

University Quantum Mechanics I · Final Examination

The whole of Quantum Mechanics I, weighted towards the second half of the course: the harmonic oscillator by series and by ladder operators, Dirac notation and two-state systems, angular momentum and spin- $\frac{1}{2}$, and the Schrödinger equation in three dimensions through to hydrogen. The wavefunction, operators, measurement, wells and tunnelling from the first half are sampled throughout.

SUGGESTED TIME 150 minutes	TOTAL MARKS 118	QUESTIONS 19	MULTIPLE CHOICE 10
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SECTION	UQMI · Final · University Quantum Mechanics I — Final examination
IN THE FORMAT OF	A cumulative university final: conceptual multiple choice, structured problems, a data-analysis question and two extended derivations
DEMAND	Recall 1 · Routine 8 · Demanding 7 · Discriminating 3 <i>Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.</i>
COURSE UNITS	UQMI-01 · UQMI-02 · UQMI-03 · UQMI-04 · UQMI-05 · UQMI-06 · UQMI-07 · UQMI-08 · UQMI-09 · UQMI-10 · UQMI-11 · UQMI-12 · UQMI-13 · UQMI-14 · UQMI-15

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Quantum Mechanics I · Final Examination

Answer all questions. 118 marks in 150 minutes.

1 MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 [1]
CORE COURSE

$\psi(x)$ is a normalised wavefunction for a particle moving along the x-axis. A second state of the particle has a position probability density of the same shape as $|\psi(x)|^2$, but compressed by a factor of 2 towards $x = 0$, so that every feature of the density lies at half its original distance from $x = 0$. Which wavefunction describes the second state?

- A $\psi(2x)$
- B $2\psi(2x)$
- C $\sqrt{2} \psi(x/2)$
- D $\sqrt{2} \psi(2x)$

Answer

2 MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.06 [1]
CORE COURSE

Each option refers to a particle of mass m in one dimension. Which pair of operators commutes, so that the two observables are compatible and can share a complete set of eigenfunctions?

- A $\hat{p}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle
- B $\hat{x}$ and $\hat{p}$
- C $\hat{p}$ and $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$, for a harmonic oscillator
- D $\hat{x}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle

Answer

3 MULTIPLE CHOICE · ROUTINE · UQMI-05.06 · UQMI-05.07 [1]
CORE COURSE

The simplest model of a wave packet for a free neutron ($m = 1.675 \times 10^{-27}$ kg) is the sum of two plane waves, $\Psi(x,t) = e^{i(k_1x - \omega_1t)} + e^{i(k_2x - \omega_2t)}$, with $k_1 = 2.90 \times 10^{10} \text{ m}^{-1}$ and $k_2 = 3.10 \times 10^{10} \text{ m}^{-1}$, and each frequency given by the free-particle dispersion relation $\omega = \hbar k^2/(2m)$. The probability density $|\Psi(x,t)|^2$ is then a row of fringes about 3.1 nm apart. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$. At what speed does the fringe pattern of $|\Psi(x,t)|^2$ move?

- A $9.4 \times 10^2 \text{ m s}^{-1}$
- B $1.9 \times 10^3 \text{ m s}^{-1}$
- C $3.8 \times 10^3 \text{ m s}^{-1}$
- D $1.2 \times 10^4 \text{ m s}^{-1}$

Answer

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-07.07

[1]

An F-centre is an electron trapped at the site of a missing negative ion in an alkali-halide crystal. Along one axis, model it as an electron ($m = 9.109 \times 10^{-31}$ kg) in a symmetric finite square well of total width 0.56 nm and depth V_0 . In this model the centre's optical absorption lifts the electron from the ground state to the first excited bound state, so the well must bind at least two states. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J. What is the smallest depth V_0 for which the well binds a second state?

- A Any depth greater than zero
- B 0.30 eV
- C 1.2 eV
- D 4.8 eV

Answer

5

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-08.04 · UQMI-08.05

[1]

An electron with kinetic energy E meets a rectangular barrier of height $V_0 > E$ and width w . The thick-barrier estimate $T \approx \exp(-2\kappa w)$, with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$ and the prefactor ignored, gives $T = 1.0 \times 10^{-3}$. A proton with the same kinetic energy meets the same barrier. Taking $m_p/m_e = 1836$, what does the same estimate give for the proton's transmission probability?

- A 2.3×10^{-5}
- B 5.4×10^{-7}
- C 3×10^{-129}
- D 10^{-5508}

Answer

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-09.08 · UQMI-09.09

[1]

For a one-dimensional harmonic oscillator with energy eigenstates $|n\rangle$, the position operator can be written $\hat{x} = \sqrt{\hbar/2m\omega}(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Which of the following matrix elements is non-zero?

- A $\langle 3|\hat{x}|2\rangle$
- B $\langle 2|\hat{x}|2\rangle$
- C $\langle 0|\hat{x}|2\rangle$
- D $\langle 4|\hat{x}|2\rangle$

Answer

7

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQMI-10.07 · UQMI-10.08

[1]

An observable $\hat{A}$ is represented in one orthonormal basis by the Hermitian matrix A , and in a second orthonormal basis by $A' = U^\dagger A U$, where U is the unitary matrix of overlaps between the two bases. Which of the following is in general different for A' than for A ?

- A The trace of the matrix
- B The set of eigenvalues
- C The expectation value $\langle \psi | \hat{A} | \psi \rangle$ for a given state, computed from that state's components in the same basis as the matrix
- D The first diagonal element, A_{11}

Answer

8

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-11.02 · UQMI-11.05

[1]

A two-state system with orthonormal basis states $|1\rangle$ and $|2\rangle$ is prepared in $|\psi\rangle = (|1\rangle + i|2\rangle)/\sqrt{2}$. Which of the following is the same physical state, giving identical probabilities for every possible measurement?

- A $(|1\rangle - i|2\rangle)/\sqrt{2}$
- B $(i|1\rangle - |2\rangle)/\sqrt{2}$
- C $(|1\rangle + |2\rangle)/\sqrt{2}$
- D $(i|1\rangle + |2\rangle)/\sqrt{2}$

Answer

9

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-14.02

[1]

Near the centre of an optical trap, an atom of mass m moves in the potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2) + \frac{1}{2}m(2\omega)^2z^2$, so that its classical oscillation frequency along z is twice that along x and y . How many independent energy eigenstates share the energy $E = 5\hbar\omega$?

- A 4
- B 6
- C 10
- D 12

Answer

10

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-15.05

[1]

The lithium ion Li^{2+} ($Z = 3$) has a single electron. The energy levels of hydrogen are $E_n = -13.6 \text{ eV}/n^2$. How much energy is needed to ionise Li^{2+} from its first excited state?

- A 3.40 eV
- B 10.2 eV
- C 30.6 eV
- D 122 eV

Answer

A single ^{87}Rb atom of mass $m = 1.443 \times 10^{-25} \text{ kg}$ moves along an optical box trap whose repulsive light walls are steep and high enough to be treated as an infinite square well on $0 < x < L$, with $L = 20.0 \text{ }\mu\text{m}$. The stationary states are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2 E_1$. At $t = 0$ the atom is prepared near the left wall in the state $\Psi(x, 0) = (1/\sqrt{2})\psi_1(x) + (1/2)\psi_2(x) + (1/2)\psi_3(x)$, for which $\langle x \rangle = 0.275L$. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$, $h = 6.626 \times 10^{-34} \text{ J s}$ and $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$.

(a) CALCULATE

[2]

Calculate the atom's zero-point energy E_1 , and express it as a temperature E_1/k_B .

(b) STATE

[2]

State the possible results of a measurement of the atom's energy and the probability of each, and state whether these probabilities change as time passes.

(c) CALCULATE

[2]

Every phase factor $e^{(-iE_n t/\hbar)}$ in $\Psi(x, t)$ returns to 1 at the same instant, after which the atom's state repeats exactly. Calculate the earliest such time.

(d) EXPLAIN

[2]

At $t = h/(2E_1)$ the probability density is $|\Psi(L - x, 0)|^2$, the mirror image of the initial density about the centre of the trap. Explain why, and state $\langle x \rangle$ at that instant.

(e) PREDICT

[2]

Suppose instead that the atom's energy is measured at $t = 0$ and measured again at $t = h/(2E_1)$. Predict the probability that the two results agree, and whether the density at $t = h/(2E_1)$ is then the mirror image of the initial density.

A beam of spin- $\frac{1}{2}$ atoms is prepared in the spin state $\chi = (1/\sqrt{3})(1, 1 + i)^T$, written as a column in the S_z basis $\{|+z\rangle, |-z\rangle\}$. The spin operators are $S_i = (\hbar/2)\sigma_i$, where σ_x , σ_y and σ_z are the Pauli matrices.

(a) STATE

[2]

State the three Pauli matrices σ_x , σ_y and σ_z in the S_z basis.

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(b) CALCULATE

[2]

Calculate the probability that a measurement of S_z gives $+\hbar/2$, and the expectation value $\langle S_z \rangle$.

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(c) DETERMINE

[3]

Determine the probability that a measurement of S_x gives $+\hbar/2$.

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(d) DETERMINE

[3]

The atoms pass through an S_x analyser, and only those leaving its $+\hbar/2$ port go on to an S_z analyser. Determine the fraction of the original atoms that leave the $-\hbar/2$ port of the S_z analyser, and give the reason it is not related to the $2/3$ found in (b).

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Very slow electrons are strongly reflected by a crystal surface even though the crystal attracts them. Inside the crystal an electron's potential energy is lower than in the vacuum outside by U , the inner potential, so the surface at $x = 0$ is modelled as a downward potential step: $V(x) = 0$ for $x < 0$ (vacuum) and $V(x) = -U$ for $x > 0$ (crystal). Electrons of energy $E = 0.50$ eV arrive from the vacuum, moving in the $+x$ direction at normal incidence. Their stationary scattering state is $\psi(x) = e^{ikx} - (2/3)e^{-ikx}$ for $x < 0$ and $\psi(x) = (1/3)e^{iqx}$ for $x > 0$, where $\hbar k = \sqrt{2m_e E}$ and $\hbar q = \sqrt{2m_e(E + U)}$, but the value of U is not known. The probability current is $j = (\hbar/m_e) \text{Im}(\psi^* d\psi/dx)$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

- (a) **DERIVE** [3]
Stating the matching conditions that ψ must satisfy at $x = 0$, and why each holds here, derive the ratio q/k for this state and hence the inner potential U in electronvolts.
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- (b) **SHOW THAT** [2]
Show that the probability current is the same on both sides of the surface, as conservation of probability requires in a stationary state.
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- (c) **CALCULATE** [3]
A beam of these electrons carrying a current of 2.0 nA strikes the surface. Calculate the reflected and transmitted electron currents, and the speed of the transmitted electrons inside the crystal.
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- (d) **CALCULATE** [2]
Calculate the electron energy at which the reflected fraction at this surface falls to 1.0%.
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- (e) **EXPLAIN** [2]
A classical electron crossing the surface would simply speed up, and none would be reflected. Explain why the quantum electron can be reflected by a drop in potential, and why far fewer electrons would be reflected if V fell from 0 to $-U$ smoothly over several nanometres instead of abruptly.
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An electron shared between two neighbouring semiconductor quantum dots, L and R, is modelled as a two-state system in the basis $\{|L\rangle, |R\rangle\}$ of states localised on one dot. Energies are measured from the mean of the two on-dot energies, and in that basis the Hamiltonian is $H = [[\epsilon/2, -\tau], [-\tau, -\epsilon/2]]$. The detuning $\epsilon = 30 \mu\text{eV}$ is set by a gate voltage, and the tunnel coupling is $\tau = 20 \mu\text{eV}$. The next single-electron level in each dot lies about 1.5 meV above these states. If the electron starts on dot L at $t = 0$, the probability of finding it on dot R is $P_R(t) = [4\tau^2/(E_+ - E_-)^2] \sin^2[(E_+ - E_-)t/(2\hbar)]$, where E_+ and E_- are the eigenvalues of H . Take $\hbar = 6.626 \times 10^{-34} \text{ J s}$ and $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$.

(a) JUSTIFY

[2]

Justify the two-state model of the electron, using the energies given.

(b) DETERMINE

[3]

Determine the eigenvalues E_+ and E_- of H , and hence the splitting $E_+ - E_-$.

(c) DETERMINE

[3]

Determine the probability of finding the electron on dot R when it is in the ground state.

(d) CALCULATE

[2]

The electron is placed on dot L at $t = 0$. Calculate the maximum probability of later finding it on dot R, and the period of the oscillation of $P_R(t)$.

(e) PREDICT

[2]

The gate voltage is adjusted until $\epsilon = 0$. Predict the maximum probability of transfer to dot R, and the time after $t = 0$ at which it is first reached.

For orbital angular momentum with $l = 1$, use the ordered basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, in which $L_z = \hbar \text{diag}(1, 0, -1)$. The raising and lowering operators $L_{\pm} = L_x \pm iL_y$ act as $L_{\pm}|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l,m\pm 1\rangle$. The uncertainty relation for L_x and L_y is $\Delta L_x \Delta L_y \geq (\hbar/2)|\langle L_z \rangle|$.

(a) SHOW THAT

[2]

Show that the only non-zero matrix elements of L_+ in this basis are $\langle 1,1|L_+|1,0\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\hbar$.

(b) DETERMINE

[2]

Determine the 3×3 matrix that represents L_x in this basis.

(c) DETERMINE

[3]

Determine the eigenvalues of L_x .

(d) CALCULATE

[3]

A particle is prepared in $|1,1\rangle$. Calculate the probabilities of obtaining $\hbar$, 0 and $-\hbar$ in a measurement of L_x .

(e) SHOW THAT

[2]

Show that in the state $|1,1\rangle$ the product $\Delta L_x \Delta L_y$ is exactly equal to the lower bound $(\hbar/2)|\langle L_z \rangle|$.

For a hydrogen atom, the radial function $u(r) = rR(r)$ satisfies $-(\hbar^2/2m) d^2u/dr^2 + [-e^2/(4\pi\epsilon_0 r) + \hbar^2 l(l+1)/(2mr^2)] u = Eu$. A proposed solution is $u(r) = N r^3 e^{-(r/(3a_0))}$, where N is a normalisation constant and $a_0 = 4\pi\epsilon_0\hbar^2/(me^2) = 5.29 \times 10^{-11} \text{ m}$ is the Bohr radius, so that $e^2/(4\pi\epsilon_0) = \hbar^2/(ma_0)$ and $\hbar^2/(2ma_0^2) = 13.6 \text{ eV}$. Treat the proton as infinitely heavy. You may use $\int_0^\infty r^n e^{-\beta r} dr = n!/\beta^{n+1}$ for a non-negative integer n and $\beta > 0$.

(a) DETERMINE

[4]

Determine the value of l for which $u(r)$ satisfies the radial equation, and the corresponding energy E in electronvolts.

(b) STATE

[2]

State the principal quantum number n of this state, its number of radial nodes, and its spectroscopic label.

(c) DETERMINE

[2]

Determine the most probable distance of the electron from the proton, in nanometres.

(d) CALCULATE

[2]

Calculate the expectation value $\langle r \rangle$ in nanometres.

(e) COMPARE

[2]

Compare the degeneracy of this energy level in hydrogen with the degeneracy an $l = 2$ level would have in a generic central potential, such as a finite spherical well, giving the cause of each.

Gate leakage in a transistor is caused by electrons tunnelling directly through the thin silicon dioxide layer between the gate and the silicon. To study it, an engineer measures the gate current I of six test capacitors made with different oxide thicknesses t_{ox} and different gate areas A , all at the same small gate voltage. The model to be tested is $J = J_0 \exp(-2\kappa t_{\text{ox}})$, where $J = I/A$ is the current density, J_0 is the same for every capacitor, and $\kappa = \sqrt{(2m^*\phi_B)/\hbar}$, with ϕ_B the height of the oxide barrier above the energy of the tunnelling electrons and m^* the effective mass of an electron in the oxide. Internal photoemission measurements, made independently, give $\phi_B = 3.1$ eV. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, $1 \text{ eV} = 1.602 \times 10^{-19}$ J and $1 \mu\text{m}^2 = 1 \times 10^{-12} \text{ m}^2$.

Gate current of six test capacitors at the same gate voltage

Capacitor	Oxide thickness t_{ox} / nm	Gate area A / μm^2	Gate current I / A
1	1.40	1.0×10^2	2.00×10^{-6}
2	1.60	2.5×10^2	5.26×10^{-7}
3	1.80	1.0×10^3	2.03×10^{-7}
4	2.00	2.5×10^3	5.44×10^{-8}
5	2.20	1.0×10^4	2.14×10^{-8}
6	2.40	2.5×10^4	5.63×10^{-9}

(a) DESCRIBE

[2]

Describe how the readings should be processed and plotted so that the model predicts a straight line, including why the currents themselves should not be plotted and what the gradient measures.

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(b) DETERMINE

[3]

Determine κ from the readings, and the reduction in oxide thickness that multiplies the leakage current density by ten.

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(c) ESTIMATE

[3]

Estimate the effective mass m^* of the tunnelling electrons in the oxide, as a multiple of m_e , using $\phi_B = 3.1$ eV, and state the assumptions on which your estimate rests.

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(d) EVALUATE

[2]

A colleague analyses the same readings taking the electron in the oxide to have the free-electron mass m_e , and reports the barrier height that follows. Evaluate this analysis.

A particle of mass m moves in the potential $V(x) = \frac{1}{2}m\omega^2x^2$, so that $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$. Define $\hat{a} = \sqrt{m\omega/2\hbar}(\hat{x} + i\hat{p}/(m\omega))$ and its adjoint $\hat{a}^\dagger = \sqrt{m\omega/2\hbar}(\hat{x} - i\hat{p}/(m\omega))$, and write $\hat{N} = \hat{a}^\dagger\hat{a}$. The canonical commutator is $[\hat{x}, \hat{p}] = i\hbar$. Once $[\hat{a}, \hat{a}^\dagger] = 1$ has been established, you may use its consequences $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(a) DERIVE

[3]

Derive $[\hat{a}, \hat{a}^\dagger] = 1$ from the canonical commutator.

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(b) SHOW THAT

[2]

Show that $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$.

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(c) DERIVE

[4]

Derive the energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$ with $n = 0, 1, 2, \dots$, using only the commutators and the fact that no vector has a negative squared norm. Do not evaluate any integral.

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(d) DETERMINE

[3]

Determine $\langle \hat{x}^2 \rangle$ in the eigenstate $|n\rangle$ using ladder operators alone.

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(e) PREDICT

[3]

The particle is in the ground state $|0\rangle$ when the potential is suddenly tightened to $V'(x) = \frac{1}{2}m(2\omega)^2x^2$, so quickly that the wavefunction has no time to change. Predict which energy eigenstates of the new oscillator the particle can then be found in, and its mean energy $\langle \hat{H}' \rangle$ immediately after the change, in terms of $\hbar\omega$.

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A simple model of the deuteron treats the neutron-proton pair as a single particle of reduced mass $\mu = 8.37 \times 10^{-28}$ kg moving in a spherical square well, with $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, where $a = 2.0 \times 10^{-15}$ m. Only states with $l = 0$ are considered, so ψ depends on r alone and $\nabla^2 \psi = (1/r^2) d/dr (r^2 d\psi/dr)$. Primes denote d/dr . Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ MeV} = 1.602 \times 10^{-13}$ J. The measured binding energy of the deuteron is 2.22 MeV, and it has no bound excited state.

(a) DERIVE

[3]

Starting from the time-independent Schrödinger equation in three dimensions, derive the equation $-(\hbar^2/2\mu)u'' + V(r)u = Eu$ satisfied by $u(r) = r\psi(r)$.

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(b) JUSTIFY

[2]

Justify the boundary condition $u(0) = 0$. Note that a wavefunction behaving as $1/r$ near the origin is square-integrable in three dimensions.

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(c) SHOW THAT

[3]

Show that a bound state with energy $E = -E_B$, where $0 < E_B < V_0$, must satisfy $k \cot(ka) = -\kappa$, where $\hbar k = \sqrt{2\mu(V_0 - E_B)}$ and $\hbar \kappa = \sqrt{2\mu E_B}$.

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(d) DERIVE

[3]

Derive, in terms of μ , a and $\hbar$, the minimum depth $V_{0,\min}$ for which the well has any bound state at all.

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(e) CALCULATE

[2]

Calculate $V_{0,\min}$ in MeV for the values given.

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(f) SKETCH

[2]

A well depth of 36.6 MeV reproduces the measured binding energy, giving $ka = 1.82$ and $\kappa a = 0.46$. Sketch $u(r)$ for the deuteron's ground state from $r = 0$ to $r = 5a$, marking $r = a$ on the axis.

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University Quantum Mechanics I · Final Examination — mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

1 MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**

CORE
COURSE

$\psi(x)$ is a normalised wavefunction for a particle moving along the x -axis. A second state of the particle has a position probability density of the same shape as $|\psi(x)|^2$, but compressed by a factor of 2 towards $x = 0$, so that every feature of the density lies at half its original distance from $x = 0$. Which wavefunction describes the second state?

KEY D $\sqrt{2} \psi(2x)$

Replacing x by $2x$ compresses the shape by a factor of 2, because every feature of $\psi(2x)$ lies at half its former distance from the origin. The substitution $u = 2x$ gives $\int |\psi(2x)|^2 dx = \frac{1}{2} \int |\psi(u)|^2 du = \frac{1}{2}$, so the compressed function must be multiplied by $\sqrt{2}$ to restore a total probability of 1, and $\sqrt{2} \psi(2x)$ then meets every requirement on a wavefunction. A is the most tempting wrong answer. It has the right shape, is finite and continuous, and falls off at large $|x|$, but its total probability is only $\frac{1}{2}$. The particle must be found somewhere, so a function whose $|\psi|^2$ integrates to $\frac{1}{2}$ does not describe a state until it is normalised. B doubles the amplitude on the reasoning that a density half as wide must be twice as tall. It is $|\psi|^2$ that must be twice as tall, and doubling ψ makes it four times as tall, so the total probability becomes 2. C has the right factor but the wrong substitution: $\psi(x/2)$ is stretched to twice the width, and $2 \int |\psi(x/2)|^2 dx = 4$. Normalisation fixes the scale of ψ , so any change of width has to be paid for by a change of height.

U1

2 MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.06 **[1]**

CORE
COURSE

Each option refers to a particle of mass m in one dimension. Which pair of operators commutes, so that the two observables are compatible and can share a complete set of eigenfunctions?

KEY A $\hat{p}$ and $\hat{H} = \hat{p}^2/2m$, for a free particle

When $V = 0$ the Hamiltonian is a function of $\hat{p}$ alone, and an operator always commutes with a function of itself. The plane waves e^{ikx} are therefore eigenfunctions of both operators, with $p = \hbar k$ and $E = \hbar^2 k^2/2m$. B is the canonical incompatible pair, $[\hat{x}, \hat{p}] = i\hbar$. It is offered to the candidate who reads 'both Hermitian' as 'compatible'. C is tempting because the kinetic term does commute with $\hat{p}$. The potential term does not: $[\hat{p}, \frac{1}{2}m\omega^2 \hat{x}^2] = -i\hbar m\omega^2 \hat{x}$, which is non-zero, so no oscillator stationary state has a definite momentum. D fails for the mirror-image reason. $[\hat{x}, \hat{p}^2/2m] = i\hbar \hat{p}/m$, so even for a free particle a state of definite position cannot have definite energy. Compatibility is decided by the commutator, never by the fact that each operator on its own represents an observable.

U1

3

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-05.06 · UQMI-05.07

[1]

The simplest model of a wave packet for a free neutron ($m = 1.675 \times 10^{-27}$ kg) is the sum of two plane waves, $\Psi(x,t) = e^{i(k_1x - \omega_1t)} + e^{i(k_2x - \omega_2t)}$, with $k_1 = 2.90 \times 10^{10} \text{ m}^{-1}$ and $k_2 = 3.10 \times 10^{10} \text{ m}^{-1}$, and each frequency given by the free-particle dispersion relation $\omega = \hbar k^2/(2m)$. The probability density $|\Psi(x,t)|^2$ is then a row of fringes about 3.1 nm apart. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$. At what speed does the fringe pattern of $|\Psi(x,t)|^2$ move?

KEY **B** $1.9 \times 10^3 \text{ m s}^{-1}$

Multiplying out gives $|\Psi|^2 = 2 + 2\cos[(k_2 - k_1)x - (\omega_2 - \omega_1)t]$, so the fringes move at $(\omega_2 - \omega_1)/(k_2 - k_1) = \hbar(k_2^2 - k_1^2)/(2m(k_2 - k_1)) = \hbar(k_1 + k_2)/(2m) = 1.9 \times 10^3 \text{ m s}^{-1}$. For two components this result is exact, and it equals the classical speed $\hbar k/m$ of a neutron with the mean wavenumber $3.00 \times 10^{10} \text{ m}^{-1}$. A, $9.4 \times 10^2 \text{ m s}^{-1}$, is the phase velocity ω/k of a component at the centre of the band, half the correct value. It is chosen by a candidate who takes the speed of the crests inside the fringes for the speed of the fringes themselves. C doubles the correct answer: the candidate writes $\hbar(k_1 + k_2)/m$, treating $k_1 + k_2$ as the central wavenumber when it is twice that. D uses h in place of $\hbar$, which makes the speed 2π times too large. The probability is carried by the envelope, which moves at the group velocity, and that is why the group velocity and not the phase velocity is the speed of the particle.

U3

4

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-07.07

[1]

An F-centre is an electron trapped at the site of a missing negative ion in an alkali-halide crystal. Along one axis, model it as an electron ($m = 9.109 \times 10^{-31}$ kg) in a symmetric finite square well of total width 0.56 nm and depth V_0 . In this model the centre's optical absorption lifts the electron from the ground state to the first excited bound state, so the well must bind at least two states. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$ and $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$. What is the smallest depth V_0 for which the well binds a second state?

KEY **C** 1.2 eV

With half-width $a = 0.28 \text{ nm}$ and $z_0 = (a/\hbar)\sqrt{2mV_0}$, bound states appear one at a time, even and odd alternately, the n -th once z_0 exceeds $(n - 1)\pi/2$. The second state, the first odd one, needs $z_0 > \pi/2$, so $V_0 > \pi^2\hbar^2/(8ma^2) = 1.92 \times 10^{-19} \text{ J} = 1.2 \text{ eV}$. A is true of the ground state but not of the second state. An even bound state exists in a one-dimensional well of any depth, and a candidate who remembers that result without its restriction to the ground state chooses A. B puts the full width, 0.56 nm, into z_0 in place of the half-width, which divides the required depth by 4. D sets $z_0 = \pi$, which is the threshold for a third state. It comes from counting as if the n -th state needed $z_0 > n\pi/2$, forgetting that the ground state has no threshold, and it would wrongly rule out the absorption for depths between 1.2 eV and 4.8 eV, where two bound states do exist.

U3

5

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-08.04 · UQMI-08.05

[1]

An electron with kinetic energy E meets a rectangular barrier of height $V_0 > E$ and width w . The thick-barrier estimate $T \approx \exp(-2\kappa w)$, with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$ and the prefactor ignored, gives $T = 1.0 \times 10^{-3}$. A proton with the same kinetic energy meets the same barrier. Taking $m_p/m_e = 1836$, what does the same estimate give for the proton's transmission probability?

KEY **C** 3×10^{-129}

κ grows as $\sqrt{m}$, so the proton's exponent is $\sqrt{1836} = 42.8$ times the electron's. That gives $T_p = (1.0 \times 10^{-3})^{42.8} = 10^{(-128.5)} \approx 3 \times 10^{-129}$. A divides T by $\sqrt{1836}$ and B divides it by 1836. Both treat the mass as if it entered a prefactor rather than the exponent, and that is the error this item is set to expose. Mass, width and $\sqrt{V_0 - E}$ all sit inside the exponential, so modest changes in any of them change T by orders of magnitude. D raises T to the power 1836 rather than $\sqrt{1836}$, taking κ proportional to m instead of $\sqrt{m}$. The neglected prefactor changes T by a factor of order one, so it cannot rescue the proton: at the same energy and barrier, a proton is effectively never transmitted, while one electron in a thousand gets through.

U3

6

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-09.08 · UQMI-09.09

[1]

For a one-dimensional harmonic oscillator with energy eigenstates $|n\rangle$, the position operator can be written $\hat{x} = \sqrt{(\hbar/2m\omega)}(\hat{a} + \hat{a}^\dagger)$, where $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Which of the following matrix elements is non-zero?

KEY **A** $\langle 3|\hat{x}|2\rangle$

$\hat{x}$ contains one $\hat{a}$ and one $\hat{a}^\dagger$, so acting on $|2\rangle$ it produces only $|1\rangle$ and $|3\rangle$: $\hat{x}|2\rangle = \sqrt{(\hbar/2m\omega)}(\sqrt{2}|1\rangle + \sqrt{3}|3\rangle)$. Hence $\langle 3|\hat{x}|2\rangle = \sqrt{3}\sqrt{(\hbar/2m\omega)}$. B is the expectation value of x in a stationary state. It vanishes because $\hat{x}|2\rangle$ has no $|2\rangle$ component, or equivalently because $|\psi_2|^2$ is even while x is odd. C and D connect states whose n differ by 2. Those are elements of $\hat{x}^2$, which contains $\hat{a}^2$ and $\hat{a}^{\dagger 2}$, and they catch a candidate who remembers that $\hat{x}$ changes n without remembering by how much. Parity agrees: ψ_0 , ψ_2 and ψ_4 are all even, so $\int \psi_0 x \psi_2 dx$ and $\int \psi_4 x \psi_2 dx$ are integrals of odd functions and vanish.

U1

7

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQMI-10.07 · UQMI-10.08

[1]

An observable $\hat{A}$ is represented in one orthonormal basis by the Hermitian matrix A , and in a second orthonormal basis by $A' = U^\dagger A U$, where U is the unitary matrix of overlaps between the two bases. Which of the following is in general different for A' than for A ?

KEY **D** The first diagonal element, A_{11}

A change of basis relabels the components of vectors and matrices without changing the physics. Trace and eigenvalues are unchanged because $U^\dagger A U$ has the same characteristic polynomial as A , and $\text{Tr}(U^\dagger A U) = \text{Tr}(A U U^\dagger) = \text{Tr} A$. A candidate who picks A or B has confused a matrix's entries with the operator it represents. C is also invariant: the column of components transforms as $c' = U^\dagger c$, so $c'^\dagger A' c' = c^\dagger U U^\dagger A U U^\dagger c = c^\dagger A c$. D is the quantity that changes. $A_{11} = \langle 1|\hat{A}|1\rangle$ is the expectation value in the first basis state, and the first basis vector is a different physical state in each basis. The same point is why a Hermitian matrix is diagonal only in its own eigenbasis.

U1

8

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-11.02 · UQMI-11.05

[1]

A two-state system with orthonormal basis states $|1\rangle$ and $|2\rangle$ is prepared in $|\psi\rangle = (|1\rangle + i|2\rangle)/\sqrt{2}$. Which of the following is the same physical state, giving identical probabilities for every possible measurement?

KEY **B** $(i|1\rangle - |2\rangle)/\sqrt{2}$

$(i|1\rangle - |2\rangle)/\sqrt{2} = i(|1\rangle + i|2\rangle)/\sqrt{2}$, which is $|\psi\rangle$ multiplied by the global phase i . A global phase cancels from every $|\langle\phi|\psi\rangle|^2$, so no measurement can detect it. All four options give probability $1/2$ for each basis state, which is why a candidate who checks only populations cannot choose between them. A and C change the relative phase between the components, from i to $-i$ and to 1 , and relative phase is physical. Measured in the basis $(|1\rangle \pm i|2\rangle)/\sqrt{2}$, $|\psi\rangle$ gives one outcome with certainty, A gives the other outcome with certainty, and C gives each half the time. D is the subtle one. It looks like $|\psi\rangle$ with the i moved to the other term, but it equals $i(|1\rangle - i|2\rangle)/\sqrt{2}$, the same physical state as A .

U1

9

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-14.02

[1]

Near the centre of an optical trap, an atom of mass m moves in the potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2) + \frac{1}{2}m(2\omega)^2z^2$, so that its classical oscillation frequency along z is twice that along x and y . How many independent energy eigenstates share the energy $E = 5\hbar\omega$?

KEY **B 6**

The potential is a sum of three one-dimensional oscillator potentials, so $\psi = X(x)Y(y)Z(z)$ and $E = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega + (n_z + \frac{1}{2})(2\hbar\omega) = (n_x + n_y + 2n_z + 2)\hbar\omega$, with each quantum number taking the values 0, 1, 2 and so on. $E = 5\hbar\omega$ therefore requires $n_x + n_y + 2n_z = 3$. With $n_z = 0$ the three quanta can be shared between x and y in four ways, $(n_x, n_y) = (3, 0), (2, 1), (1, 2)$ and $(0, 3)$; with $n_z = 1$ the one remaining quantum gives two more, $(1, 0)$ and $(0, 1)$; $n_z = 2$ would already need $4\hbar\omega$ above the zero-point energy. The level holds $4 + 2 = 6$ states. A counts only the states with $n_z = 0$, the answer of a candidate who assumes the stiffer direction stays in its ground state, when one quantum along z costs exactly as much as two in the plane. C applies the isotropic-oscillator count $(N + 1)(N + 2)/2$ to $N = 3$, which would be right only if every quantum cost $\hbar\omega$. D leaves out the zero-point energy, which is $2\hbar\omega$ here, and so counts the solutions of $n_x + n_y + 2n_z = 5$ instead of 3. The coincidence of states with $n_z = 0$ and $n_z = 1$ is not required by the trap's symmetry about the z -axis: it comes from the 2 : 1 ratio of the frequencies, and an irrational ratio would remove it.

U2

10

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-15.05

[1]

The lithium ion Li^{2+} ($Z = 3$) has a single electron. The energy levels of hydrogen are $E_n = -13.6 \text{ eV}/n^2$. How much energy is needed to ionise Li^{2+} from its first excited state?

KEY **C 30.6 eV**

For a one-electron ion the levels scale as Z^2 : $E_n = -13.6 \text{ eV} \times Z^2/n^2$. The first excited state, $n = 2$, lies at $-13.6 \times 9/4 = -30.6 \text{ eV}$, so 30.6 eV removes the electron. A is the hydrogen value for $n = 2$, the answer of a candidate who ignores Z . B scales by Z instead of Z^2 . The Coulomb potential carries one factor of Z and the orbit shrinks by another, so the energy goes as Z^2 . D, 122 eV, is the ionisation energy from the ground state, chosen by a candidate who reads 'first excited state' as $n = 1$.

U3

11

CORE
COURSE

STRUCTURED · ROUTINE · UQMI-04.04 · UQMI-06.03 · UQMI-06.07 · UQMI-06.08 · UQMI-06.09

[10]

A single ^{87}Rb atom of mass $m = 1.443 \times 10^{-25} \text{ kg}$ moves along an optical box trap whose repulsive light walls are steep and high enough to be treated as an infinite square well on $0 < x < L$, with $L = 20.0 \mu\text{m}$. The stationary states are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2 E_1$. At $t = 0$ the atom is prepared near the left wall in the state $\Psi(x, 0) = (1/\sqrt{2})\psi_1(x) + (1/2)\psi_2(x) + (1/2)\psi_3(x)$, for which $\langle x \rangle = 0.275L$. Take $\hbar = 1.055 \times 10^{-34} \text{ J s}$, $h = 6.626 \times 10^{-34} \text{ J s}$ and $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$.

(a) CALCULATE · U3

[2]

Calculate the atom's zero-point energy E_1 , and express it as a temperature E_1/k_B .

$$1 \quad E_1 = \pi^2 \hbar^2 / (2mL^2) = \pi^2 (1.055 \times 10^{-34})^2 / (2 \times 1.443 \times 10^{-25} \times (20.0 \times 10^{-6})^2) = 9.52 \times 10^{-34} \text{ J}$$

$$2 \quad E_1/k_B = 9.52 \times 10^{-34} / 1.381 \times 10^{-23} = 6.9 \times 10^{-11} \text{ K, about 69 pK}$$

L must be in metres before it is squared. The result is tiny because the atom is heavy and the box is wide: the level spacing is comparable with the lowest temperatures cold-atom experiments reach, which is why the discrete levels of so large a box matter only for atoms cooled that far.

- (b) **STATE** · U1 [2]
- State the possible results of a measurement of the atom's energy and the probability of each, and state whether these probabilities change as time passes.
- 1 E_1 , $4E_1$ and $9E_1$, with probabilities $|1/\sqrt{2}|^2 = 1/2$, $|1/2|^2 = 1/4$ and $|1/2|^2 = 1/4$, which sum to 1; no other energy can be obtained
 - 2 the probabilities do not change: each coefficient acquires only the phase factor $e^{(-iE_n t/\hbar)}$, which leaves $|c_n|^2$ unchanged
- Candidates who give the energies as $2E_1$ and $3E_1$ have used a level spacing proportional to n , the pattern of the oscillator rather than the box. Those who say the probabilities oscillate have confused the distribution of energy, which is fixed, with the distribution of position, which is not.*
- (c) **CALCULATE** · U3 [2]
- Every phase factor $e^{(-iE_n t/\hbar)}$ in $\Psi(x,t)$ returns to 1 at the same instant, after which the atom's state repeats exactly. Calculate the earliest such time.
- 1 $\Psi(x,t) = \sum c_n \psi_n(x) e^{(-in^2 E_1 t/\hbar)}$; since every n^2 is an integer, all three factors first equal 1 together when $E_1 t/\hbar = 2\pi$ (the relative phases $3E_1 t/\hbar$ and $8E_1 t/\hbar$ have no common factor that would allow an earlier repeat)
 - 2 $t = 2\pi\hbar/E_1 = \hbar/E_1 = 6.626 \times 10^{-34} / 9.52 \times 10^{-34} = 0.70 \text{ s}$
- The time is set by E_1 alone, because every energy is an integer multiple of E_1 . A candidate who uses $\hbar/(E_2 - E_1) = 0.23 \text{ s}$ has found when the ψ_1 and ψ_2 terms are back in step, but the ψ_3 term is not yet back in step with them.*
- (d) **EXPLAIN** · U1 [2]
- At $t = \hbar/(2E_1)$ the probability density is $|\Psi(L - x, 0)|^2$, the mirror image of the initial density about the centre of the trap. Explain why, and state $\langle x \rangle$ at that instant.
- 1 at $t = \hbar/(2E_1)$ each phase factor is $e^{(-i\pi n^2)} = (-1)^n$, while reflection about the centre gives $\psi_n(L - x) = (-1)^{(n+1)} \psi_n(x)$; so $(-1)^n \psi_n(x) = -\psi_n(L - x)$ for every n , and $\Psi(x,t) = -\Psi(L - x, 0)$
 - 2 an overall sign does not change $|\Psi|^2$, so the density is the initial one reflected about $x = L/2$: the atom is now near the right wall, with $\langle x \rangle = L - 0.275L = 0.725L$, although no energy probability has changed
- Candidates who expect $\langle x \rangle$ simply to swing about the centre miss the mechanism. The phases reverse the sign of ψ_2 , the only term that is odd about the centre, relative to the two even terms, and reversing the odd part of a function is exactly a reflection.*
- (e) **PREDICT** · U1 [2]
- Suppose instead that the atom's energy is measured at $t = 0$ and measured again at $t = \hbar/(2E_1)$. Predict the probability that the two results agree, and whether the density at $t = \hbar/(2E_1)$ is then the mirror image of the initial density.
- 1 the first measurement leaves the atom in the single eigenstate ψ_n whose energy it found; that state is stationary, so the second measurement repeats the result with probability 1
 - 2 the density is then $|\psi_n|^2$, constant in time and symmetric about $x = L/2$ with $\langle x \rangle = L/2$, so no mirror image of the initial density appears near the right wall: the measurement removed the other terms whose relative phases produced it
- Candidates who combine the probabilities as independent events, $1/2 \times 1/2 + 1/4 \times 1/4 + 1/4 \times 1/4 = 3/8$, have not updated the state after the first result. Repeatability after collapse is what the part tests.*

A beam of spin- $\frac{1}{2}$ atoms is prepared in the spin state $\chi = (1/\sqrt{3})(1, 1 + i)^T$, written as a column in the S_z basis $\{|+z\rangle, |-z\rangle\}$. The spin operators are $S_i = (\hbar/2)\sigma_i$, where σ_x , σ_y and σ_z are the Pauli matrices.

(a) STATE · U1

[2]

State the three Pauli matrices σ_x , σ_y and σ_z in the S_z basis.

1 $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2 $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, with $-i$ in the upper right

The sign in σ_y is the one most often lost. It is fixed by $\sigma_x\sigma_y = i\sigma_z$, which takes a line to check.

(b) CALCULATE · U3

[2]

Calculate the probability that a measurement of S_z gives $+\hbar/2$, and the expectation value $\langle S_z \rangle$.

1 $P(+\hbar/2) = |1/\sqrt{3}|^2 = 1/3$ (and $P(-\hbar/2) = |1 + i|^2/3 = 2/3$)

2 $\langle S_z \rangle = (\hbar/2)(1/3 - 2/3) = -\hbar/6$

$|1 + i|^2 = 2$, not $(1 + i)^2 = 2i$. A candidate who squares the complex number instead of taking its squared modulus finds probabilities that do not sum to 1 and should stop there.

(c) DETERMINE · U2

[3]

Determine the probability that a measurement of S_x gives $+\hbar/2$.

1 the $+\hbar/2$ eigenspinor of S_x is $|+x\rangle = (1/\sqrt{2})(1, 1)^T$

2 $\langle +x|\chi \rangle = (1/\sqrt{6})(1 + 1 + i) = (2 + i)/\sqrt{6}$, with the bra's components conjugated (they are real here)

3 $P = |2 + i|^2/6 = 5/6$

The probability is the squared modulus of the overlap with the S_x eigenspinor. A candidate who looks for 'the x component of χ ' is looking for something that does not exist, because spinor components are amplitudes along z only.

(d) DETERMINE · U1

[3]

The atoms pass through an S_x analyser, and only those leaving its $+\hbar/2$ port go on to an S_z analyser. Determine the fraction of the original atoms that leave the $-\hbar/2$ port of the S_z analyser, and give the reason it is not related to the $2/3$ found in (b).

1 the atoms kept after the S_x analyser are in $|+x\rangle$, and they are a fraction $5/6$ of the original beam

2 $|\langle -z|+x \rangle|^2 = 1/2$, so the fraction leaving the $-\hbar/2$ port is $(5/6)(1/2) = 5/12$

3 the S_x measurement replaces χ by $|+x\rangle$ and erases the earlier S_z amplitudes; since $[S_x, S_z] \neq 0$ no state has definite values of both, so the loss comes from the change of state, not from the magnet mechanically disturbing the atoms

Candidates multiply $5/6$ by $2/3$, carrying the original S_z statistics through a measurement that destroyed them. The cascade is set to catch exactly that.

Very slow electrons are strongly reflected by a crystal surface even though the crystal attracts them.

Inside the crystal an electron's potential energy is lower than in the vacuum outside by U , the inner potential, so the surface at $x = 0$ is modelled as a downward potential step: $V(x) = 0$ for $x < 0$ (vacuum) and $V(x) = -U$ for $x > 0$ (crystal). Electrons of energy $E = 0.50$ eV arrive from the vacuum, moving in the $+x$ direction at normal incidence. Their stationary scattering state is $\psi(x) = e^{ikx} - (2/3)e^{-ikx}$ for $x < 0$ and $\psi(x) = (1/3)e^{iqx}$ for $x > 0$, where $\hbar k = \sqrt{2m_e E}$ and $\hbar q = \sqrt{2m_e(E + U)}$, but the value of U is not known. The probability current is $j = (\hbar/m_e) \text{Im}(\psi^* d\psi/dx)$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

- (a)** DERIVE · U2 [3]
- Stating the matching conditions that ψ must satisfy at $x = 0$, and why each holds here, derive the ratio q/k for this state and hence the inner potential U in electronvolts.
- ψ is continuous at $x = 0$, because a jump in ψ would put the derivative of a delta function into ψ'' , which no finite V can balance in the Schrödinger equation: the two sides give $1 - 2/3 = 1/3$, so the amplitudes given satisfy this condition
 - $d\psi/dx$ is continuous at $x = 0$, because the step in V is finite and so ψ'' has at most a finite jump: $ik(1 + 2/3) = iq(1/3)$, giving $q = 5k$
 - $(q/k)^2 = (E + U)/E = 25$, so $U = 24E = 12.0 \text{ eV}$
The reflected amplitude is negative, so the slope condition contains $1 + 2/3 = 5/3$. A candidate who writes $1 - 2/3$ there finds $q = k$, which would mean no step at all and contradicts the reflected wave in the same state.
- (b)** SHOW THAT · U2 [2]
- Show that the probability current is the same on both sides of the surface, as conservation of probability requires in a stationary state.
- for $x < 0$ the cross terms between e^{ikx} and e^{-ikx} contribute nothing to $\text{Im}(\psi^* d\psi/dx)$, so $j = (\hbar k/m_e)(1 - 4/9) = (5/9)\hbar k/m_e$
 - for $x > 0$, $j = (\hbar q/m_e)(1/3)^2 = (5\hbar k/m_e)(1/9) = (5/9)\hbar k/m_e$, equal to the current for $x < 0$, so no probability builds up at the surface
Candidates who compare $|\psi|^2$ on the two sides, $1/9$ inside against 1 for the incident wave, conclude that probability is lost at the surface. Density is not current: the transmitted electrons move five times faster, so a smaller density carries the same flux.
- (c)** CALCULATE · U3 [3]
- A beam of these electrons carrying a current of 2.0 nA strikes the surface. Calculate the reflected and transmitted electron currents, and the speed of the transmitted electrons inside the crystal.
- $R = |-2/3|^2 = 4/9$, so the reflected current is $(4/9) \times 2.0 \text{ nA} = 0.89 \text{ nA}$
 - $T = (q/k)|t|^2 = 5 \times (1/9) = 5/9$, so the transmitted current is $(5/9) \times 2.0 \text{ nA} = 1.1 \text{ nA}$, and $R + T = 1$
 - inside the crystal $v = \hbar q/m_e = 5\sqrt{2E/m_e} = 5 \times 4.19 \times 10^5 \text{ m s}^{-1} = 2.1 \times 10^6 \text{ m s}^{-1}$
 T is not $|t|^2 = 1/9$. A candidate who uses $|t|^2$ finds only 0.22 nA transmitted and 0.89 nA reflected, and should notice that 0.9 nA of the beam has disappeared.
- (d)** CALCULATE · U3 [2]
- Calculate the electron energy at which the reflected fraction at this surface falls to 1.0% .
- $R = 0.010$ requires $|r| = (q - k)/(q + k) = 0.10$, so $q/k = 1.10/0.90 = 11/9$
 - $(q/k)^2 = 1 + U/E = 121/81$, giving $E = 81U/40 = 24 \text{ eV}$ (24.3 eV)
Candidates who set $|r| = 0.010$ instead of $|r|^2 = 0.010$ find about 290 eV . The reflection from a drop is appreciable only while E is not large compared with U .

(e) EXPLAIN · U1

[2]

A classical electron crossing the surface would simply speed up, and none would be reflected. Explain why the quantum electron can be reflected by a drop in potential, and why far fewer electrons would be reflected if V fell from 0 to $-U$ smoothly over several nanometres instead of abruptly.

- 1 at an abrupt step the wavenumber jumps from k to $q = 5k$; a transmitted wave alone would need $1 = t$ from continuity of ψ and $k = qt$ from continuity of $d\psi/dx$, which cannot both hold when $q \neq k$, so a reflected wave is required: a wave effect, like the partial reflection of light where the refractive index changes suddenly, with no counterpart for a classical particle
- 2 if V changes over a distance much larger than the reduced wavelength $1/k$ (0.28 nm in vacuum at 0.50 eV, and $1/q = 0.055$ nm inside), the local wavelength adjusts gradually and ψ follows it without a mismatch at any point, so the reflected amplitude becomes exponentially small and is negligible for a change spread over several nanometres; reflection is appreciable only if V changes within a distance of order $1/k$ or less, a fraction of a nanometre

Candidates who say the electron bounces off the edge of the crystal are using a classical picture, and that picture predicts no reflection at all. The reflection comes from the abruptness of the change in wavelength, not from a force pushing the electron back.

14

CORE COURSE

STRUCTURED · DEMANDING · UQMI-11.01 · UQMI-11.04 · UQMI-11.07 · UQMI-11.08

[12]

An electron shared between two neighbouring semiconductor quantum dots, L and R, is modelled as a two-state system in the basis $\{|L\rangle, |R\rangle\}$ of states localised on one dot. Energies are measured from the mean of the two on-dot energies, and in that basis the Hamiltonian is $H = [[\epsilon/2, -\tau], [-\tau, -\epsilon/2]]$. The detuning $\epsilon = 30 \mu\text{eV}$ is set by a gate voltage, and the tunnel coupling is $\tau = 20 \mu\text{eV}$. The next single-electron level in each dot lies about 1.5 meV above these states. If the electron starts on dot L at $t = 0$, the probability of finding it on dot R is $P_R(t) = [4\tau^2/(E_+ - E_-)^2] \sin^2[(E_+ - E_-)t/(2\hbar)]$, where E_+ and E_- are the eigenvalues of H . Take $\hbar = 6.626 \times 10^{-34} \text{ J s}$ and $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$.

(a) JUSTIFY · U4

[2]

Justify the two-state model of the electron, using the energies given.

- 1 the omitted levels lie about 1.5 meV away, some 30 times the $50 \mu\text{eV}$ scale of ϵ , τ and the level splitting, so they mix only weakly into the two localised states
- 2 nothing supplies the 1.5 meV needed to reach those levels: the gate voltage must change slowly compared with $\hbar/(1.5 \text{ meV}) \approx 3 \text{ ps}$, and the thermal energy k_{BT} must be much less than 1.5 meV (either condition, stated as a comparison with the 1.5 meV spacing, earns the mark)

A justification must compare energies. 'Two dots, so two states' earns nothing, because each dot has many states; what matters is that only two of them are close together.

(b) DETERMINE · U2

[3]

Determine the eigenvalues E_+ and E_- of H , and hence the splitting $E_+ - E_-$.

- 1 $\det(H - EI) = 0$ gives $(\epsilon/2 - E)(-\epsilon/2 - E) - \tau^2 = 0$, so $E^2 = \epsilon^2/4 + \tau^2$
- 2 $E_{\pm} = \pm\sqrt{(\epsilon^2/4 + \tau^2)} = \pm\sqrt{(225 + 400) \mu\text{eV}} = \pm 25 \mu\text{eV}$
- 3 $E_+ - E_- = \sqrt{(\epsilon^2 + 4\tau^2)} = 50 \mu\text{eV}$, larger than the $30 \mu\text{eV}$ detuning alone because the coupling pushes the levels apart

Candidates who take the eigenvalues to be the diagonal elements $\pm 15 \mu\text{eV}$ have ignored the coupling, and those who add ϵ and 2τ get $70 \mu\text{eV}$. The sum in quadrature is the avoided crossing.

(c) DETERMINE · U2

[3]

Determine the probability of finding the electron on dot R when it is in the ground state.

- 1 for the ground state, $E_- = -25 \mu\text{eV}$, writing $\psi = a|L\rangle + b|R\rangle$, $(H - E_-)\psi = 0$ gives $(15 + 25)a - 20b = 0$ with energies in μeV
- 2 so $b = 2a$, and the normalised ground state is $(|L\rangle + 2|R\rangle)/\sqrt{5}$
- 3 $P_R = 4/5 = 0.80$ (equivalently, the mixing angle with $\tan 2\chi = 2\tau/\epsilon = 4/3$ gives $\sin^2\chi = 0.20$ on dot L)

The ground state leans towards dot R, which has the lower on-dot energy, but not completely. A candidate who answers 1 has ignored the coupling; one who answers 1/2 has ignored the detuning.

(d) CALCULATE · U3

[2]

The electron is placed on dot L at $t = 0$. Calculate the maximum probability of later finding it on dot R, and the period of the oscillation of $P_R(t)$.

- 1 maximum $P_R = 4\tau^2/(E_+ - E_-)^2 = 1600/2500 = 0.64$
- 2 $\sin^2$ repeats when $(E_+ - E_-)t/(2\hbar)$ advances by π , so the period is $2\pi\hbar/(E_+ - E_-) = \hbar/(E_+ - E_-) = 6.626 \times 10^{-34}/(50 \times 10^{-6} \times 1.602 \times 10^{-19}) = 8.3 \times 10^{-11} \text{ s}$

$\sin^2\theta$ has period π , not 2π . A candidate who misses this doubles the period to $1.7 \times 10^{-10} \text{ s}$.

(e) PREDICT · U1

[2]

The gate voltage is adjusted until $\epsilon = 0$. Predict the maximum probability of transfer to dot R, and the time after $t = 0$ at which it is first reached.

- 1 with $\epsilon = 0$ the eigenstates are $(|L\rangle \pm |R\rangle)/\sqrt{2}$, with equal weight on each dot, and $E_+ - E_- = 2\tau = 40 \mu\text{eV}$, so the maximum $P_R = 4\tau^2/(2\tau)^2 = 1$: complete transfer
- 2 P_R first reaches 1 when $(2\tau)t/(2\hbar) = \pi/2$, at $t = \pi\hbar/(2\tau) = \hbar/(4\tau) = 5.2 \times 10^{-11} \text{ s}$

Removing the detuning makes the transfer complete but the oscillation slower, because the splitting falls from $50 \mu\text{eV}$ to $40 \mu\text{eV}$. Candidates who expect a more complete transfer to be a faster one have confused the amplitude with the frequency.

For orbital angular momentum with $l = 1$, use the ordered basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, in which $L_z = \hbar \text{diag}(1, 0, -1)$. The raising and lowering operators $L_{\pm} = L_x \pm iL_y$ act as $L_{\pm}|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle$. The uncertainty relation for L_x and L_y is $\Delta L_x \Delta L_y \geq (\hbar/2)|\langle L_z \rangle|$.

(a) SHOW THAT · U2 [2]

Show that the only non-zero matrix elements of L_+ in this basis are $\langle 1,1|L_+|1,0\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\hbar$.

1 $L_+|1,0\rangle = \hbar\sqrt{2-0}|1,1\rangle = \sqrt{2}\hbar|1,1\rangle$ and $L_+|1,-1\rangle = \hbar\sqrt{2-(-1)(0)}|1,0\rangle = \sqrt{2}\hbar|1,0\rangle$

2 $L_+|1,1\rangle = \hbar\sqrt{2-2} = 0$, and each $L_+|1,m\rangle$ is proportional to $|1,m+1\rangle$, so by orthonormality every other element $\langle 1,m'|L_+|1,m\rangle$ vanishes

The ladder stops at $m = l$ because the square root vanishes there. Candidates should see the zero come out of the formula rather than assert that $m = 2$ is not allowed.

(b) DETERMINE · U2 [2]

Determine the 3×3 matrix that represents L_x in this basis.

1 $L_- = L_+^\dagger$, with non-zero elements $\langle 1,0|L_-|1,1\rangle = \langle 1,-1|L_-|1,0\rangle = \sqrt{2}\hbar$

2 $L_x = (L_+ + L_-)/2 = (\hbar/\sqrt{2})[[0, 1, 0], [1, 0, 1], [0, 1, 0]]$

Candidates who forget the $1/2$ in $L_x = (L_+ + L_-)/2$ carry a matrix with eigenvalues $\pm 2\hbar$ into (c), which cannot be right for a component of $l = 1$ angular momentum.

(c) DETERMINE · U2 [3]

Determine the eigenvalues of L_x .

1 sets up $\det(L_x - \lambda I) = 0$ using the matrix from (b)

2 expanding gives $-\lambda^3 + \lambda\hbar^2 = 0$, that is $-\lambda(\lambda^2 - \hbar^2) = 0$

3 $\lambda = \hbar, 0$ and $-\hbar$, the same values as L_z , as they must be, since nothing distinguishes the x axis from the z axis

The comparison with L_z is a built-in check. A candidate who finds any other eigenvalues should look for a lost factor in (b) before going on.

(d) CALCULATE · U3 [3]

A particle is prepared in $|1,1\rangle$. Calculate the probabilities of obtaining $\hbar, 0$ and $-\hbar$ in a measurement of L_x .

1 the normalised eigenvectors of L_x are $(1, \sqrt{2}, 1)/2$ for $\hbar$, $(1, 0, -1)/\sqrt{2}$ for 0 , and $(1, -\sqrt{2}, 1)/2$ for $-\hbar$

2 their overlaps with $|1,1\rangle = (1, 0, 0)$ are the first components, $1/2, 1/\sqrt{2}$ and $1/2$

3 the probabilities are $1/4, 1/2$ and $1/4$, which sum to 1

A state with definite L_z has no definite L_x . Candidates who answer ' $\hbar$ with certainty' have assumed that the largest z component forces the largest x component too.

(e) SHOW THAT · U2 [2]

Show that in the state $|1,1\rangle$ the product $\Delta L_x \Delta L_y$ is exactly equal to the lower bound $(\hbar/2)|\langle L_z \rangle|$.

1 from (d), $\langle L_x \rangle = 0$ and $\langle L_x^2 \rangle = (1/4)\hbar^2 + (1/2)(0) + (1/4)\hbar^2 = \hbar^2/2$, so $\Delta L_x = \hbar/\sqrt{2}$; the state is unchanged except for a phase by rotation about z , so $\Delta L_y = \hbar/\sqrt{2}$ as well (or $\langle L_x^2 \rangle + \langle L_y^2 \rangle = \langle L^2 \rangle - \langle L_z^2 \rangle = 2\hbar^2 - \hbar^2 = \hbar^2$)

2 $\Delta L_x \Delta L_y = \hbar^2/2$, and $(\hbar/2)|\langle L_z \rangle| = (\hbar/2)(\hbar) = \hbar^2/2$, so the bound is met with equality

Candidates who say that $|1,1\rangle$ has no uncertainty because it is an eigenstate are thinking of L_z alone. The state is exactly as sharp in L_x and L_y as the commutator permits, and no sharper.

For a hydrogen atom, the radial function $u(r) = rR(r)$ satisfies $-(\hbar^2/2m) d^2u/dr^2 + [-e^2/(4\pi\epsilon_0 r) + \hbar^2 l(l+1)/(2mr^2)] u = Eu$. A proposed solution is $u(r) = N r^3 e^{-(r/(3a_0))}$, where N is a normalisation constant and $a_0 = 4\pi\epsilon_0 \hbar^2/(me^2) = 5.29 \times 10^{-11} \text{ m}$ is the Bohr radius, so that $e^2/(4\pi\epsilon_0) = \hbar^2/(ma_0)$ and $\hbar^2/(2ma_0^2) = 13.6 \text{ eV}$. Treat the proton as infinitely heavy. You may use $\int_0^\infty r^n e^{-\beta r} dr = n!/\beta^{n+1}$ for a non-negative integer n and $\beta > 0$.

(a) DETERMINE · U2

[4]

Determine the value of l for which $u(r)$ satisfies the radial equation, and the corresponding energy E in electronvolts.

- $d^2u/dr^2 = N[6r - 2r^2/a_0 + r^3/(9a_0^2)] e^{-(r/(3a_0))}$
- writes the Coulomb term as $-(\hbar^2/2m)(2/(a_0 r))u$, whose r^2 contribution $-(\hbar^2/2m)(2r^2/a_0)Ne^{-(r/(3a_0))}$ cancels the $+(\hbar^2/2m)(2r^2/a_0)Ne^{-(r/(3a_0))}$ from $-(\hbar^2/2m)u''$
- the terms linear in r are $-(\hbar^2/2m)(6r)$ from the kinetic term and $+(\hbar^2/2m)l(l+1)r$ from the centrifugal term (each times $Ne^{-(r/(3a_0))}$), which cancel only if $l(l+1) = 6$, so $l = 2$
- the surviving r^3 term gives $-(\hbar^2/2m)(1/(9a_0^2))u = Eu$, so $E = -\hbar^2/(18ma_0^2) = -13.6 \text{ eV}/9 = -1.51 \text{ eV}$

l is not given; it is found only by insisting that the equation hold at every r , power by power. A candidate who tries $l = 1$ finds an 'energy' that depends on r , which means the guess is wrong, not that the energy varies.

(b) STATE · U1

[2]

State the principal quantum number n of this state, its number of radial nodes, and its spectroscopic label.

- $E = -13.6 \text{ eV}/n^2 = -1.51 \text{ eV}$ gives $n = 3$
- u has no zero for $r > 0$, so there are no radial nodes ($n - l - 1 = 0$), and the state is 3d

Candidates who count the zero of u at $r = 0$ as a node report one radial node. That zero is imposed at the origin for every state and is never counted.

(c) DETERMINE · U3

[2]

Determine the most probable distance of the electron from the proton, in nanometres.

- the radial probability density is $P(r) = |u|^2 \propto r^6 e^{-(2r/(3a_0))}$, and $dP/dr = 0$ gives $6/r = 2/(3a_0)$
- $r = 9a_0 = 0.476 \text{ nm}$

The function to maximise is $|u|^2 = r^2|R|^2$, which includes the r^2 from the volume element. Maximising $|R|^2$ instead gives $6a_0$, which is the peak of the density per unit volume along a line, not of the probability per unit radius.

(d) CALCULATE · U3

[2]

Calculate the expectation value $\langle r \rangle$ in nanometres.

- $\langle r \rangle = \int_0^\infty r^7 e^{-(\beta r)} dr / \int_0^\infty r^6 e^{-(\beta r)} dr$ with $\beta = 2/(3a_0)$, which is $(7!/\beta^8)/(6!/\beta^7) = 7/\beta$
- $\langle r \rangle = 21a_0/2 = 10.5a_0 = 0.555 \text{ nm}$, larger than the most probable distance because $P(r)$ has a long tail at large r

N cancels in the ratio and never needs to be found. A candidate who normalises first loses no marks but loses several minutes.

(e) COMPARE · U1

[2]

Compare the degeneracy of this energy level in hydrogen with the degeneracy an $l = 2$ level would have in a generic central potential, such as a finite spherical well, giving the cause of each.

- 1 in hydrogen E depends on n alone, so the level contains every $n = 3$ state, $3s$, $3p$ and $3d$: $1 + 3 + 5 = n^2 = 9$ spatial states (18 with spin); the extra degeneracy between different l belongs to the $1/r$ form of the Coulomb potential, not to spherical symmetry
- 2 in a generic central potential such as a finite spherical well, E depends on l as well as on the radial quantum number, so only the $2l + 1 = 5$ states with $m = -2$ to 2 share the energy; that degeneracy is guaranteed by rotational symmetry, because m does not appear in the radial equation

Candidates often attribute the ninefold degeneracy to spherical symmetry. Spherical symmetry accounts only for the fivefold degeneracy in m , which every central potential shares; the degeneracy between $3s$, $3p$ and $3d$ is special to $1/r$. The comparison is made with a generic well because the isotropic harmonic oscillator has an accidental degeneracy of its own.

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CORE
COURSE

DATA ANALYSIS · DEMANDING · UQMI-08.04 · UQMI-08.08

[10]

Gate leakage in a transistor is caused by electrons tunnelling directly through the thin silicon dioxide layer between the gate and the silicon. To study it, an engineer measures the gate current I of six test capacitors made with different oxide thicknesses t_{ox} and different gate areas A , all at the same small gate voltage. The model to be tested is $J = J_0 \exp(-2\kappa t_{ox})$, where $J = I/A$ is the current density, J_0 is the same for every capacitor, and $\kappa = \sqrt{(2m^*\phi_B)/\hbar}$, with ϕ_B the height of the oxide barrier above the energy of the tunnelling electrons and m^* the effective mass of an electron in the oxide. Internal photoemission measurements, made independently, give $\phi_B = 3.1$ eV. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s, 1 eV $= 1.602 \times 10^{-19}$ J and $1 \mu\text{m}^2 = 1 \times 10^{-12}$ m².

(a) DESCRIBE · U4

[2]

Describe how the readings should be processed and plotted so that the model predicts a straight line, including why the currents themselves should not be plotted and what the gradient measures.

- 1 divide each current by its gate area, converted to m², to obtain $J = I/A$: the capacitors have different areas and the model predicts current per unit area, so $\ln I$ would contain a different $\ln A$ for each capacitor and would not follow the model
- 2 plot $\ln(J / \text{A m}^{-2})$ against t_{ox} ; the model gives $\ln J = \ln J_0 - 2\kappa t_{ox}$, a straight line of gradient -2κ , with J_0 affecting only the intercept

Plotting $\ln I$ against t_{ox} gives a gradient of about -5.8 nm^{-1} , half the true value, because the larger capacitors were used for the thicker oxides. A candidate who does not remove the area has measured the design of the test chip, not the barrier.

(b) DETERMINE · U4

[3]

Determine κ from the readings, and the reduction in oxide thickness that multiplies the leakage current density by ten.

- 1 $J = 2.00 \times 10^4$, 2.10×10^3 , 203, 21.8, 2.14 and 0.225 A m^{-2} , so $\ln(J / \text{A m}^{-2}) = 9.90$, 7.65, 5.31, 3.08, 0.76 and -1.49 for capacitors 1 to 6
- 2 the best-fit gradient of $\ln J$ against t_{ox} is $-1.14 \times 10^{10} \text{ m}^{-1}$ (accept -1.10×10^{10} to $-1.18 \times 10^{10} \text{ m}^{-1}$, or -11.4 nm^{-1}), so $\kappa = 5.7 \times 10^9 \text{ m}^{-1}$ (accept 5.5×10^9 to $5.9 \times 10^9 \text{ m}^{-1}$)
- 3 a tenfold rise in J needs a thinning of $\Delta t_{ox} = \ln 10 / (2\kappa) = 2.303 / (1.14 \times 10^{10} \text{ m}^{-1}) = 2.0 \times 10^{-10} \text{ m}$, so each 0.20 nm removed from the oxide raises the leakage tenfold

Candidates who leave the areas in μm^2 shift every $\ln J$ by the same amount, which does not change the gradient; those who leave the areas out altogether halve it. Checking that $\ln J$ falls by about 2.3 for each 0.20 nm of oxide catches the second slip.

(c) ESTIMATE · U4

[3]

Estimate the effective mass m^* of the tunnelling electrons in the oxide, as a multiple of m_e , using $\phi_B = 3.1$ eV, and state the assumptions on which your estimate rests.

- 1 rearranges $\kappa = \sqrt{(2m^*\phi_B)/\hbar}$ to $m^* = \hbar^2\kappa^2/(2\phi_B)$
- 2 $m^* = (1.055 \times 10^{-34})^2 \times (5.7 \times 10^9)^2 / (2 \times 3.1 \times 1.602 \times 10^{-19}) = 3.6 \times 10^{-31} \text{ kg} \approx 0.40m_e$ (accept $0.37m_e$ to $0.43m_e$)
- 3 states at least two assumptions, for example a rectangular barrier of fixed height ϕ_B that the small gate voltage does not tilt appreciably, a single effective mass independent of energy and thickness, the same J_0 for every capacitor, or oxide thicknesses that are accurate and uniform across each gate

A number without its assumptions earns two of the three marks. The assumption about J_0 matters most, because any prefactor that varies with t_{ox} changes the gradient and therefore m^* .

(d) EVALUATE · U4

[2]

A colleague analyses the same readings taking the electron in the oxide to have the free-electron mass m_e , and reports the barrier height that follows. Evaluate this analysis.

- 1 with $m^* = m_e$ the gradient gives $\phi = \hbar^2\kappa^2/(2m_e) = 2.0 \times 10^{-19} \text{ J} \approx 1.2 \text{ eV}$ (accept 1.15 to 1.33 eV), less than half the 3.1 eV measured independently by photoemission
- 2 the gradient fixes only the product $m^*\phi_B$, so the readings alone cannot decide between the two analyses; judged against the independent barrier height the free-electron analysis must be rejected, and the data require an effective mass near $0.4m_e$ rather than a barrier near 1.2 eV

Candidates who accept 1.2 eV because the straight line fits well have tested the exponential form, not the parameters. A good fit cannot separate two quantities that enter the model only as a product.

18 CORE COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-09.04 · UQMI-09.07 · UQMI-09.08 · UQMI-09.09

[15]

A particle of mass m moves in the potential $V(x) = \frac{1}{2}m\omega^2x^2$, so that $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$. Define $\hat{a} = \sqrt{(m\omega/2\hbar)}(\hat{x} + i\hat{p}/(m\omega))$ and its adjoint $\hat{a}^\dagger = \sqrt{(m\omega/2\hbar)}(\hat{x} - i\hat{p}/(m\omega))$, and write $\hat{N} = \hat{a}^\dagger\hat{a}$. The canonical commutator is $[\hat{x}, \hat{p}] = i\hbar$. Once $[\hat{a}, \hat{a}^\dagger] = 1$ has been established, you may use its consequences $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(a) DERIVE · U2

[3]

Derive $[\hat{a}, \hat{a}^\dagger] = 1$ from the canonical commutator.

- 1 $[\hat{a}, \hat{a}^\dagger] = (m\omega/2\hbar)[\hat{x} + i\hat{p}/(m\omega), \hat{x} - i\hat{p}/(m\omega)]$, and $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$ leaves only the cross terms
- 2 these are $-(i/(m\omega))[\hat{x}, \hat{p}] + (i/(m\omega))[\hat{p}, \hat{x}] = -(2i/(m\omega))[\hat{x}, \hat{p}]$
- 3 substituting $[\hat{x}, \hat{p}] = i\hbar$ gives $(m\omega/2\hbar)(2\hbar/(m\omega)) = 1$

The result is a pure number. A candidate who finds $i\hbar$ or $\hbar/(m\omega)$ has dropped a factor in the definition of $\hat{a}$; $\hat{a}$ is dimensionless, and checking that catches the slip at once.

(b) SHOW THAT · U2

[2]

Show that $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$.

- 1 $\hat{a}^\dagger\hat{a} = (m\omega/2\hbar)[\hat{x}^2 + \hat{p}^2/(m^2\omega^2) + (i/(m\omega))(\hat{x}\hat{p} - \hat{p}\hat{x})] = (m\omega/2\hbar)[\hat{x}^2 + \hat{p}^2/(m^2\omega^2)] - \frac{1}{2}$, using $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$
- 2 multiplying by $\hbar\omega$ gives $\hbar\omega\hat{a}^\dagger\hat{a} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{1}{2}\hbar\omega = \hat{H} - \frac{1}{2}\hbar\omega$, so $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$

The $\frac{1}{2}$ comes entirely from the operators failing to commute. A candidate who factorises $x^2 + p^2$ as if they were numbers loses the zero-point energy.

(c) DERIVE · U2

[4]

Derive the energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$ with $n = 0, 1, 2, \dots$, using only the commutators and the fact that no vector has a negative squared norm. Do not evaluate any integral.

- 1 if $\hat{N}|\nu\rangle = \nu|\nu\rangle$, then $\hat{N}(\hat{a}|\nu\rangle) = (\hat{a}\hat{N} - \hat{a})|\nu\rangle = (\nu - 1)\hat{a}|\nu\rangle$, so $\hat{a}|\nu\rangle$ is either zero or an eigenstate with eigenvalue $\nu - 1$; likewise $\hat{a}^\dagger|\nu\rangle$ has eigenvalue $\nu + 1$
- 2 for normalised $|\nu\rangle$, $\nu = \langle\nu|\hat{a}^\dagger\hat{a}|\nu\rangle = \|\hat{a}|\nu\rangle\|^2 \geq 0$, so no eigenvalue of $\hat{N}$ is negative
- 3 repeated lowering would reach a negative eigenvalue unless it ends on a state $|0\rangle$ with $\hat{a}|0\rangle = 0$, and that state has $\nu = \|\hat{a}|0\rangle\|^2 = 0$; a non-integer ν would step past zero without ending, so ν must be a non-negative integer
- 4 $\|\hat{a}^\dagger|n\rangle\|^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle = n + 1 > 0$, so raising from $\nu = 0$ never terminates and reaches every $\nu = n = 0, 1, 2, \dots$; then $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$ gives $E_n = (n + \frac{1}{2})\hbar\omega$, with ground-state energy $\frac{1}{2}\hbar\omega$

The positive norm is what makes the argument work. Candidates who announce that the ladder has a bottom rung have assumed the result, and the terminating state must be shown to have $\nu = 0$, not merely to exist.

(d) DETERMINE · U2

[3]

Determine $\langle\hat{x}^2\rangle$ in the eigenstate $|n\rangle$ using ladder operators alone.

- 1 $\hat{x} = \sqrt{(\hbar/2m\omega)}(\hat{a} + \hat{a}^\dagger)$, so $\hat{x}^2 = (\hbar/2m\omega)(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$
- 2 $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ change n by -2 and $+2$, so $\langle n|\hat{a}^2|n\rangle = \langle n|\hat{a}^{\dagger 2}|n\rangle = 0$
- 3 $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$ gives $\langle n|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle = 2n + 1$, so $\langle\hat{x}^2\rangle = (2n + 1)\hbar/(2m\omega) = E_n/(m\omega^2)$

Candidates who expand $(\hat{a} + \hat{a}^\dagger)^2$ as $\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}$ have commuted $\hat{a}$ past $\hat{a}^\dagger$ and lost the $+1$ that keeps $\langle x^2\rangle$ non-zero in the ground state. Writing the answer as $E_n/(m\omega^2)$ shows half the energy is potential, a check from the virial theorem.

(e) PREDICT · U1

[3]

The particle is in the ground state $|0\rangle$ when the potential is suddenly tightened to $V(x) = \frac{1}{2}m(2\omega)^2x^2$, so quickly that the wavefunction has no time to change. Predict which energy eigenstates of the new oscillator the particle can then be found in, and its mean energy $\langle\hat{H}'\rangle$ immediately after the change, in terms of $\hbar\omega$.

- 1 V and V' are both even, so the new eigenstates have parity $(-1)^n$; the unchanged state $|0\rangle$ is even, so its overlap with every odd- n state of the new oscillator vanishes, and only $n = 0, 2, 4, \dots$ can be found, with energies $(n + \frac{1}{2})(2\hbar\omega) = \hbar\omega, 5\hbar\omega, 9\hbar\omega, \dots$
- 2 the state is still the old $|0\rangle$, so from (d) $\langle\hat{x}^2\rangle = \hbar/(2m\omega)$; then $\langle V\rangle = \frac{1}{2}m\omega^2\langle\hat{x}^2\rangle = \frac{1}{4}\hbar\omega$ is half of $E_0 = \frac{1}{2}\hbar\omega$, so the kinetic energy is $\langle\hat{p}^2\rangle/2m = \frac{1}{4}\hbar\omega$, and neither expectation changes at the instant of the change
- 3 $\langle\hat{H}'\rangle = \langle\hat{p}^2\rangle/2m + \frac{1}{2}m(2\omega)^2\langle\hat{x}^2\rangle = \frac{1}{4}\hbar\omega + \hbar\omega = (5/4)\hbar\omega$, above the new ground energy $\hbar\omega$, so the particle is not certain to be found in the new ground state

Candidates who say the energy is still $\frac{1}{2}\hbar\omega$, or that the particle simply occupies the new ground state, have confused an unchanged wavefunction with an unchanged energy: the Hamiltonian has changed while the state has not. A candidate who allows the new $|1\rangle$ has not used the parity that both potentials share.

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CORE
COURSE

FREE RESPONSE · MODEL, SOLVE AND CHECK THE LIMITS · DISCRIMINATING · UQMI-14.01 · UQMI-14.04 · UQMI-14.08 · UQMI-14.09

[15]

A simple model of the deuteron treats the neutron-proton pair as a single particle of reduced mass $\mu = 8.37 \times 10^{-28}$ kg moving in a spherical square well, with $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, where $a = 2.0 \times 10^{-15}$ m. Only states with $l = 0$ are considered, so ψ depends on r alone and $\nabla^2\psi = (1/r^2) d/dr (r^2 d\psi/dr)$. Primes denote d/dr . Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ MeV} = 1.602 \times 10^{-13}$ J. The measured binding energy of the deuteron is 2.22 MeV, and it has no bound excited state.

- (a) DERIVE · U2 [3]**
 Starting from the time-independent Schrödinger equation in three dimensions, derive the equation $-(\hbar^2/2\mu)u'' + V(r)u = Eu$ satisfied by $u(r) = r\psi(r)$.
- 1 writes $-(\hbar^2/2\mu)\nabla^2\psi + V(r)\psi = E\psi$, with $\nabla^2\psi = (1/r^2)(r^2\psi')'$ for $l = 0$
 - 2 with $\psi = u/r$, $\psi' = u'/r - u/r^2$, so $r^2\psi' = ru' - u$ and $(r^2\psi')' = u' + ru'' - u' = ru''$, giving $\nabla^2\psi = u''/r$
 - 3 substituting and multiplying through by r gives $-(\hbar^2/2\mu)u'' + V(r)u = Eu$
- The point of $u = r\psi$ is that the first-derivative terms cancel exactly, turning a three-dimensional problem into a one-dimensional one. Candidates who drop the u' terms without showing that they cancel have asserted the result.*
- (b) JUSTIFY · U1 [2]**
 Justify the boundary condition $u(0) = 0$. Note that a wavefunction behaving as $1/r$ near the origin is square-integrable in three dimensions.
- 1 if $u(0) \neq 0$ then $\psi \approx u(0)/r$ near $r = 0$, and the volume element $r^2 dr$ keeps $\int |\psi|^2 r^2 dr$ finite there, so normalisability alone does not exclude it
 - 2 but $\nabla^2(1/r) = -4\pi\delta^3(r)$, so such a ψ satisfies the Schrödinger equation at the origin only if V contains a delta function there; the finite well contains none, so $u(0) = 0$ is required
- The tempting answer is that ψ would blow up. It does, but harmlessly for normalisation; the real objection is that $1/r$ is not a solution at the origin, which is why the prompt supplies the hint.*
- (c) SHOW THAT · U2 [3]**
 Show that a bound state with energy $E = -E_B$, where $0 < E_B < V_0$, must satisfy $k \cot(ka) = -\kappa$, where $\hbar k = \sqrt{2\mu(V_0 - E_B)}$ and $\hbar \kappa = \sqrt{2\mu E_B}$.
- 1 inside, $u'' = -k^2u$ with $u(0) = 0$ gives $u = A \sin(kr)$; outside, $u'' = \kappa^2u$ and normalisability removes $e^{+\kappa r}$, leaving $u = B e^{-\kappa r}$
 - 2 u and u' are continuous at $r = a$ because V is finite there: $A \sin(ka) = B e^{-\kappa a}$ and $Ak \cos(ka) = -\kappa B e^{-\kappa a}$
 - 3 dividing the second condition by the first gives $k \cot(ka) = -\kappa$
- This is the odd-parity condition of a one-dimensional finite well of width $2a$, with $u(0) = 0$ playing the part of odd symmetry. Candidates who write $u = A \cos(kr)$ inside have lost the result of (b).*
- (d) DERIVE · U2 [3]**
 Derive, in terms of μ , a and $\hbar$, the minimum depth $V_{0,\min}$ for which the well has any bound state at all.
- 1 with $k > 0$ and $\kappa > 0$, $k \cot(ka) = -\kappa$ needs $\cot(ka) < 0$, so the lowest state requires $ka > \pi/2$; for $ka \leq \pi/2$ there is no solution
 - 2 the bound state first appears at threshold, $E_B \rightarrow 0$, where $\kappa \rightarrow 0$ and the condition becomes $\cot(ka) = 0$, with $ka = \pi/2$
 - 3 at $E_B = 0$, $\hbar k = \sqrt{2\mu V_0}$, so $V_{0,\min} = \pi^2 \hbar^2 / (8\mu a^2)$
- In one dimension any attractive well binds a state. In three dimensions $u(0) = 0$ removes the even solutions and with them that guarantee, and candidates who carry the one-dimensional result over never look for a minimum depth.*
- (e) CALCULATE · U3 [2]**
 Calculate $V_{0,\min}$ in MeV for the values given.
- 1 $V_{0,\min} = \pi^2(1.055 \times 10^{-34})^2 / (8 \times 8.37 \times 10^{-28} \times (2.0 \times 10^{-15})^2) = 4.10 \times 10^{-12} \text{ J}$
 - 2 $V_{0,\min} = 25.6 \text{ MeV}$
- Squaring $2.0 \times 10^{-15} \text{ m}$ is where powers of ten go astray. Tens of MeV is the scale of nuclear binding, and an answer in eV or GeV signals a slip.*

(f) SKETCH · U1

[2]

A well depth of 36.6 MeV reproduces the measured binding energy, giving $ka = 1.82$ and $ka = 0.46$. Sketch $u(r)$ for the deuteron's ground state from $r = 0$ to $r = 5a$, marking $r = a$ on the axis.

- 1 for $r < a$, u rises from $u(0) = 0$ as a sine, reaches its maximum just inside the wall (at $kr = \pi/2$, $r \approx 0.86a$) and has begun to fall by $r = a$
- 2 at $r = a$, u and its slope join smoothly onto a slowly decaying exponential with decay length $1/\kappa \approx 2.2a$, still clearly non-zero at $r = 5a$, so that most of the area under u^2 lies outside the well

Candidates tend to draw a bound state as a hump confined to the well. A barely bound state spends most of its time outside the range of the force that binds it (here about 64% of the probability lies beyond $r = a$), and the sketch must show that tail.

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 31 MARKS · 26%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 49 MARKS · 42%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

26 MARKS · 22%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

12 MARKS · 10%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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