- 2 The cut gives $2 \text{Im } M = \frac{1}{2}\lambda^2 \int d\Omega = \frac{1}{2}\lambda^2 |p_{\text{cm}}|/(4\pi\sqrt{s})$, and $|p_{\text{cm}}| = \beta\sqrt{s}/2$, so $\text{Im } M = \lambda^2\beta/(32\pi)$; with $\beta = \sqrt{1 - 0.64/4.0} = 0.9165$, $\text{Im } M = 0.64 \times 0.9165/(32\pi) = 5.83 \times 10^{-3}$
- 3 $2E_{\text{cm}}|p_{\text{cm}}|\sigma_{\text{tot}} = 2 \times 2.0 \text{ GeV} \times 0.9165 \text{ GeV} \times 1.59 \times 10^{-3} \text{ GeV}^{-2} = 5.83 \times 10^{-3}$, equal to $\text{Im } M$: the loop's symmetry factor $\frac{1}{2}$ does the work of the $\frac{1}{2}$ for identical particles in σ_{tot}

The factor $\frac{1}{2}$ appears twice and must be matched: once as the loop's symmetry factor and once in the identical-particle phase space. A candidate who drops either finds the optical theorem out by a factor of 2 and should read that as a counting error, not as a failure of unitarity.

(d) EXPLAIN · U1

[1]

Explain why $\text{Im } M$ is finite although the loop integral itself diverges logarithmically.

- 1 The imaginary part comes only from the region in which both internal particles are on shell with total four-momentum k , a compact two-body phase space; the ultraviolet divergence comes from large loop momenta far off shell, is a real constant independent of s , and is absorbed into δ_λ

A candidate who says the divergence cancels in $\text{Im } M$ has the right conclusion for the wrong reason: nothing cancels, the divergence is simply real.

(e) CALCULATE · U3

[2]

Taking $\lambda(\mu_0) = 0.80$ at $\mu_0 = 0.40$ GeV in minimal subtraction, calculate λ at $\mu = 100$ GeV from the one-loop β function, and the scale at which the one-loop coupling would diverge.

- 1 Integrating $\mu d\lambda/d\mu = 3\lambda^2/(16\pi^2)$ gives $1/\lambda(\mu) = 1/\lambda(\mu_0) - (3/16\pi^2)\ln(\mu/\mu_0) = 1.250 - 0.01900 \times \ln 250 = 1.250 - 0.105 = 1.145$, so $\lambda(100 \text{ GeV}) = 0.873$
- 2 $1/\lambda$ reaches zero when $\ln(\mu/\mu_0) = 16\pi^2/(3\lambda(\mu_0)) = 65.8$, at $\mu = 0.40 \text{ GeV} \times e^{(65.8)} \approx 1.5 \times 10^{28} \text{ GeV}$, the Landau pole: far beyond the Planck mass, and long before it the coupling is too large for one loop to be trusted

Candidates who use $\log_{10}$ for the natural logarithm, or $\ln(\mu^2/\mu_0^2)$ with the coefficient for $d/d \ln \mu$, are out by factors of 2.3 or 2 in the running.

13

CORE
COURSE

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[12]

Work in natural units with the metric $(+, -, -, -)$. In QED with an electron and a muon, $e^+e^- \rightarrow \mu^+\mu^-$ proceeds at lowest order through one-photon exchange in the s channel, with vertex $-ie\gamma^\mu$ and Feynman-gauge photon propagator $-ig_{\mu\nu}/(q^2 + i\epsilon)$. Label the momenta $e^-(p_1)e^+(p_2) \rightarrow \mu^-(p_3)\mu^+(p_4)$, neglect the electron mass, keep the muon mass $m_\mu = 0.10566 \text{ GeV}$, and take $\alpha = e^2/4\pi = 1/137.036$ and $(\hbar c)^2 = 0.3894 \text{ GeV}^2 \text{ mb}$. The tree amplitude is $M = (e^2/s)[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)]$, with $s = (p_1 + p_2)^2$. Use $\sum u \bar{u} = \not{p} + m$ and $\sum v \bar{v} = \not{p} - m$ over spins, $\text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$ and $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$; traces of an odd number of gamma matrices vanish.

(a) SHOW THAT · U2

[3]

Show that the spin-averaged squared amplitude is $\frac{1}{4}\Sigma|M|^2 = (8e^4/s^2)[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_\mu^2(p_1 \cdot p_2)]$.

- 1 Casimir's trick turns the spin sums into traces: $\frac{1}{4}\Sigma|M|^2 = (e^4/4s^2)\text{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu]\text{Tr}[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu]$
- 2 The electron trace is $4[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu}(p_1 \cdot p_2)]$ and the muon trace $4[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - g^{\mu\nu}(p_3 \cdot p_4 + m_\mu^2)]$, the mass term coming from $-m_\mu^2\text{Tr}(\gamma_\mu\gamma_\nu)$ and the terms linear in m_μ vanishing as odd traces
- 3 Contracting, with $g^{\mu\nu}g_{\mu\nu} = 4$, the $(p_1 \cdot p_2)(p_3 \cdot p_4)$ terms cancel and the product of traces is $32[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_\mu^2(p_1 \cdot p_2)]$, so $\frac{1}{4}\Sigma|M|^2 = (e^4/4s^2) \times 32[\dots] = (8e^4/s^2)[\dots]$

Candidates who write the antiparticle spin sum as $\not{p} + m$ reverse the sign of the mass term, and those who drop $-m_\mu^2\text{Tr}(\gamma_\mu\gamma_\nu)$ lose it. The massless limit, $(s^2/8)(1 + \cos^2\theta)$, cannot catch either slip, because both errors vanish with m_μ ; threshold can: at $\beta = 0$ the correct bracket is $s^2/4$ at every angle, whereas the reversed sign makes it vanish and the dropped term halves it.

(b) SHOW THAT · U2 [3]

In the centre-of-mass frame, with θ the angle between p_3 and p_1 and $\beta = \sqrt{1 - 4m_\mu^2/s}$ the muon speed, show that $d\sigma/d\Omega = (\alpha^2\beta/4s)(2 - \beta^2\sin^2\theta)$, and hence that $\sigma = (4\pi\alpha^2/3s) \times \beta(3 - \beta^2)/2$.

- 1 With every energy $\sqrt{s}/2$ and muon momentum $\beta\sqrt{s}/2$, $p_1 \cdot p_2 = s/2$, $p_1 \cdot p_3 = p_2 \cdot p_4 = (s/4)(1 - \beta \cos \theta)$ and $p_1 \cdot p_4 = p_2 \cdot p_3 = (s/4)(1 + \beta \cos \theta)$, so the bracket is $(s^2/8)(1 + \beta^2 \cos^2 \theta) + m_\mu^2 s/2 = (s^2/8)(2 - \beta^2 \sin^2 \theta)$, using $4m_\mu^2/s = 1 - \beta^2$
- 2 $\frac{1}{4}\Sigma|M|^2 = e^4(2 - \beta^2 \sin^2 \theta) = 16\pi^2\alpha^2(2 - \beta^2 \sin^2 \theta)$, and $d\sigma/d\Omega = (|p_f|/|p_i|)^{1/4}\Sigma|M|^2/(64\pi^2s)$ with $|p_f|/|p_i| = \beta$ gives $(\alpha^2\beta/4s)(2 - \beta^2 \sin^2 \theta)$
- 3 Integrating, $\int(2 - \beta^2 + \beta^2 \cos^2 \theta)d\Omega = 2\pi[2(2 - \beta^2) + (2/3)\beta^2] = (8\pi/3)(3 - \beta^2)$, so $\sigma = (\alpha^2\beta/4s)(8\pi/3)(3 - \beta^2) = (4\pi\alpha^2/3s)\beta(3 - \beta^2)/2$, which tends to $4\pi\alpha^2/3s$ as $\beta \rightarrow 1$

The ratio $|p_f|/|p_i| = \beta$ is the factor most often left out above threshold, and with it the cross-section's rise from zero at threshold.

(c) CALCULATE · U3 [2]

Calculate σ in nanobarns at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, and, at the lower energy, the ratio of σ to the value $4\pi\alpha^2/3s$ that massless muons would give.

- 1 $4\pi\alpha^2/3 = 2.2306 \times 10^{-4}$, which with 0.3894 GeV² mb is 86.86 nb GeV²; at $\sqrt{s} = 0.300$ GeV, $s = 0.0900$ GeV² and $\beta = \sqrt{1 - 0.04466/0.0900} = 0.7098$, so $\beta(3 - \beta^2)/2 = 0.886$ and $\sigma = (86.86/0.0900)$ nb $\times 0.886 = 965$ nb $\times 0.886 = 855$ nb
- 2 At $\sqrt{s} = 3.00$ GeV, $\beta = 0.9975$ and $\beta(3 - \beta^2)/2 = 1.0000$ to four figures, so $\sigma = 86.86/9.00$ nb $= 9.65$ nb; the ratio at 0.300 GeV is 0.886 , an 11% suppression by the muon mass that has disappeared by 3 GeV

Candidates who forget $(\hbar c)^2$, or use it upside down, are wrong by factors of order 10^6 ; 86.9 nb divided by s in GeV² is the number to check against.

(d) SKETCH · U1 [2]

Sketch on one set of axes the angular distribution at $\sqrt{s} = 0.300$ GeV and at $\sqrt{s} = 3.00$ GeV, each normalised to its own value at $\cos \theta = \pm 1$, that is $(d\sigma/d\Omega)/((d\sigma/d\Omega) \text{ at } \cos \theta = 1)$ against $\cos \theta$ from -1 to $+1$, marking the value of each curve at $\cos \theta = 0$.

- 1 Both normalised curves equal 1 at $\cos \theta = \pm 1$, are symmetric about $\cos \theta = 0$ and have their minimum there; at 3.00 GeV the curve is $(1 + \cos^2 \theta)/2$ to within 0.5%, marked 0.50 at $\cos \theta = 0$
- 2 At 0.300 GeV the curve is $(2 - \beta^2 + \beta^2 \cos^2 \theta)/2$ with $\beta^2 = 0.504$, flatter and lying above the 3.00 GeV curve at every angle between the ends, marked $(2 - \beta^2)/2 = 0.75$ at $\cos \theta = 0$; at threshold it would become flat, the pair being produced in an s wave

A sketch that shows the $1 + \cos^2 \theta$ shape at both energies has missed what the mass term does: near threshold the muons are produced mostly in an s wave and the distribution flattens.

(e) DETERMINE · U2 [2]

Determine by crossing the spin-averaged $|M|^2$ for $e^-\mu^- \rightarrow e^-\mu^-$ at energies where both masses can be neglected, in the Mandelstam variables of that process, and state the sign that crossing introduces.

- 1 With the masses neglected, $p_1 \cdot p_3 = p_2 \cdot p_4 = -t/2$ and $p_1 \cdot p_4 = p_2 \cdot p_3 = -u/2$, so $\frac{1}{4}\Sigma|M|^2$ for $e^+e^- \rightarrow \mu^+\mu^-$ is $2e^4(t^2 + u^2)/s^2$
- 2 Crossing the incoming e^+ into an outgoing e^- and the outgoing μ^+ into an incoming μ^- turns the old s , t and u into the new t , u and s ; each crossed fermion multiplies $\Sigma|M|^2$ by -1 and two are crossed, so the sign is $(-1)^2 = +1$ and $\frac{1}{4}\Sigma|M|^2 = 2e^4(s^2 + u^2)/t^2$, positive as it must be and rising as $1/t^2$ in the forward direction

Crossing one fermion only would give a negative squared amplitude: the sign rule is what keeps the crossed result physical, and checking the sign of the answer catches the error.

Work in natural units with the metric (+, −, −, −). Compton scattering $\gamma(k)e^-(p) \rightarrow \gamma(k')e^-(p')$ has, at tree level, an s-channel diagram, in which the electron absorbs k before emitting k' , and a u-channel diagram, in which it emits k' first. For an electron initially at rest and a photon of energy ω , the spin- and polarisation-summed result integrates to the Klein–Nishina total cross-section $\sigma = 2\pi r_e^2 \{[(1 + \kappa)/\kappa^2][2(1 + \kappa)/(1 + 2\kappa) - \ln(1 + 2\kappa)/\kappa] + \ln(1 + 2\kappa)/(2\kappa) - (1 + 3\kappa)/(1 + 2\kappa)^2\}$, with $\kappa = \omega/m$ and $r_e = \alpha/m$. Crossing the same amplitude into $e^+e^- \rightarrow \gamma\gamma$, with the factor $1/2$ for identical photons, gives for a slow pair the spin-averaged $\sigma_{v_{\text{rel}}} \rightarrow \pi\alpha^2/m^2$ as the relative speed $v_{\text{rel}} \rightarrow 0$. Take $m_e = 0.51100$ MeV, $m_\mu = 105.658$ MeV, $\alpha = 1/137.036$, $\hbar c = 197.327$ MeV fm, $\hbar = 6.582 \times 10^{-22}$ MeV s and $1 \text{ b} = 100 \text{ fm}^2$.

(a) EXPLAIN · U1

[2]

Replacing the polarisation vector of the incoming photon by its momentum k , explain why neither Compton diagram alone satisfies the Ward identity $k_\mu M^{\mu\nu} = 0$, while their sum does.

- 1 In the s-channel diagram $\not{k}$ stands next to $u(p)$: $(\not{p} + m)\not{k}u(p) = [2p \cdot k - \not{k}(\not{p} - m)]u(p) = 2p \cdot k u(p)$ and $k^2 = 0$, so the numerator cancels the denominator $(p + k)^2 - m^2 = 2p \cdot k$ and leaves a term proportional to $\bar{u}(p')\gamma^\mu v u(p)$; in the u-channel diagram $\not{k}$ stands next to $\bar{u}(p')$, and the same steps with $(p' - k)^2 - m^2 = -2p' \cdot k$ leave the same term with the opposite sign
- 2 So each diagram alone gives $\pm e^2 \bar{u}(p')\gamma^\mu v u(p)$, which does not vanish, and only their sum is zero: the Ward identity holds for the gauge-invariant set, and only for the sum can both photon polarisation sums be replaced by $-g_{\mu\nu}$

Candidates who check one diagram, find a non-zero result and conclude that the Ward identity fails have missed that it is a statement about the complete amplitude; the two diagrams are not separately gauge invariant.

(b) CALCULATE · U3

[1]

Calculate the Thomson cross-section $(8\pi/3)r_e^2$ in barns.

- 1 $r_e = \alpha\hbar c/m_e = 197.327 \text{ MeV fm}/(137.036 \times 0.51100 \text{ MeV}) = 2.818 \text{ fm}$, so $(8\pi/3)r_e^2 = 66.52 \text{ fm}^2 = 0.6652 \text{ b}$

The classical radius is α times the reduced Compton wavelength, 386 fm; candidates who put that wavelength itself in place of r_e are out by 137^2 in the cross-section.

(c) CALCULATE · U3

[3]

A 511 keV photon from positron annihilation enters the scintillator of a positron-emission tomograph. Calculate the Klein–Nishina cross-section per electron in barns, its ratio to the Thomson value, and the largest kinetic energy the photon can give to a free electron at rest.

- 1 $\kappa = 0.511/0.51100 = 1.000$, so the bracket is $2(4/3 - \ln 3) + (\ln 3)/2 - 4/9 = 0.4694 + 0.5493 - 0.4444 = 0.5743$, and $\sigma = 2\pi(2.818 \text{ fm})^2 \times 0.5743 = 28.65 \text{ fm}^2 = 0.287 \text{ b}$
- 2 $\sigma/\sigma_T = 0.287/0.665 = 0.431$: at $\kappa = 1$ the cross-section has already fallen to less than half its classical value
- 3 The largest transfer is in back-scattering, $\theta = \pi$: $\omega' = \omega/(1 + 2\kappa) = 170.3 \text{ keV}$, so $T_{\text{max}} = \omega - \omega' = 2\kappa\omega/(1 + 2\kappa) = 340.7 \text{ keV}$, the Compton edge of the 511 keV line

Candidates who evaluate the bracket with $\log_{10}$ obtain a cross-section larger than Thomson's, which should stop them at once: Klein–Nishina never exceeds the Thomson value.

(d) DETERMINE · U3

[3]

True muonium is the atom formed by a μ^+ and a μ^- . Treating it as a hydrogen-like atom bound by the Coulomb potential, determine the lifetime of its para ground state, 1^1S_0 , against annihilation into two photons, which is its only significant decay.

- 1 Only the spin singlet, one of the four spin states, has $C = +1$ and can annihilate into two photons, so it carries the whole of the spin-averaged rate and its σv_{rel} is four times the average: $4\pi\alpha^2/m_\mu^2$
- 2 The rate is $\Gamma = |\psi(0)|^2 \sigma v_{\text{rel}}$; with the reduced mass $m_\mu/2$ the Bohr radius is $a = 2/(m_\mu\alpha)$, so $|\psi(0)|^2 = 1/(\pi a^3) = (m_\mu\alpha)^3/(8\pi)$ and $\Gamma = (m_\mu\alpha)^3/(8\pi) \times 4\pi\alpha^2/m_\mu^2 = \alpha^5 m_\mu/2$
- 3 $\Gamma = (1/137.036)^5 \times 105.658 \text{ MeV}/2 = 2.069 \times 10^{-11} \times 52.83 \text{ MeV} = 1.093 \times 10^{-9} \text{ MeV}$, so $\tau = \hbar/\Gamma = (6.582 \times 10^{-22} \text{ MeV s})/(1.093 \times 10^{-9} \text{ MeV}) = 6.02 \times 10^{-13} \text{ s} = 0.602 \text{ ps}$, para-positronium's 124.5 ps scaled by m_e/m_μ

Two factors are lost most often: the 4 that comes from the spin singlet carrying the whole two-photon rate, and the reduced mass, which makes the atom's radius $2/(m\alpha)$ rather than $1/(m\alpha)$ and changes $|\psi(0)|^2$ by a factor of 8.

(e) EXPLAIN · U1

[1]

Explain why the two Compton diagrams enter with the same sign, whereas the two diagrams of Bhabha scattering, $e^+e^- \rightarrow e^+e^-$, enter with a relative minus sign.

- 1 The Compton diagrams differ only in which photon field is contracted with which external photon, and photon operators commute, so no sign arises; Bhabha's annihilation and exchange diagrams differ in the order in which the fermion operators are contracted with the external states, and moving one anticommuting operator past another brings a factor -1

Candidates who attribute the sign to the s- and t-channel propagators having opposite signs confuse a kinematic factor with the statistics of the fields.

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[10]

Work in natural units $\hbar = c = 1$ with the metric $(+, -, -, -)$. Charge conjugation is a unitary operator C on the Fock space of QED. On the Dirac field it acts as $C\psi(x)C^{-1} = M_C\bar{\psi}^T(x)$ and $C\bar{\psi}(x)C^{-1} = -\psi^T(x)M_C^{-1}$, where ^{T} is the transpose in spinor indices and the 4×4 matrix M_C satisfies $M_C^{-1}\gamma^\mu M_C = -(\gamma^\mu)^T$; on the photon field $CA_\mu(x)C^{-1} = -A_\mu(x)$. Write $(-1)^\mu = +1$ for $\mu = 0$ and -1 for $\mu = 1, 2, 3$. Under parity the Hermitian bilinear $i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi$ is multiplied by $-(-1)^\mu(-1)^\nu$ and $F_{\mu\nu}$ by $(-1)^\mu(-1)^\nu$; under time reversal, which is antiunitary, they are multiplied by $(-1)^\mu(-1)^\nu$ and $-(-1)^\mu(-1)^\nu$ respectively; under C both are multiplied by -1 . Each is evaluated at the transformed point. An electric dipole moment d of the electron would add $L_{\text{EDM}} = -(d/2)(i\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi)F_{\mu\nu}$ to the Lagrangian, with d real, whose non-relativistic limit is the energy $-d\sigma \cdot E$. An experiment on polar molecules has found $|d| < 4.1 \times 10^{-30} \text{ e cm}$. Take $m_e = 0.511 \text{ MeV}$ and $\hbar c = 1.973 \times 10^{-11} \text{ MeV cm}$.

(a) SHOW THAT · U2

[3]

Show that the current $\bar{\psi}\gamma^\mu\psi$ is odd under C while $\bar{\psi}\psi$ is even, identifying the step at which the anticommutation of the fields supplies a sign, and say what would become of the electron's coupling to the photon if the Dirac field were quantised with commutators.

- 1 $C(\bar{\psi}\gamma^\mu\psi)C^{-1} = -\psi^T M_C^{-1} \gamma^\mu M_C \bar{\psi}^T = +\psi^T (\gamma^\mu)^T \bar{\psi}^T = \psi_a (\gamma^\mu)_{ba} \bar{\psi}_b$, using $M_C^{-1} \gamma^\mu M_C = -(\gamma^\mu)^T$
- 2 Reordering $\psi_a \bar{\psi}_b = -\bar{\psi}_b \psi_a$, by anticommutation up to a c-number that normal ordering removes, gives $-\bar{\psi}_b (\gamma^\mu)_{ba} \psi_a = -\bar{\psi} \gamma^\mu \psi$, odd; for $\bar{\psi}\psi$ the same steps give $-\psi^T M_C^{-1} M_C \bar{\psi}^T = -\psi_a \bar{\psi}_a = +\bar{\psi}\psi$, even
- 3 With commuting fields the reordering would bring no sign, the current would be C -even, particle and antiparticle would carry the same charge, and the coupling $-e\bar{\psi}\gamma^\mu\psi A_\mu$ to the C -odd photon could not be C -invariant: the C -odd current, and with it the positron's opposite charge, rests on anticommutation

Candidates who move ψ past $\bar{\psi}$ without a sign find the current even and must then conclude, wrongly, that QED violates C . The sign from anticommutation and the sign in $M_C^{-1} \gamma^\mu M_C = -(\gamma^\mu)^T$ are two different signs, and both are needed.

- (b) DETERMINE · U2 [2]
- Determine whether L_{EDM} is even or odd under each of P, C, T and CP, and check that it is even under CPT.
- Under P the signs multiply to $-(-1)^\mu(-1)^\nu \times (-1)^\mu(-1)^\nu = -1$, since $[(-1)^\mu(-1)^\nu]^2 = 1$; under T to $(-1)^\mu(-1)^\nu \times [(-1)^\mu(-1)^\nu] = -1$, the real coefficient $-d/2$ being untouched by the antiunitary T; under C to $(-1)(-1) = +1$: L_{EDM} is P-odd, T-odd and C-even
 - Hence it is CP-odd, and under CPT the signs multiply to $(-1)(+1)(-1) = +1$, even, as every Hermitian, Lorentz-invariant local term must be; the non-relativistic form $-\bar{\psi}\sigma\cdot E\psi$ agrees, the spin being P-even and T-odd and E being P-odd and T-even
- Candidates who let T complex-conjugate the coefficient as well count the i twice: it is already inside the Hermitian bilinear whose sign is given, and d is real.*
- (c) EXPLAIN · U1 [2]
- Explain, both from C invariance and from the diagrams, why the sum of the one-loop electron-loop diagrams with three external photons vanishes in QED, and what this means for the counterterms QED needs.
- QED is C-invariant and A_μ is C-odd, so $\langle\Omega|T A_\mu A_\nu A_\rho|\Omega\rangle = (-1)^3\langle\Omega|T A_\mu A_\nu A_\rho|\Omega\rangle = 0$; diagrammatically, the two orientations of the electron loop are related by transposing the trace with M_C , which turns each γ^μ into $-(\gamma^\mu)^T$ and reverses each propagator, so they differ by $(-1)^3$ and cancel in pairs (Furry's theorem)
 - The three-photon function, superficially divergent with $D = 4 - B = 1$, therefore needs no counterterm and no A^3 term is generated; with gauge invariance this leaves the electron self-energy, the photon self-energy and the vertex as QED's divergent functions
- A common slip is to say that each loop vanishes separately; only the two orientations together cancel, and for an even number of photons the same pairing doubles the result instead.*
- (d) ESTIMATE · U4 [2]
- A new interaction at a mass scale Λ could generate d at one loop. Estimate the lowest Λ consistent with the experimental bound, stating your assumptions.
- Assume new particles of mass Λ coupling to the electron with couplings g of order 1, a CP-violating phase of order 1, the photon attached with charge e, a loop factor $g^2/16\pi^2$ and one power of m_e for the chirality flip that $\sigma^{\mu\nu}\gamma^5$ requires, so that $d \approx e m_e / (16\pi^2 \Lambda^2)$
 - In natural units $d/e < 4.1 \times 10^{-30} \text{ cm} / (1.973 \times 10^{-11} \text{ MeV cm}) = 2.08 \times 10^{-19} \text{ MeV}^{-1}$, so $\Lambda^2 > 0.511 \text{ MeV} / (16\pi^2 \times 2.08 \times 10^{-19} \text{ MeV}^{-1}) = 1.6 \times 10^{16} \text{ MeV}^2$ and $\Lambda > 1.2 \times 10^8 \text{ MeV} \approx 10^5 \text{ GeV}$, about 100 TeV, beyond the direct reach of any collider
- An estimate earns nothing without its assumptions. A candidate who writes $d \sim e/\Lambda$, without the chirality flip and the loop factor, finds about $5 \times 10^{15} \text{ GeV}$, a scale that means nothing here.*
- (e) PREDICT · U1 [1]
- Predict, from CPT invariance alone, the electric dipole moment of the positron if the electron's were found to be d.
- CPT relates the electron to the positron with every charge and moment reversed, so the positron's dipole moment is $-d$, equal in size and opposite in sign, just as its magnetic moment is opposite to the electron's; a measured difference in size would violate CPT
- Candidates who reason from C alone predict the same relation but cannot defend it: C is not a symmetry of nature, being broken maximally by the weak interaction. Only CPT, which holds for every local, Lorentz-invariant, Hermitian theory, is guaranteed to survive, and it alone fixes the positron's moment.*

Work in natural units with the metric $(+, -, -, -)$. An electron of mass m bound to a nucleus of charge Ze , treated as infinitely heavy, feels at lowest order the Coulomb potential $V(r) = -Z\alpha/r$, whose Fourier transform is $\tilde{V}(q) = -4\pi Z\alpha/|q|^2$. The renormalised one-loop vertex, computed with a photon mass μ_γ , has to first order in the spacelike momentum transfer $q^2 = -|q|^2$ the form factors $F_1(q^2) = 1 + (\alpha/3\pi)(q^2/m^2)[\ln(m/\mu_\gamma) - 3/8]$ and $F_2(0) = \alpha/2\pi$. For an ns state $|\psi_{ns}(0)|^2 = (Z\alpha m)^3/(\pi n^3)$, and states with $l \geq 1$ vanish at the origin. Write $U = 4\alpha(Z\alpha)^4 m/(3\pi n^3)$. In these units, at order $\alpha(Z\alpha)^4$: the F_2 (Pauli) term shifts ns levels by $+(3/8)U$ and $2P_{1/2}$ by $-(1/8)U$; the Uehling (vacuum-polarisation) term shifts ns levels by $-(1/5)U$; Bethe's non-relativistic treatment of photons of energy below a cutoff K shifts ns levels by $U \ln(K/(E)_{ns})$ and $2P_{1/2}$ by $-U \ln((E)_{2p}/(Z^2 Ry))$, where $Z^2 Ry = (Z\alpha)^2 m/2$ and the Bethe logarithms are $\ln((E)_{2s}/(Z^2 Ry)) = 2.8118$ and $\ln((E)_{2p}/(Z^2 Ry)) = -0.0300$. Matching across $(Z\alpha)^2 m \ll K \ll m$ replaces $\ln(m/\mu_\gamma)$ in the high-energy part by $\ln(m/2K) + 5/6$. For hydrogen-like helium, ${}^4\text{He}^+$ ($Z = 2$), take $\alpha = 1/137.036$, $m = 0.511$ MeV and, for $n = 2$, $U = 2170$ MHz, reduced-mass corrections neglected. The same assembly for hydrogen gives 1051.9 MHz for the $2S_{1/2}-2P_{1/2}$ splitting against the measured 1057.8 MHz; for He^+ the measured splitting is 14 041 MHz.

(a) SHOW THAT · U2

[3]

Show that the q^2 term of F_1 adds to the Coulomb potential a contact term $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, and that this raises an ns level by $U[\ln(m/\mu_\gamma) - 3/8]$ while leaving P levels unshifted.

- 1 In a static field the vertex multiplies the Coulomb amplitude by $F_1(q^2)$, F_2 giving the separate Pauli term, so $\tilde{V}(q) = -(4\pi Z\alpha/|q|^2)F_1(-|q|^2) = -4\pi Z\alpha/|q|^2 + (4\pi Z\alpha)(\alpha/3\pi m^2)[\ln(m/\mu_\gamma) - 3/8]$
- 2 The correction does not depend on q , so it transforms to a delta function, $\delta V(r) = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8]\delta^3(r)$, positive, a repulsive contact term that raises the level
- 3 $\Delta E_{ns} = \langle \delta V \rangle = (4Z\alpha^2/3m^2)[\ln(m/\mu_\gamma) - 3/8](Z\alpha m)^3/(\pi n^3) = (4\alpha(Z\alpha)^4 m/3\pi n^3)[\ln(m/\mu_\gamma) - 3/8] = U[\ln(m/\mu_\gamma) - 3/8]$, and zero for $l \geq 1$ because those wavefunctions vanish at the origin

The sign is where candidates go wrong: q^2 is negative for a spacelike transfer, so the q^2 term of F_1 weakens the attraction at large $|q|$, which is a repulsive contact term. A candidate who finds the s levels lowered has lost that sign.

(b) EXPLAIN · U1

[2]

Explain why the result of (a) cannot be used on its own, and why a cutoff K with $(Z\alpha)^2 m \ll K \ll m$ can be found for He^+ , giving the two ends of the window in electronvolts.

- 1 The result depends on the photon mass μ_γ , an infrared regulator: the free-electron vertex treats virtual photons of every energy as though the electron were free, but photons with energies of order the binding, $(Z\alpha)^2 m$, see a bound electron whose intermediate states are atomic levels, and there the free vertex is wrong; Bethe's dipole treatment is right there but fails for photon energies approaching m , where the electron becomes relativistic
- 2 The two descriptions overlap where $(Z\alpha)^2 m \ll K \ll m$: for He^+ , $(Z\alpha)^2 m = (2/137.036)^2 \times 0.511$ MeV = 109 eV and $m = 511\,000$ eV, a window of more than three decades, so both hold at some K and the matching is controlled

Candidates who say that the μ_γ dependence is a divergence to be renormalised away have confused the infrared with the ultraviolet: no counterterm removes μ_γ here; the binding does, by cutting off the soft photons at atomic energies.

(c) DETERMINE · U3

[3]

Using the matching relation, determine the total order- $\alpha(Z\alpha)^4$ shift of the $\text{He}^+ 2S_{1/2}$ level in units of U and in megahertz, showing that K cancels.

- 1 The high-energy part with the matching is $U[\ln(m/2K) + 5/6 - 3/8]$ and the low-energy part $U \ln(K/(E_{2s}))$; their sum, $U[\ln(m/(2(E_{2s}))) + 5/6 - 3/8]$, is free of K , and since $2(E_{2s}) = (Z\alpha)^2 m c^2 (2.8118)$ the logarithm is $\ln(1/(Z\alpha)^2) - 2.8118 = 8.4542 - 2.8118 = 5.6424$
- 2 Adding the Pauli term $+3/8$ and the Uehling term $-1/5$ gives $5.6424 + 5/6 - 3/8 + 3/8 - 1/5 = 6.2757$ in units of U
- 3 $\Delta E(2S_{1/2}) = 6.2757 \times 2170 \text{ MHz} = 13\,618 \text{ MHz}$, of which the Uehling term contributes -434 MHz and the Pauli term $+814 \text{ MHz}$

Candidates who keep K or μ_γ in a final answer have not finished the matching. One who measures $\langle E \rangle_{2s}$ in R_y rather than $Z^2 R_y$ makes an error of $\ln 4 = 1.39 U$, about 3000 MHz .

(d) CALCULATE · U3

[2]

Calculate the shift of $2P_{1/2}$ and hence the $2S_{1/2}-2P_{1/2}$ splitting of He^+ in megahertz.

- 1 $2P_{1/2}$ receives no contact terms, so its shift is $-U \ln(\langle E \rangle_{2p}/(Z^2 R_y)) - U/8 = (0.0300 - 0.1250)U = -0.0950U = -206 \text{ MHz}$
- 2 The splitting is $(6.2757 + 0.0950)U = 6.3707 \times 2170 \text{ MHz} = 13\,824 \text{ MHz}$

The $2P_{1/2}$ level moves down, so its shift adds to the splitting; candidates who subtract its magnitude find $13\,412 \text{ MHz}$.

(e) EVALUATE · U4

[2]

Evaluate the agreement with the measured splittings of He^+ and of hydrogen, and say what the comparison suggests about the largest term omitted.

- 1 For He^+ , $14\,041 - 13\,824 = 217 \text{ MHz}$, 1.5% of the measured value; for hydrogen, $1057.8 - 1051.9 = 5.9 \text{ MHz}$, 0.56%: in both the order- $\alpha(Z\alpha)^4$ assembly falls short, by a fraction that grows by a factor of about 2.8 from $Z = 1$ to $Z = 2$
- 2 A fractional shortfall that grows at least in proportion to Z points to an omitted term one power of $Z\alpha$ higher, of order $\alpha(Z\alpha)^5 m$, from a further Coulomb exchange inside the loop, about $+7 \text{ MHz}$ in hydrogen and $+230 \text{ MHz}$ in He^+ ; two-loop terms are suppressed by a further α/π and do not grow with Z , and the neglected reduced-mass correction (about -1.7 MHz in hydrogen, which widens the gap) and the nuclear size (about $+0.1 \text{ MHz}$ in hydrogen) are too small to account for it

An evaluation that stops at 'agrees to 1.5%' earns one mark. The second is for using the two atoms together: the Z dependence of the shortfall identifies the order of the missing term.

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[12]

An electron-positron collider measures the total cross-section for $e^+e^- \rightarrow \mu^+\mu^-$ at six centre-of-mass energies well below the Z boson. Z exchange and initial- and final-state radiation have been corrected for, so that what remains is one-photon exchange through the vacuum-polarisation-corrected photon propagator, for which $\sigma = 4\pi\alpha(s)^2/(3s)$, with $\alpha(s)$ the effective coupling at momentum transfer squared s ; the muon mass is negligible at these energies. Work in natural units with $\alpha(0) = 1/137.036$ and $(\hbar c)^2 = 0.3894 \text{ GeV}^2 \text{ mb}$, so that $4\pi\alpha(0)^2/3 = 86.86 \text{ nb GeV}^2$. At one loop a fermion of charge $Q_f e$ with N_c colours makes α run as $\mu d\alpha/d\mu = 2N_c Q_f^2 \alpha^2/(3\pi)$ while μ is well above its mass, with $\mu^2 = s$. Every quoted uncertainty is 0.2% of σ , statistical and systematic combined. Lepton loops alone, run from $q^2 = 0$ with the measured lepton masses, give $1/\alpha = 134.02$ at $\sqrt{s} = 12.0 \text{ GeV}$ and 132.73 at $\sqrt{s} = 91.19 \text{ GeV}$.

(a) **SHOW THAT** · U2 [2]

Show that if every fermion lighter than $\sqrt{s}$ contributes at one loop, $1/\alpha(s)$ is a straight line in $\ln s$ with gradient $-\sum_f N_{cQ_f^2}/(3\pi)$, and that $s\sigma/(86.86 \text{ nb GeV}^2) = [\alpha(s)/\alpha(0)]^2$.

1 $d(1/\alpha)/d \ln \mu = -(1/\alpha^2)\mu d\alpha/d\mu = -2\sum_f N_{cQ_f^2}/(3\pi)$, and since $\ln \mu = \frac{1}{2} \ln s$, $d(1/\alpha)/d \ln s = -\sum_f N_{cQ_f^2}/(3\pi)$, constant while no threshold is crossed, so $1/\alpha(s) = 1/\alpha(s_0) - [\sum_f N_{cQ_f^2}/(3\pi)]\ln(s/s_0)$

2 $\sigma = 4\pi\alpha(s)^2/(3s)$ gives $s\sigma = [4\pi\alpha(0)^2/3][\alpha(s)/\alpha(0)]^2$, and $4\pi\alpha(0)^2/3 = 2.2306 \times 10^{-4} \times 0.3894 \text{ GeV}^2 \text{ mb} = 8.686 \times 10^{-5} \text{ GeV}^2 \text{ mb} = 86.86 \text{ nb GeV}^2$

Candidates who differentiate with respect to $\ln \mu$ and then plot against $\ln s$ are out by a factor of 2, and would read the data as requiring only half as many charged species, $\Sigma \approx 3.3$, which looks like the leptons alone.

(b) **DESCRIBE** · U4 [2]

Describe how you would process the readings to test this prediction and extract the gradient, including how the uncertainty in each σ becomes an uncertainty in $1/\alpha$.

- 1 For each run form $R = s\sigma/(86.86 \text{ nb GeV}^2)$ and $1/\alpha(s) = 137.036/\sqrt{R}$, then plot $1/\alpha$ against $\ln(s/\text{GeV}^2)$ and fit a straight line, whose gradient is $-\sum_f N_{cQ_f^2}/(3\pi)$; a straight line tests one-loop running with a fixed set of active species
- 2 Since $1/\alpha \propto \sigma^{(-1/2)}$, $\delta(1/\alpha) = \frac{1}{2}(1/\alpha)(\delta\sigma/\sigma) = \frac{1}{2} \times 131 \times 0.002 \approx 0.13$ at every point, equal error bars, so an unweighted fit is appropriate; the intercept at $\ln s = 0$ has no direct meaning, since it absorbs all the running below 12 GeV

Fitting σ itself against $1/s$, or plotting α rather than $1/\alpha$, gives curves rather than a line; the linear variable is $1/\alpha$, because one-loop running sums a geometric series in the logarithm.

(c) **DETERMINE** · U4 [3]

Determine $1/\alpha$ at each energy, the gradient and intercept of the best straight line with an estimate of the uncertainty in the gradient, and hence $\sum_f N_{cQ_f^2}$.

- 1 $s\sigma = 93.96, 94.54, 95.44, 96.17, 96.51$ and 97.09 nb GeV^2 , so $1/\alpha = 131.76, 131.35, 130.73, 130.23, 130.00$ and 129.62 at $\ln(s/\text{GeV}^2) = 4.970, 5.781, 6.438, 7.111, 7.613$ and 8.121
- 2 Gradient = -0.695 (accept -0.66 to -0.73), with an uncertainty of about 0.05 from 0.13 error bars over the spread of $\ln s$ (accept 0.03 to 0.06), and intercept 135.25 at $\ln(s/\text{GeV}^2) = 0$ (accept 135.0 to 135.5)
- 3 $\sum_f N_{cQ_f^2} = -3\pi \times \text{gradient} = 6.6 \pm 0.5$ (accept 6.2 to 6.9)

Candidates who take the gradient from the first and last points alone get -0.68 , which is acceptable, but must then estimate its uncertainty from the error bars; a gradient quoted without one cannot be used in (d).

(d) **EVALUATE** · U4 [3]

Evaluate which of these the data support: the three charged leptons alone; the leptons with the five quarks u, d, s, c and b each counted once; or the leptons with those quarks in three colours.

- 1 Leptons alone give $\Sigma = 3$ and gradient $-3/(3\pi) = -0.318$; adding u and c ($Q = 2/3$) and d, s and b ($Q = -1/3$) once each gives $\Sigma = 3 + 8/9 + 3/9 = 4.22$ and gradient -0.448 ; with three colours $\Sigma = 3 + 11/3 = 20/3$ and gradient -0.707
- 2 The measured -0.695 ± 0.05 lies about 7.5 standard deviations from the leptons-only value and 5 from uncoloured quarks, and within 0.3 standard deviations of the three-colour value, so the data require quarks in three colours
- 3 The verdict survives what was neglected: the strong-interaction correction multiplies the quark term by about $1 + \alpha_s/\pi \approx 1.05$, moving the prediction to about -0.73 , within the error, and the top quark, at 173 GeV, is not yet active

A comparison that does not use the uncertainty in the gradient is not an evaluation; the point of (c) was to have one.

(e) PREDICT · U3

[2]

Predict $1/\alpha$ at $\sqrt{s} = 91.19$ GeV by extrapolating your line, and say what its difference from the lepton-only running requires of hadrons, already at 12 GeV and at the Z.

- 1 $\ln(s/\text{GeV}^2) = 9.026$ at 91.19 GeV, so $1/\alpha \approx 135.25 - 0.695 \times 9.026 = 128.98$, about 129.0 (accept 128.8 to 129.2), against 132.73 from leptons alone
- 2 Hadrons must supply $\Delta(1/\alpha) \approx 3.7$ at the Z, of which 2.3 (134.02 – 131.76) is already present at 12 GeV and comes from hadronic states below 12 GeV, resonances included, where quark loops with ill-defined masses cannot be trusted; it is obtained instead from a dispersion relation over the measured $e^+e^- \rightarrow \text{hadrons}$ cross-section, which the optical theorem ties to $\text{Im } \Pi$

Candidates who extrapolate from the 58 GeV point with the lepton-only gradient reach about 129.3 and have discarded the slope they measured; the extrapolation must use the fitted line.

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[15]

Work in natural units with the metric $(+, -, -, -)$, in QED with an electron of physical mass m and bare mass m_0 , in Feynman gauge with a small photon mass μ_γ . At one loop the electron self-energy is $-i\Sigma_2(p) = (-ie)^2 \int d^d k / (2\pi)^d \gamma^\mu [i(\not{k} + m)/(k^2 - m^2 + i\epsilon)] \gamma_\mu [-i/((p - k)^2 - \mu_\gamma^2 + i\epsilon)]$, continued to $d = 4 - \epsilon$ dimensions with $e^2 \rightarrow e^2 \mu^\epsilon$ to keep e dimensionless. You may use: in d dimensions $\gamma^\mu \gamma_\mu = d$ and $\gamma^\mu \not{k} \gamma_\mu = -(d - 2)\not{k}$; Feynman's parametrisation $1/(AB) = \int_0^1 dx \, 1/[xA + (1 - x)B]^2$; the Minkowski master integral $\int d^d l / (2\pi)^d \, 1/(l^2 - \Delta + i\epsilon)^2 = i\Gamma(2 - d/2)\Delta^{(d/2 - 2)}/(4\pi)^{(d/2)}$, integrals odd in l vanishing; and, as $\epsilon \rightarrow 0$, $\Gamma(\epsilon/2)(4\pi\mu^2/\Delta)^{\epsilon/2} = 2/\epsilon - \gamma_E + \ln(4\pi\mu^2/\Delta) + O(\epsilon)$. The physical mass is the pole of the full propagator, $m = m_0 + \Sigma_2(\not{p} = m)$ at this order. With the full vertex Γ^μ (its factor $-ie$ removed) and the full inverse propagator $\tilde{S}^{-1}(p) = \not{p} - m_0 - \Sigma(p)$, the Ward-Takahashi identity reads $q_\mu \Gamma^\mu(p + q, p) = \tilde{S}^{-1}(p + q) - \tilde{S}^{-1}(p)$. Take $\alpha = 1/137.036$, $m = 0.511$ MeV and the Planck mass 1.22×10^{19} GeV.

(a) DERIVE · U2

[2]

Derive, by counting powers of loop momentum, the superficial degree of divergence D of a QED diagram in terms of the number B of its external photons and the number F of its external electron lines, and hence the value of D for the electron self-energy.

- 1 With L loops, P_e internal electron and P_γ internal photon lines and V vertices, $D = 4L - P_e - 2P_\gamma$ and $L = P_e + P_\gamma - V + 1$, and each vertex ends two electron lines and one photon line: $2V = 2P_e + F$ and $V = 2P_\gamma + B$
- 2 Substituting, $D = 3P_e + 2P_\gamma - 4V + 4 = 3(V - F/2) + (V - B) - 4V + 4 = 4 - B - (3/2)F$, independent of V , so finitely many functions diverge; the self-energy, with $B = 0$ and $F = 2$, has $D = 1$

Candidates who count external lines as propagators, or forget that every vertex ends two electron lines, lose the independence of D from V , which is the whole point.

(b) EXPLAIN · U1

[2]

Explain why, although $D = 1$, the mass shift can diverge only logarithmically and must be proportional to m , and contrast this with what happens to a scalar's mass.

- 1 At $m_0 = 0$ the QED Lagrangian is invariant under the chiral rotation $\psi \rightarrow e^{i\theta\gamma^5}\psi$, which forbids a $\bar{\psi}\psi$ term, so Σ at $\not{p} = m$ must vanish as $m \rightarrow 0$: δm is proportional to m , which takes one power of the cutoff out of $D = 1$ and leaves a logarithm
- 2 No symmetry protects a scalar's mass, so its shift is of order $(\text{coupling}/16\pi^2)\Lambda^2$, quadratically sensitive to high scales; the electron mass is natural because a small m is protected by the symmetry that appears when it vanishes

The answer 'because the integral happens to be logarithmic' earns nothing: a symmetry forces the power of m , and the power of m forces the logarithm.

- (c) **SHOW THAT** · U2 [4]
- Show that $\Sigma_2(p) = [e^2 \mu^\epsilon / (4\pi)^{d/2}] \int_0^1 dx [(4 - \epsilon)m - (2 - \epsilon)x \not{p}] \Gamma(2 - d/2) \Delta^{d/2 - 2}$, with $\Delta = (1 - x)m^2 + x\mu_\gamma^2 - x(1 - x)p^2$.
- 1 In d dimensions the numerator is $\gamma^\mu (\not{k} + m) \gamma_\mu = -(d - 2)\not{k} + dm = -(2 - \epsilon)\not{k} + (4 - \epsilon)m$
 - 2 With x multiplying the photon denominator, $x[(p - k)^2 - \mu_\gamma^2] + (1 - x)(k^2 - m^2) = (k - xp)^2 - \Delta$, with $\Delta = (1 - x)m^2 + x\mu_\gamma^2 - x(1 - x)p^2$
 - 3 Shifting to $l = k - xp$, the term linear in l vanishes and the numerator becomes $(4 - \epsilon)m - (2 - \epsilon)x \not{p}$
 - 4 The master integral gives $i\Gamma(2 - d/2)\Delta^{d/2 - 2}/(4\pi)^{d/2}$, and the vertices and propagator numerators give $(-ie)^2 \times i \times (-i) = -e^2$, with $e^2 \rightarrow e^2 \mu^\epsilon$, so $-i\Sigma_2 = -e^2 \mu^\epsilon \times i[\dots]/(4\pi)^{d/2}$ and $\Sigma_2 = [e^2 \mu^\epsilon / (4\pi)^{d/2}] \int_0^1 dx [(4 - \epsilon)m - (2 - \epsilon)x \not{p}] \Gamma(2 - d/2) \Delta^{d/2 - 2}$
- Using the four-dimensional identities $\gamma^\mu \gamma_\mu = 4$ and $\gamma^\mu \not{k} \gamma_\mu = -2\not{k}$ here loses the $O(\epsilon)$ terms that, multiplied by the $2/\epsilon$ pole, supply finite parts of δm ; the result would then depend on the order in which the limits are taken.*
- (d) **DERIVE** · U2 [3]
- Derive the self-mass $\delta m = m - m_0$ at one loop, showing that it is proportional to m , logarithmic and finite as $\mu_\gamma \rightarrow 0$, and find its finite part.
- 1 On shell $\not{p} = m$ and $p^2 = m^2$, and as $\mu_\gamma \rightarrow 0$, $\Delta = (1 - x)^2 m^2$, whose logarithm is integrable at $x = 1$, so $\delta m = \Sigma_2(\not{p} = m)$ needs no photon mass; with $e^2 \mu^\epsilon / (4\pi)^{d/2} \Gamma(\epsilon/2) \Delta^{-\epsilon/2} = (\alpha/4\pi) \Gamma(\epsilon/2) (4\pi\mu^2/\Delta)^{\epsilon/2}$, $\delta m = (\alpha/4\pi) m \int_0^1 dx [(4 - \epsilon) - (2 - \epsilon)x] [2/\epsilon - \gamma_E + \ln(4\pi\mu^2/((1 - x)^2 m^2))]$
 - 2 The pole term gives $(2/\epsilon) \int_0^1 (4 - 2x) dx = 6/\epsilon$; the $O(\epsilon)$ part of the numerator, $-\epsilon(1 - x)$, times $2/\epsilon$ gives $-2 \int_0^1 (1 - x) dx = -1$; and $-\int_0^1 (4 - 2x) \ln[(1 - x)^2] dx = -2 \int_0^1 (2 + 2y) \ln y dy = +5$
 - 3 So $\delta m = (3\alpha/4\pi) m [2/\epsilon - \gamma_E + \ln(4\pi\mu^2/m^2) + 4/3]$: proportional to m and only logarithmically dependent on the regulator, the $1/\epsilon$ pole standing for $\ln \Lambda^2$
- The -1 from the $O(\epsilon)$ numerator times the pole is the finite part the d -dimensional Dirac algebra supplies; candidates who drop it get $5/3$ for the constant instead of $4/3$.*
- (e) **COMPARE** · U3 [2]
- Interpreting the pole as $\ln(\Lambda^2/m^2)$ for a cutoff Λ , so that $\delta m \approx (3\alpha/4\pi) m \ln(\Lambda^2/m^2)$, compare $\delta m/m$ for Λ equal to the Planck mass with the shift of order $\alpha\Lambda$ that a linearly divergent electromagnetic mass would give.
- 1 $\ln(\Lambda^2/m^2) = 2 \ln(1.22 \times 10^{19} \text{ GeV}/5.11 \times 10^{-4} \text{ GeV}) = 103.1$, so $\delta m/m = [3/(4\pi \times 137.036)] \times 103.1 = 0.0017421 \times 103.1 = 0.18$: an 18% shift even with the Planck mass as cutoff
 - 2 A linear shift $\alpha\Lambda = 1.22 \times 10^{19} \text{ GeV}/137.036 = 8.9 \times 10^{16} \text{ GeV}$ would be 1.7×10^{20} times m , so the observed mass would need a cancellation to twenty decimal places; the relativistic theory, with its antiparticles, turns the linear divergence into a logarithm and makes the electron's small mass natural
- Candidates who report $\delta m/m \approx \alpha$ are close in size but have missed the comparison: it is the dependence on Λ , logarithmic against linear, that separates a natural mass from a fine-tuned one.*

(f) SHOW THAT · U2

[2]

From the Ward-Takahashi identity show that $Z_1 = Z_2$, where near the mass shell $\tilde{S}^{-1}(p) \rightarrow Z_2^{-1}(\not{p} - m)$ and, between on-shell spinors as $q \rightarrow 0$, $\Gamma^\mu(p, p) \rightarrow Z_1^{-1}\gamma^\mu$.

- 1 Expanding the identity to first order in q , $q_\mu \Gamma^\mu(p + q, p) = q_\mu \partial \tilde{S}^{-1}(p) / \partial p_\mu + O(q^2)$, so $\Gamma^\mu(p, p) = \partial \tilde{S}^{-1}(p) / \partial p_\mu$, Ward's original identity
- 2 Near the mass shell $\partial \tilde{S}^{-1} / \partial p_\mu \rightarrow Z_2^{-1} \gamma^\mu$, so $Z_1^{-1} \gamma^\mu = Z_2^{-1} \gamma^\mu$ and $Z_1 = Z_2$: the vertex and field renormalisations cancel in $e = Z_2 Z_3^{1/2} e_0 / Z_1$, and the charge is renormalised by Z_3 alone

Candidates who argue from the one-loop expressions show only $\delta_1 = \delta_2$ at this order; the identity holds to all orders and for any charged particle, which is what makes charge universal.

19
CORE
COURSE

FREE RESPONSE · DERIVE, CONFRONT WITH MEASUREMENT AND CANCEL THE INFRARED · DISCRIMINATING · UQFTI-14.01 · UQFTI-14.02 · UQFTI-14.03 · UQFTI-14.04 · UQFTI-14.07 · UQFTI-14.08 · UQFTI-14.09

[16]

Work in natural units with the metric $(+, -, -, -)$. Between on-shell spinors the full electron-photon vertex is $\bar{u}(p') \Gamma^\mu u(p)$, with $q = p' - p$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$; the Gordon identity reads $\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p')[(p' + p)^\mu / 2m + i\sigma^{\mu\nu} q_\nu / 2m] u(p)$. At one loop in Feynman gauge, after Feynman parameters, the shift $\ell = k + yq - zp$ and use of the Dirac equation, the correction is $\bar{u}(p') \delta \Gamma^\mu u(p) = 2ie^2 \int d^4\ell / (2\pi)^4 \int_0^1 dx dy dz \delta(x + y + z - 1) (2/D^3) \bar{u}(p') N^\mu u(p)$, with $D = \ell^2 - \Delta + i\epsilon$, $\Delta = -xyq^2 + (1 - z)^2 m^2$ and $N^\mu = \gamma^\mu [-\ell^2/2 + (1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2] + (i\sigma^{\mu\nu} q_\nu / 2m)[2m^2 z(1 - z)]$. Use $\int d^4\ell / (2\pi)^4 1/(\ell^2 - \Delta)^3 = -i/(2(4\pi)^2 \Delta)$. The measured electron anomaly is $a_e = 1.159\,652\,180\,59(13) \times 10^{-3}$, and the QED series is $a_e = \alpha/2\pi + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots$, with $C_2 = -0.328\,478\,965$, $C_3 = 1.181\,241\,456$ and $C_4 = -1.912\,245\,764$; in this question take $\alpha^{-1} = 137.035\,999$, from atom-recoil measurements. For elastic scattering of an electron of energy E through angle θ by a recoil-free nucleus, $-q^2 = 4E^2 \sin^2(\theta/2)$; the order- α virtual correction multiplies the Born cross-section by $1 - (\alpha/\pi) f_{\text{IR}}(q^2) \ln(-q^2/\mu_\gamma^2)$, μ_γ being a photon mass, and the probability of emitting one soft photon with energy between μ_γ and E_γ is $(\alpha/\pi) f_{\text{IR}}(q^2) \ln(E_\gamma^2/\mu_\gamma^2)$, where for $-q^2 \gg m^2$, $f_{\text{IR}}(q^2) = \ln(-q^2/m^2) - 1$. Take $m = 0.511$ MeV.

(a) JUSTIFY · U1

[2]

Justify writing the on-shell vertex as $\Gamma^\mu = \gamma^\mu F_1(q^2) + (i\sigma^{\mu\nu} q_\nu / 2m) F_2(q^2)$.

- 1 Between on-shell spinors the vertex can be built only from γ^μ , p^μ and p'^μ , that is γ^μ , $(p' + p)^\mu$ and q^μ , multiplied by functions of the one invariant q^2 , since $p^2 = p'^2 = m^2$; structures with γ^5 are excluded because QED conserves parity, and the Gordon identity trades $(p' + p)^\mu$ for γ^μ and $i\sigma^{\mu\nu} q_\nu$
- 2 A q^μ term is excluded by the Ward identity $q_\mu \Gamma^\mu = 0$: $q_\mu \gamma^\mu$ vanishes between on-shell spinors by the Dirac equation and $q_\mu \sigma^{\mu\nu} q_\nu = 0$ by antisymmetry, but $q_\mu q^\mu = q^2$ does not, so its coefficient must vanish and two form factors remain

Candidates who list structures without using current conservation are left with three form factors; it is the Ward identity, not Lorentz invariance, that removes q^μ .

(b) SHOW THAT · U2

[2]

Show that in a slowly varying magnetic field the vertex gives the electron a magnetic moment with $g = 2[F_1(0) + F_2(0)]$.

- 1 In a static vector potential with Fourier transform $A^i(q)$ the amplitude contains $\bar{u}(p') \Gamma^i u(p) A^i(q)$; in the non-relativistic expansion of the spinors the spin-dependent part of $\bar{u}(p') \gamma^i u(p)$ is $2m \xi^\dagger (-i\epsilon^{ijk} q^j \sigma^k / 2m) \xi$, through the lower components, and $\bar{u}(p') (i\sigma^{i\nu} q_\nu / 2m) u(p)$ gives the same structure, so the spin term is weighted by $F_1(0) + F_2(0)$
- 2 Since $\epsilon^{ijk} q^j A^k(q)$ is, up to a factor $\pm i$ set by the Fourier convention, the transform of the magnetic field $B = \nabla \times A$, the spin term is the Born amplitude of the energy $-\mu \cdot B$ with $\mu = (e/m)[F_1(0) + F_2(0)]S$, that is $g = 2[F_1(0) + F_2(0)]$; $F_1(0) = 1$ defines the charge, so $g - 2 = 2F_2(0)$ and $a_e = F_2(0)$

Candidates who take the moment from the $\sigma^{\mu\nu}$ term alone obtain $g = 2F_2(0)$; the Dirac moment $g = 2$ is already in γ^μ , through the lower components of the spinors.

(c) DERIVE · U2

[3]

Derive $F_2(q^2)$ as an integral over Feynman parameters, evaluate $F_2(0)$, and explain why F_2 needs neither an ultraviolet nor an infrared regulator.

- 1 Only the $\sigma^{\mu\nu}$ term of N^μ contributes to F_2 , giving $2ie^2 \times 2 \times 2m^2 z(1-z) \times [-i/(2(4\pi)^2 \Delta)]$ under the parameter integrals, that is $F_2(q^2) = (\alpha/2\pi) \int dx dy dz \delta(x+y+z-1) 2m^2 z(1-z)/[(1-z)^2 m^2 - xyq^2]$
- 2 At $q^2 = 0$ the integrand depends only on z , and $\int dx dy \delta(x+y+z-1) = 1-z$, so $F_2(0) = (\alpha/2\pi) \int_0^1 (1-z) \times 2z(1-z)/(1-z)^2 dz = (\alpha/2\pi) \int_0^1 2z dz = \alpha/2\pi$
- 3 The $\sigma^{\mu\nu}$ term carries no ℓ^2 in its numerator, so $\int d^4\ell/D^3$ converges in the ultraviolet and F_2 has no counterterm; at $z \rightarrow 1$ the factor $z(1-z)$ and the measure's $1-z$ cancel the $(1-z)^2$ of Δ , so the integral is finite without a photon mass: $F_2(0)$ is a prediction

The factor 2 in $2/D^3$ from the Feynman parametrisation is the one most often dropped, giving $\alpha/4\pi$. A second check is that a prediction cannot depend on a counterterm, and F_2 has none to depend on.

(d) CALCULATE · U3

[3]

Calculate the residual, the measured a_e minus the QED series truncated after one, two and three terms, and compare each residual with the size of the next term.

- 1 $\alpha/2\pi = 1.161\,409\,73 \times 10^{-3}$, and $a_e - \alpha/2\pi = -1.7576 \times 10^{-6}$; divided by $(\alpha/\pi)^2 = 5.3955 \times 10^{-6}$ this is -0.326 , close to C_2
- 2 $C_2(\alpha/\pi)^2 = -1.7723 \times 10^{-6}$, leaving $+1.475 \times 10^{-8}$; divided by $(\alpha/\pi)^3 = 1.2533 \times 10^{-8}$ this is 1.18 , close to C_3
- 3 $C_3(\alpha/\pi)^3 = 1.4804 \times 10^{-8}$, leaving -5.2×10^{-11} , the size of the four-loop term $C_4(\alpha/\pi)^4 = -5.57 \times 10^{-11}$: each order shrinks the residual by a factor of order α/π and confirms the next coefficient

The subtraction must be carried to ten significant figures; candidates who round $\alpha/2\pi$ to 1.1614×10^{-3} find a residual dominated by their own rounding.

(e) ESTIMATE · U4

[1]

Estimate the fractional precision with which this measurement of a_e would determine α if the QED series were known exactly, stating your assumption.

- 1 Assuming the theoretical series, hadronic and weak terms included, is exact, $a_e \approx \alpha/2\pi$ to leading order, so $\delta\alpha/\alpha \approx \delta a_e/(\alpha/2\pi) = 1.3 \times 10^{-13}/1.1614 \times 10^{-3} = 1.1 \times 10^{-10}$, about 0.1 parts per billion

The assumption is what matters: the theory must add no uncertainty of its own, since dropping even the three-loop term would shift α^{-1} by $2\pi \times 1.48 \times 10^{-8} \times 137.036^2 = 0.0017$, some 10^5 times this precision.

(f) SHOW THAT · U2

[2]

Show that in the cross-section measured by a detector that cannot register photons below E_I the photon mass cancels at order α , leaving the factor $1 - (\alpha/\pi)f_{\text{IR}}(q^2)\ln(-q^2/E_I^2)$.

- 1 The detector counts elastic events and events with one undetected photon below E_I together, so at order α the measured cross-section is the Born value times $[1 - (\alpha/\pi)f_{\text{IR}}\ln(-q^2/\mu_\gamma^2)] + (\alpha/\pi)f_{\text{IR}}\ln(E_I^2/\mu_\gamma^2)$
- 2 Since $\ln(-q^2/\mu_\gamma^2) - \ln(E_I^2/\mu_\gamma^2) = \ln(-q^2/E_I^2)$, μ_γ cancels and the factor is $1 - (\alpha/\pi)f_{\text{IR}}(q^2)\ln(-q^2/E_I^2)$: the infrared divergences of the virtual and real corrections cancel in the measured rate, as Bloch and Nordsieck showed

Candidates who try to remove μ_γ with a counterterm have confused the infrared with the ultraviolet; the cancellation is between final states that the detector cannot tell apart.

(g) CALCULATE · U3

[2]

Calculate this correction factor for 4.0 GeV electrons scattered through 20° with $E_I = 20$ MeV, and compare it with its exponentiated form $\exp[-(\alpha/\pi)f_{IR} \ln(-q^2/E_I^2)]$.

- 1 $-q^2 = 4 \times (4.0 \text{ GeV})^2 \times \sin^2 10^\circ = 1.930 \text{ GeV}^2$, so $\ln(-q^2/m^2) = \ln[1.930/(5.11 \times 10^{-4})^2] = 15.82$ and $f_{IR} = 14.82$, and $\ln(-q^2/E_I^2) = \ln(1.930/0.0004) = 8.48$
- 2 $(\alpha/\pi)f_{IR} \ln(-q^2/E_I^2) = 0.0023228 \times 14.82 \times 8.48 = 0.292$, so the order- α factor is 0.708 while the exponentiated form gives $e^{-0.292} = 0.747$: the correction is so large that the difference of about 0.04 between them, the α^2 terms, matters at this resolution

A correction of 29% is no sign of error but the double logarithm doing its work; candidates who stop at the order- α factor have ignored α^2 terms that exponentiation shows to be about 0.04.

(h) PREDICT · U1

[1]

Predict what happens, as $E_I \rightarrow 0$, to the rate of scattering with no photon above E_I and to the inclusive rate.

- 1 The exponentiated factor $(-q^2/E_I^2)^{-(\alpha/\pi)f_{IR}}$ tends to zero as $E_I \rightarrow 0$, so scattering with no photon at all is impossible, every scattered electron radiating a cloud of soft photons; the inclusive rate, summed over all emissions, stays finite and close to the Born value

Candidates who conclude from (g) that the correction becomes infinitely negative as $E_I \rightarrow 0$ have kept only the first order of a series that sums to a power tending to zero.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 26 MARKS · 22%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 47 MARKS · 40%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

31 MARKS · 26%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

13 MARKS · 11%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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