

University Quantum Mechanics I · Midterm Examination

The first eight units on one paper: the experiments that forced a wavefunction, the Schrödinger equation and its probability current, operators, measurement and uncertainty, the free particle, the infinite and finite square wells, and tunnelling at steps and barriers. Every probability is an integral of $|\psi|^2$, and every coefficient is matched region by region before it is quoted.

SUGGESTED TIME 90 minutes	TOTAL MARKS 74	QUESTIONS 14	MULTIPLE CHOICE 8
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SECTION **UQMI · Midterm · University Quantum Mechanics I — Midterm examination**

IN THE FORMAT OF A university midterm: conceptual multiple choice, structured problems, a data-analysis question and an extended derivation

DEMAND **Recall 1 · Routine 7 · Demanding 5 · Discriminating 1**

Pitched at what a strong course grade requires rather than at what a homework set rehearses. Routine substitutions carry few marks; most sit on problems that have to be set up before they can be solved, on derivations from a named law, and on checking a result against its limits. Every paper still opens with entry points, so you can tell which marks you should be securing first.

COURSE UNITS UQMI-01 · UQMI-02 · UQMI-03 · UQMI-04 · UQMI-05 · UQMI-06 · UQMI-07 · UQMI-08

INSTRUCTIONS

- Answer all questions in the spaces provided.
- Show your working. Method marks are awarded even where the final answer is wrong.
- Give your answers to a sensible number of significant figures, with a unit.
- The mark scheme for this paper follows the last question.

Written by GioPhysics from its own University Physics course maps. These are practice examinations in a university style: they are not the examinations of any university, contain no past-paper questions, and carry no academic credit. For a course you are taking, your instructor's syllabus and past papers remain the authority.

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University Quantum Mechanics I · Midterm Examination

Answer all questions. 74 marks in 90 minutes.

1 MULTIPLE CHOICE · ROUTINE · UQMI-01.04 [1]
CORE COURSE

The Compton shift for a photon scattered through an angle θ by a free electron at rest is $\Delta\lambda = (h/m_e c)(1 - \cos \theta)$, with $h/m_e c = 2.43$ pm. Photons of wavelength 71.0 pm and photons of wavelength 500 nm are each scattered through 90° . Which statement best explains why the effect was established with X-rays rather than with visible light?

- A** The shift is 2.43 pm in both cases, which is 3.4% of 71.0 pm but only 4.9×10^{-6} of 500 nm, a fractional change about 7000 times smaller and far harder to detect.
- B** A visible photon carries no momentum, because its energy is far below the electron's rest energy, so it cannot make the electron recoil.
- C** The shift is proportional to the incident wavelength, so for 500 nm light it is so large that the scattered light leaves the visible band and goes undetected.
- D** The shift is 2.43 pm for X-rays but smaller by the factor 71.0 pm/500 nm for visible light, about 0.35 fm, which is too small to resolve.

Answer

2 MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 [1]
CORE COURSE

Each function in the options is finite and continuous for all x , and takes the form stated at large $|x|$, where C and k are nonzero real constants. Which one could be normalised, and so represent the state of a particle?

- A** $\psi(x) \approx C/\sqrt{|x|}$ at large $|x|$
- B** $\psi(x) \approx C e^{(ikx)}$ at large $|x|$
- C** $\psi(x) \approx C/|x|$ at large $|x|$
- D** $\psi(x) \approx C \cos(kx)/|x|^{1/4}$ at large $|x|$

Answer

3 MULTIPLE CHOICE · ROUTINE · UQMI-02.05 [1]
CORE COURSE

A particle in a time-independent potential is in the state $\Psi(x,t) = [\psi_1(x)e^{(-iE_1t/\hbar)} + \psi_2(x)e^{(-iE_2t/\hbar)}]/\sqrt{2}$, where ψ_1 and ψ_2 are real, normalised, mutually orthogonal eigenfunctions of $\hat{H}$ with eigenvalues $E_1 \neq E_2$. Which quantity is independent of time whatever the forms of ψ_1 and ψ_2 ?

- A** $|\Psi(x,t)|^2$ at a point x where ψ_1 and ψ_2 are both nonzero
- B** $\langle x \rangle$
- C** $|\int \Psi^*(x,0)\Psi(x,t) dx|^2$, the probability of finding the particle still in its initial state
- D** $\langle H \rangle$

Answer

4

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-03.05

[1]

Acting on square-integrable functions on $-\infty < x < \infty$ that vanish, together with their derivatives, as $|x| \rightarrow \infty$, which one of these operators is Hermitian?

- A d/dx
- B $x d/dx$
- C $i d^2/dx^2$
- D $i d/dx$

Answer

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.05

[1]

A particle has Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, and $[\hat{x}, \hat{p}] = i\hbar$. What is the commutator $[\hat{x}, \hat{H}]$?

- A $i\hbar\hat{p}/m$
- B 0, because $\hat{x}$ commutes with $V(\hat{x})$
- C $i\hbar\hat{p}/(2m)$
- D $-i\hbar\hat{p}/m$

Answer

6

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQMI-04.09

[1]

Position is measured once on each of N electrons, all prepared in the same state ψ , with a detector whose resolution is far finer than the spread of the results. The readings have sample standard deviation s , and their mean has standard error $s/\sqrt{N}$. N is increased from 100 to 10 000. Which statement is correct?

- A s and the standard error both fall by a factor of 10, because more data reduces every uncertainty.
- B s stays close to σ_x of the state, while the standard error falls by a factor of 10.
- C s and the standard error both stay the same, because the uncertainty principle fixes both.
- D s grows, because each extra measurement disturbs the ensemble, while the standard error falls by a factor of 10.

Answer

7CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.09 · UQMI-05.01 · UQMI-05.04

[1]

A particle of mass m has $V(x) = 0$ everywhere on $-\infty < x < \infty$. Which statement about the solutions of its time-independent Schrödinger equation is correct?

- A** Only discrete energies are allowed, because ψ must be normalisable and only certain wavelengths fit.
- B** Every $E > 0$ is allowed, and each is twofold degenerate, with solutions e^{ikx} and e^{-ikx} where $k = \sqrt{2mE}/\hbar$.
- C** Every $E > 0$ is allowed, and each is non-degenerate, because states in one dimension are never degenerate.
- D** Negative energies are also allowed, with solutions $e^{\kappa x}$ and $e^{-\kappa x}$ where $\kappa = \sqrt{-2mE}/\hbar$.

Answer

8CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-06.03

[1]

A particle is in the ground state $\psi_1(x) = \sqrt{2/L} \sin(\pi x/L)$ of an infinite square well occupying $0 < x < L$. What is the probability that a measurement of position finds it in $0 < x < L/4$?

- A** 0.41
- B** 0.25
- C** 0.091
- D** 0.045

Answer

A particle of mass m moving along the x -axis is in a stationary state of some potential $V(x)$. Its wavefunction is $\psi(x) = 0$ for $x < 0$; $\psi(x) = Ax/a$ for $0 \leq x \leq a$; and $\psi(x) = Ae^{-(x-a)/a}$ for $x > a$, where a is a positive length and A is a positive real constant. ψ is continuous everywhere, but its slope jumps at $x = 0$ and at $x = a$.

(a) SHOW THAT

[3]

Show that $A = \sqrt{6/(5a)}$.

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.....

(b) CALCULATE

[2]

Calculate the probability that a measurement of position finds the particle in $0 < x < a/2$, and the probability that it finds the particle at $x > a$.

.....

.....

(c) EXPLAIN

[2]

The slope of ψ must be continuous wherever V is finite. Explain why, and explain why the jump in slope at $x = 0$ is nevertheless admissible.

.....

.....

(d) DETERMINE

[2]

The jump in slope at $x = a$ requires $V(x)$ to contain a delta-function term $g\delta(x - a)$. Determine g in terms of $\hbar$, m and a , and say whether the spike is attractive or repulsive.

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.....

An electron ($m_e = 9.109 \times 10^{-31}$ kg) moves freely along the x-axis, $V = 0$. At $t = 0$ it is prepared as a Gaussian wave packet centred on wavenumber k_0 , with position standard deviation $\sigma_0 = 0.40$ nm and $\sigma_x \sigma_p = \hbar/2$. The kinetic energy corresponding to the central wavenumber is $E_0 = \hbar^2 k_0^2 / (2m_e) = 1.50$ eV. The width of the packet then evolves as $\sigma_x(t) = \sigma_0 \sqrt{1 + (\hbar t / (2m_e \sigma_0^2))^2}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) EXPLAIN

[2]

Explain why no single plane wave $e^{ik_0 x}$ can describe the electron, and how a superposition of plane waves overcomes the difficulty.

(b) SHOW THAT

[2]

Using the dispersion relation of a free particle, show that the centre of the packet moves at $v_g = \hbar k_0 / m_e$, twice the phase velocity of its central component.

(c) CALCULATE

[3]

Calculate the group velocity, the time at which σ_x has grown to $2\sigma_0$, and the distance the centre of the packet travels in that time.

(d) PREDICT

[2]

The electron is instead prepared with $\sigma_0 = 0.80$ nm, still with $\sigma_x \sigma_p = \hbar/2$ at $t = 0$. Predict, with a reason, the new time for the width to double.

An electron ($m_e = 9.109 \times 10^{-31}$ kg) is confined along one axis of a nanostructure modelled as an infinite square well on $0 < x < L$, with $L = 1.20$ nm. Its energy eigenfunctions are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m_eL^2)$. At $t = 0$ its state is $\Psi(x,0) = \sqrt{2/3} \psi_1(x) + (i/\sqrt{3}) \psi_2(x)$. You may use $\int_0^L \psi_n \times \psi_n dx = L/2$ for every n , and $\int_0^L \psi_1 \times \psi_2 dx = -16L/(9\pi^2)$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) CALCULATE

[2]

Calculate the probabilities of obtaining E_1 and E_2 in a measurement of energy, and the expectation value $\langle H \rangle$ in electronvolts.

(b) SHOW THAT

[3]

Write down $\Psi(x,t)$, and show that $\langle x \rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2)) \sin(\omega t)$, where $\omega = (E_2 - E_1)/\hbar$.

(c) CALCULATE

[2]

Calculate the period of the oscillation of $\langle x \rangle$, and its amplitude in nanometres.

(d) PREDICT

[2]

Suppose instead that $\Psi(x,0) = \sqrt{2/3} \psi_1(x) + (1/\sqrt{3}) \psi_2(x)$. Predict which of $P(E_1)$, $P(E_2)$, $\langle H \rangle$, and the period and amplitude of $\langle x \rangle(t)$ change, and what does change.

(e) DESCRIBE

[2]

Return to the original state. At $t = 2.0$ fs a measurement of energy on the electron returns E_2 . Describe the state immediately afterwards, and how $\langle x \rangle$ and the result of a second energy measurement behave from then on.

An electron is bound in a thin layer of material modelled as a symmetric finite square well: $V(x) = 0$ for $|x| < a$ and $V(x) = V_0$ for $|x| > a$, with well width $2a = 0.80$ nm and $V_0 = 2.00$ eV. Treat the electron as having the free-electron mass $m_e = 9.109 \times 10^{-31}$ kg. For a bound state with $0 < E < V_0$, write $k = \sqrt{2m_e E}/\hbar$, $\kappa = \sqrt{2m_e(V_0 - E)}/\hbar$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{2m_e V_0}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) JUSTIFY

[2]

Justify looking for bound states that are either even or odd in x , rather than of general form.

(b) DERIVE

[4]

Starting from the solutions of the Schrödinger equation in each region, derive the condition that the energies of the even bound states must satisfy, written as a single equation in z and z_0 alone.

(c) DETERMINE

[2]

Calculate z_0 , and hence determine how many bound states the well has and the parity of each.

(d) COMPARE

[3]

The even condition has its lowest root at $z = 1.159$. Compare the ground-state energy with that of an infinite square well of the same width, 0.80 nm, and account for the difference.

In a scanning tunnelling microscope a sharp tungsten tip is held above a clean metal surface with a small bias voltage between them, and the tunnelling current I is recorded as a piezoelectric actuator withdraws the tip in equal steps. Δd is the increase in the tip-to-surface gap from its starting value d_0 , which is not known accurately but is estimated to be about 0.5 nm. The gap is modelled as a rectangular barrier of height ϕ above the energy of the tunnelling electrons. In the thick-barrier limit $I = I_0 \exp(-2\kappa d)$, with $\kappa = \sqrt{(2m_e\phi)/\hbar}$, $d = d_0 + \Delta d$, and I_0 independent of d . Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

Tunnelling current as the tip is withdrawn

$\Delta d / \text{pm}$	0	25	50	75	100	125
I / nA	3.20	1.85	1.13	0.644	0.388	0.222

(a) **EXPLAIN** [1]

(i) Explain why a plot of $\ln(I/\text{nA})$ against Δd should give a straight line even though d_0 is not known.

(a) **STATE** [1]

(ii) State what the gradient of that line represents.

(b) **DETERMINE** [3]

Determine κ from the data.

(c) **CALCULATE** [2]

Calculate the barrier height ϕ in electronvolts.

(d) **EVALUATE** [2]

Evaluate whether the thick-barrier form is adequate for analysing these readings.

(e) **ESTIMATE** [2]

Estimate, stating your assumptions, the change in gap that would alter the current by 5%.

A beam of electrons of mass m_e and energy E travels in the $+x$ direction towards a potential step: $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, with V_0 real and positive. The electrons are described by stationary scattering states of the time-dependent Schrödinger equation $i\hbar \partial\Psi/\partial t = -(\hbar^2/2m_e)\partial^2\Psi/\partial x^2 + V(x)\Psi$. Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) **DERIVE**

[4]

Derive, from the time-dependent Schrödinger equation and its complex conjugate, the continuity equation $\partial\rho/\partial t + \partial j/\partial x = 0$ for $\rho = |\Psi|^2$, obtaining an expression for the probability current j and stating where the reality of V is used.

(b) **SHOW THAT**

[4]

For $E > V_0$, take $\psi(x) = e^{ik_1x} + r e^{-ik_1x}$ for $x < 0$ and $\psi(x) = t e^{ik_2x}$ for $x > 0$, where $k_1 = \sqrt{2m_e E}/\hbar$ and $k_2 = \sqrt{2m_e(E - V_0)}/\hbar$. Explaining why there is no e^{-ik_2x} term, show that the reflection and transmission probabilities are $R = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = 4k_1k_2/(k_1 + k_2)^2$.

(c) **CALCULATE**

[2]

For electrons with $E = 6.0$ eV incident on a step of $V_0 = 4.5$ eV, calculate R , T and $|t|^2$, and confirm that $R + T = 1$.

(d) **SKETCH**

[3]

The electrons now have $E = 3.0$ eV, below the step. Sketch $|\psi(x)|^2$ against x from about $x = -1$ nm to $x = +0.5$ nm, for an incident wave of unit amplitude, marking the spacing of the maxima for $x < 0$, the value of $|\psi|^2$ at $x = 0$, and the length over which $|\psi|^2$ falls by a factor of e for $x > 0$.

(e) **EXPLAIN**

[2]

Explain why every electron is reflected when $E < V_0$, even though $|\psi|^2$ is not zero inside the step.

University Quantum Mechanics I · Midterm Examination

— mark scheme

One line per mark, as an examiner would award them. A note under a part is the error that part is set to catch.

- 1** MULTIPLE CHOICE · ROUTINE · UQMI-01.04 **[1]**
 CORE COURSE The Compton shift for a photon scattered through an angle θ by a free electron at rest is $\Delta\lambda = (h/m_e c)(1 - \cos \theta)$, with $h/m_e c = 2.43$ pm. Photons of wavelength 71.0 pm and photons of wavelength 500 nm are each scattered through 90° . Which statement best explains why the effect was established with X-rays rather than with visible light?

KEY A The shift is 2.43 pm in both cases, which is 3.4% of 71.0 pm but only 4.9×10^{-6} of 500 nm, a fractional change about 7000 times smaller and far harder to detect.

D is the most tempting, because it reaches the right conclusion that visible light is a poor probe of the effect. It gets there by scaling the shift itself with wavelength, but the formula contains no λ at all: at 90° the absolute shift is 2.43 pm whatever the incident wavelength, and what differs is the fraction $\Delta\lambda/\lambda$. C makes the opposite scaling error, treating the shift as proportional to λ , which again contradicts the formula in the stem. B confuses energy with momentum: every photon carries $p = h/\lambda$, about 1.3×10^{-27} kg m s $^{-1}$ at 500 nm, and the size of its energy compared with $m_e c^2$ decides how large the fractional shift is, not whether recoil happens. A is correct because $2.43/71.0 = 0.034$, while 2.43×10^{-12} m / 500×10^{-9} m = 4.9×10^{-6} , about 7000 times smaller. A fractional change of that size is not beyond every optical instrument, since a good interferometer can resolve it; in practice the further obstacle is that a visible photon's few electronvolts are comparable with atomic binding energies, so the electrons do not recoil as free particles at rest.

U1

- 2** MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-01.09 **[1]**
 CORE COURSE Each function in the options is finite and continuous for all x , and takes the form stated at large $|x|$, where C and k are nonzero real constants. Which one could be normalised, and so represent the state of a particle?

KEY C $\psi(x) \approx C/|x|$ at large $|x|$

Normalisability needs $\int |\psi|^2 dx$ to be finite, so the test is the tail of $|\psi|^2$, not the tail of ψ . A is the trap: ψ does fall to zero, but $|\psi|^2 = C^2/|x|$, and $\int dx/|x|$ grows as $\ln|x|$ without limit, so the integral diverges however slowly. D is tempting because the function both decays and oscillates, and oscillation looks as though it should cancel; but $|\psi|^2 = C^2 \cos^2(kx)/\sqrt{|x|}$ is never negative and averages to $C^2/(2\sqrt{|x|})$, whose integral grows as $\sqrt{|x|}$. B is a plane wave: $|\psi|^2 = C^2$ everywhere, so the integral grows linearly and no choice of C normalises it. C is correct: $|\psi|^2 = C^2/x^2$, and $\int dx/x^2$ converges at large $|x|$.

U1

3

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.05

[1]

A particle in a time-independent potential is in the state $\Psi(x,t) = [\psi_1(x)e^{(-iE_1t/\hbar)} + \psi_2(x)e^{(-iE_2t/\hbar)}]/\sqrt{2}$, where ψ_1 and ψ_2 are real, normalised, mutually orthogonal eigenfunctions of $\hat{H}$ with eigenvalues $E_1 \neq E_2$. Which quantity is independent of time whatever the forms of ψ_1 and ψ_2 ?

KEY D (H)

A is what a candidate expects if each phase factor is thought to cancel in the modulus squared, as it does for a single stationary state. With two terms the cross term $\psi_1\psi_2\cos((E_2 - E_1)t/\hbar)$ survives, so the density at any point where both functions are nonzero oscillates. B moves with the same cross term, weighted by $\int \psi_1\psi_2 dx$; it is constant only in the special case where that integral vanishes, for example when ψ_1 and ψ_2 have the same parity, so it fails the words whatever the forms. C looks conserved because the state stays normalised, but normalisation is not overlap: up to an overall phase the integral is $[1 + e^{(-i(E_2-E_1)t/\hbar)}]/2$, whose squared modulus is $\cos^2((E_2 - E_1)t/2\hbar)$. D is correct: $\langle H \rangle = (E_1 + E_2)/2$ at all times, because $\hat{H}$ returns each term multiplied by its own energy and orthogonality removes the cross terms.

U1

4

CORE
COURSE

MULTIPLE CHOICE · DEMANDING · UQMI-03.05

[1]

Acting on square-integrable functions on $-\infty < x < \infty$ that vanish, together with their derivatives, as $|x| \rightarrow \infty$, which one of these operators is Hermitian?

KEY D $i d/dx$

Each option is tested by moving the operator across $\int f^*(\hat{O}g) dx$ with integration by parts, the boundary terms vanishing because the functions do. A gives $\int f^*g' dx = -\int (f')^*g dx$: one integration by parts flips the sign, so d/dx is anti-Hermitian. It is tempting because it is $\hat{p}$ with the constant $-i\hbar$ removed, and that factor of i is exactly what restores Hermiticity. B fails twice over: moving d/dx across gives a minus sign, and differentiating the x produces an extra term, so its adjoint is $-1 - x d/dx$. C needs two integrations by parts, which leave d^2/dx^2 itself Hermitian, but the i is complex-conjugated on the way across, so the adjoint is $-i d^2/dx^2$. D is correct: $\int f^*(ig') dx = -i \int (f')^*g dx = \int (if')^*g dx$, the sign from the integration by parts cancelled by the conjugation of i . It is $-\hat{p}/\hbar$.

U2

5

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-03.09 · UQMI-04.05

[1]

A particle has Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, and $[\hat{x}, \hat{p}] = i\hbar$. What is the commutator $[\hat{x}, \hat{H}]$?

KEY A $i\hbar\hat{p}/m$

B stops at the potential term. $V(\hat{x})$ does commute with $\hat{x}$, but the kinetic term does not, and it is the kinetic term that makes position and energy incompatible. C expands $[\hat{x}, \hat{p}^2]$ as $\hat{p}[\hat{x}, \hat{p}]$ alone and loses the second term; the product rule gives $[\hat{x}, \hat{p}^2] = \hat{p}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{p} = 2i\hbar\hat{p}$, and dividing by $2m$ leaves $i\hbar\hat{p}/m$. D is the sign error made by starting from $[\hat{p}, \hat{x}] = -i\hbar$ with the operators in the wrong order. A is correct, and it is consistent with $d\langle x \rangle/dt = (i/\hbar)\langle [\hat{H}, \hat{x}] \rangle = \langle \hat{p} \rangle/m$.

U2

6

CORE
COURSE

MULTIPLE CHOICE · RECALL · UQMI-04.09

[1]

Position is measured once on each of N electrons, all prepared in the same state ψ , with a detector whose resolution is far finer than the spread of the results. The readings have sample standard deviation s , and their mean has standard error $s/\sqrt{N}$. N is increased from 100 to 10 000. Which statement is correct?

KEY B s stays close to σ_x of the state, while the standard error falls by a factor of 10.

A treats the spread of the readings as a measurement error that averaging removes. It is not: s estimates σ_x , a property of the prepared state, and more readings only pin that estimate down more precisely. C is right about s but wrong about the mean, whose standard error does fall as $1/\sqrt{N}$, by $\sqrt{(10\,000/100)} = 10$, so $\langle x \rangle$ can be located as precisely as patience allows without any conflict with $\sigma_x\sigma_p \geq \hbar/2$. D pictures the measurements acting on one another, but each electron is measured once and then discarded, so every reading comes from a fresh copy of ψ . B is correct.

U4

7

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-02.09 · UQMI-05.01 · UQMI-05.04

[1]

A particle of mass m has $V(x) = 0$ everywhere on $-\infty < x < \infty$. Which statement about the solutions of its time-independent Schrödinger equation is correct?

KEY B Every $E > 0$ is allowed, and each is twofold degenerate, with solutions e^{ikx} and e^{-ikx} where $k = \sqrt{2mE}/\hbar$.

A carries over the logic of the bound-state problems, where a condition at the walls or at infinity picks out discrete energies. With $V = 0$ there are no walls and no solution vanishes at infinity, so nothing selects particular values of k and the spectrum is continuous. C quotes a genuine theorem outside its scope: non-degeneracy in one dimension is proved for normalisable bound states, and plane waves are not normalisable, so e^{ikx} and e^{-ikx} share each energy. D writes down correct solutions of the differential equation, but every combination of real exponentials grows without bound in at least one direction, so none can be normalised even in the delta-function sense or superposed into a state, and $E < 0$ is excluded. B is correct.

U1

8

CORE
COURSE

MULTIPLE CHOICE · ROUTINE · UQMI-01.08 · UQMI-06.03

[1]

A particle is in the ground state $\psi_1(x) = \sqrt{2/L} \sin(\pi x/L)$ of an infinite square well occupying $0 < x < L$. What is the probability that a measurement of position finds it in $0 < x < L/4$?

KEY C 0.091

B is the classical answer for a particle equally likely to be anywhere in the well, and also what comes from replacing $\sin^2$ by its average value of $1/2$; but ψ_1 is small near a wall, so the quarter of the well beside the wall holds much less than a quarter of the probability. A comes from a sign slip in $\int \sin^2 u \, du = u/2 - (\sin 2u)/4$, which gives $1/4 + 1/(2\pi)$. D uses $1/L$ instead of $2/L$ for $|\psi_1|^2$, forgetting what the normalisation constant is for. C is correct: $P = (2/L) \int_0^{L/4} \sin^2(\pi x/L) \, dx = 1/4 - 1/(2\pi) = 0.091$.

U3

A particle of mass m moving along the x -axis is in a stationary state of some potential $V(x)$. Its wavefunction is $\psi(x) = 0$ for $x < 0$; $\psi(x) = Ax/a$ for $0 \leq x \leq a$; and $\psi(x) = A e^{-(x-a)/a}$ for $x > a$, where a is a positive length and A is a positive real constant. ψ is continuous everywhere, but its slope jumps at $x = 0$ and at $x = a$.

(a) SHOW THAT · U2

[3]

Show that $A = \sqrt{6/(5a)}$.

- 1 splits the normalisation integral at $x = a$ and evaluates the linear piece: $\int_0^a A^2 x^2/a^2 dx = A^2 a/3$
- 2 evaluates the tail: $\int_a^\infty A^2 e^{-2(x-a)/a} dx = A^2 a/2$
- 3 sets $A^2(a/3 + a/2) = 5A^2 a/6 = 1$, so $A = \sqrt{6/(5a)}$

(b) CALCULATE · U3

[2]

Calculate the probability that a measurement of position finds the particle in $0 < x < a/2$, and the probability that it finds the particle at $x > a$.

- 1 $P(0 < x < a/2) = A^2 \int_0^{a/2} x^2/a^2 dx = A^2 a/24 = (6/5)(1/24) = 0.050$
- 2 $P(x > a) = A^2 a/2 = 3/5 = 0.60$

This part catches candidates who read a probability off $|\psi|^2$ at one point and multiply by the interval width, or who integrate ψ rather than $|\psi|^2$. The interval $0 < x < a/2$ is half of the linear region yet holds only a twentieth of the probability.

(c) EXPLAIN · U1

[2]

The slope of ψ must be continuous wherever V is finite. Explain why, and explain why the jump in slope at $x = 0$ is nevertheless admissible.

- 1 integrating $-(\hbar^2/2m)\psi'' + V\psi = E\psi$ from $x_0 - \epsilon$ to $x_0 + \epsilon$ gives $\psi'(x_0 + \epsilon) - \psi'(x_0 - \epsilon) = (2m/\hbar^2) \int (V - E)\psi dx$, which tends to zero as $\epsilon \rightarrow 0$ when V is finite, so ψ' cannot jump at such a point
- 2 $\psi = 0$ for all $x < 0$ is the wavefunction beside an infinite wall ($V = \infty$ for $x < 0$); V is not finite there, so the slope may jump from 0 to A/a at $x = 0$ while ψ itself stays continuous

Candidates who state the rule without deriving it from the Schrödinger equation cannot say when it relaxes, and so often call the kink at the wall inadmissible too.

(d) DETERMINE · U2

[2]

The jump in slope at $x = a$ requires $V(x)$ to contain a delta-function term $g\delta(x - a)$. Determine g in terms of $\hbar$, m and a , and say whether the spike is attractive or repulsive.

- 1 the slope changes from $+A/a$ to $-A/a$, so $\Delta\psi' = -2A/a$, while integrating the Schrödinger equation across $x = a$ gives $\Delta\psi' = (2m/\hbar^2)g\psi(a) = (2m/\hbar^2)gA$
- 2 $g = -\hbar^2/(ma)$: the spike is attractive, $V(x)$ containing $-(\hbar^2/(ma))\delta(x - a)$

The sign is what separates answers. A cusp where the slope falls, at a point where ψ is positive, needs an attractive (negative) spike; candidates who remember only that a delta function makes a kink often give the right magnitude with the wrong sign.

An electron ($m_e = 9.109 \times 10^{-31}$ kg) moves freely along the x -axis, $V = 0$. At $t = 0$ it is prepared as a Gaussian wave packet centred on wavenumber k_0 , with position standard deviation $\sigma_0 = 0.40$ nm and $\sigma_x \sigma_p = \hbar/2$. The kinetic energy corresponding to the central wavenumber is $E_0 = \hbar^2 k_0^2 / (2m_e) = 1.50$ eV. The width of the packet then evolves as $\sigma_x(t) = \sigma_0 \sqrt{1 + (\hbar t / (2m_e \sigma_0^2))^2}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) EXPLAIN · U1

[9]

Explain why no single plane wave $e^{i(k_0 x)}$ can describe the electron, and how a superposition of plane waves overcomes the difficulty.

- $|e^{i(k_0 x)}|^2 = 1$ at every x , so $\int |\psi|^2 dx$ over the whole line diverges: the plane wave cannot be normalised and so is not a physical state
- a Fourier superposition $\psi(x) = (1/\sqrt{2\pi}) \int \phi(k) e^{ikx} dk$ with square-integrable $\phi(k)$ is normalisable, since $\int |\psi|^2 dx = \int |\phi|^2 dk$ (Plancherel's theorem); the price is a spread of wavenumbers, and so of momenta, about k_0

The error this catches is saying only that a plane wave is not localised. The marks are for the divergent normalisation integral, and for seeing that the spread in k is what buys normalisability rather than a side effect of preparing the state.

(b) SHOW THAT · U2

[2]

Using the dispersion relation of a free particle, show that the centre of the packet moves at $v_g = \hbar k_0 / m_e$, twice the phase velocity of its central component.

- for $V = 0$, $\hbar \omega = \hbar^2 k^2 / (2m_e)$, so $\omega(k) = \hbar k^2 / (2m_e)$ and the phase velocity of the central component is $v_p = \omega/k = \hbar k_0 / (2m_e)$
- the envelope moves at $v_g = d\omega/dk$ evaluated at k_0 , which is $\hbar k_0 / m_e = 2v_p$ and equals the classical speed p_0/m_e

Candidates who set $\hbar \omega$ equal to $\frac{1}{2}mv^2$ and then take $v = \omega/k$ obtain half the particle's speed. The part rewards knowing that a detector registers the envelope, and that the envelope moves at $d\omega/dk$.

(c) CALCULATE · U3

[2]

Calculate the group velocity, the time at which σ_x has grown to $2\sigma_0$, and the distance the centre of the packet travels in that time.

- $v_g = \sqrt{2E_0/m_e} = \sqrt{(2 \times 1.50 \times 1.602 \times 10^{-19} \text{ J}) / (9.109 \times 10^{-31} \text{ kg})} = 7.26 \times 10^5 \text{ m s}^{-1}$
- $\sigma_x = 2\sigma_0$ when $(\hbar t / (2m_e \sigma_0^2))^2 = 3$, so $t = 2\sqrt{3} m_e \sigma_0^2 / \hbar = 2\sqrt{3} \times 9.109 \times 10^{-31} \text{ kg} \times (0.40 \times 10^{-9} \text{ m})^2 / 1.055 \times 10^{-34} \text{ J s} = 4.79 \times 10^{-15} \text{ s}$
- distance = $v_g t = 7.26 \times 10^5 \text{ m s}^{-1} \times 4.79 \times 10^{-15} \text{ s} = 3.5 \times 10^{-9} \text{ m}$ (3.5 nm)

Setting $\hbar t / (2m_e \sigma_0^2) = 2$, instead of solving $1 + (\hbar t / (2m_e \sigma_0^2))^2 = 4$, gives $5.5 \times 10^{-15} \text{ s}$ and is the commonest slip. The width has doubled when the squared ratio equals 3, not when the ratio equals 2.

(d) PREDICT · U1

[3]

The electron is instead prepared with $\sigma_0 = 0.80$ nm, still with $\sigma_x \sigma_p = \hbar/2$ at $t = 0$. Predict, with a reason, the new time for the width to double.

- the packet spreads because ω is quadratic in k , so its components travel with a spread of group velocities σ_p/m_e ; with $\sigma_p = \hbar/(2\sigma_0)$ halved, they separate half as fast, while the width to be doubled is twice as large
- the doubling time scales as σ_0^2 , so it becomes 4 times longer: $1.91 \times 10^{-14} \text{ s}$

The trap is the intuition that a wider packet is less confined and so should spread at least as fast. A narrower packet spreads faster because confining it in x broadens it in k , a property of the Fourier transform rather than a disturbance.

[2]

An electron ($m_e = 9.109 \times 10^{-31}$ kg) is confined along one axis of a nanostructure modelled as an infinite square well on $0 < x < L$, with $L = 1.20$ nm. Its energy eigenfunctions are $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$, with energies $E_n = n^2\pi^2\hbar^2/(2m_eL^2)$. At $t = 0$ its state is $\Psi(x,0) = \sqrt{2/3} \psi_1(x) + (i/\sqrt{3}) \psi_2(x)$. You may use $\int_0^L \psi_n \times \psi_n dx = L/2$ for every n , and $\int_0^L \psi_1 \times \psi_2 dx = -16L/(9\pi^2)$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) CALCULATE · U3

[2]

Calculate the probabilities of obtaining E_1 and E_2 in a measurement of energy, and the expectation value $\langle H \rangle$ in electronvolts.

1 $P(E_1) = |\sqrt{2/3}|^2 = 2/3$ and $P(E_2) = |i/\sqrt{3}|^2 = 1/3$, which sum to 1, so Ψ is normalised

2 $E_1 = \pi^2\hbar^2/(2m_eL^2) = 4.19 \times 10^{-20} \text{ J} = 0.261 \text{ eV}$, so $\langle H \rangle = (2/3)E_1 + (1/3)(4E_1) = 2E_1 = 0.523 \text{ eV}$

Squaring $i/\sqrt{3}$ to $-1/3$, which leaves probabilities summing to $1/3$, is the error to catch: $|c|^2$ is c^*c , not c^2 . A candidate who gives $\langle H \rangle$ as one of the eigenvalues has confused an average over identically prepared systems with the result of one measurement.

(b) SHOW THAT · U2

[3]

Write down $\Psi(x,t)$, and show that $\langle x \rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2)) \sin(\omega t)$, where $\omega = (E_2 - E_1)/\hbar$.

1 $\Psi(x,t) = \sqrt{2/3} \psi_1 e^{(-iE_1 t/\hbar)} + (i/\sqrt{3}) \psi_2 e^{(-iE_2 t/\hbar)}$, with each phase factor attached to its own term

2 $\langle x \rangle = (2/3)(L/2) + (1/3)(L/2) + 2\text{Re}[c_1^* c_2 e^{(-i\omega t)}] \int \psi_1 x \psi_2 dx = L/2 + 2\text{Re}[(i\sqrt{2/3})e^{(-i\omega t)}] \times (-16L/(9\pi^2))$

3 $\text{Re}[i e^{(-i\omega t)}] = \sin(\omega t)$, so $\langle x \rangle = L/2 - (2\sqrt{2/3})(16L/(9\pi^2)) \sin(\omega t) = L/2 - (32\sqrt{2}L/(27\pi^2)) \sin(\omega t)$

Pulling a single factor $e^{(-iEt/\hbar)}$ out in front of the whole sum freezes $|\Psi|^2$ and removes the cross term from $\langle x \rangle(t)$. The other slip is dropping the i in c_2 , which gives a cosine and the wrong value of $\langle x \rangle$ at $t = 0$.

(c) CALCULATE · U3

[2]

Calculate the period of the oscillation of $\langle x \rangle$, and its amplitude in nanometres.

1 $\omega = 3E_1/\hbar = 3 \times 4.19 \times 10^{-20} \text{ J} / 1.055 \times 10^{-34} \text{ J s} = 1.19 \times 10^{15} \text{ rad s}^{-1}$, so the period is $2\pi/\omega = 5.28 \times 10^{-15} \text{ s}$

2 amplitude $= 32\sqrt{2}L/(27\pi^2) = 0.170L = 0.204 \text{ nm}$

Using E_2 alone, or $E_1 + E_2$, in place of the difference $E_2 - E_1$ is the error this part is set to catch: only the beat between the two phases is observable.

(d) PREDICT · U1

[2]

Suppose instead that $\Psi(x,0) = \sqrt{2/3} \psi_1(x) + (1/\sqrt{3}) \psi_2(x)$. Predict which of $P(E_1)$, $P(E_2)$, $\langle H \rangle$, and the period and amplitude of $\langle x \rangle(t)$ change, and what does change.

1 $P(E_1)$, $P(E_2)$ and $\langle H \rangle$ are unchanged because they depend only on $|c_n|^2$; the period and amplitude are unchanged because they depend only on $E_2 - E_1$ and on $|c_1||c_2|$

2 only the phase of the oscillation changes: $\langle x \rangle(t) = L/2 - (32\sqrt{2}L/(27\pi^2)) \cos(\omega t)$, so $\langle x \rangle$ starts at its minimum, $0.600 - 0.204 = 0.396 \text{ nm}$, instead of at the centre of the well

Candidates who expect a relative phase to leave everything untouched, because it cancels in each $|c_n|^2$, miss that it enters the interference term. The converse error, predicting that the energy probabilities change, is just as common.

(e) DESCRIBE · U1

[2]

Return to the original state. At $t = 2.0$ fs a measurement of energy on the electron returns E_2 . Describe the state immediately afterwards, and how $\langle x \rangle$ and the result of a second energy measurement behave from then on.

- 1 the state collapses to ψ_2 , multiplied by a phase factor of no physical consequence and normalised, with no ψ_1 component remaining
- 2 it is then stationary: $\langle x \rangle = L/2 = 0.600$ nm at all later times, and a repeated energy measurement returns $E_2 = 4E_1 = 1.05$ eV with probability 1

The error to catch is carrying on with the pre-measurement superposition, so that $\langle x \rangle$ keeps oscillating, or keeping a reduced ψ_1 component and renormalising. Collapse removes the other term entirely.

An electron is bound in a thin layer of material modelled as a symmetric finite square well: $V(x) = 0$ for $|x| < a$ and $V(x) = V_0$ for $|x| > a$, with well width $2a = 0.80$ nm and $V_0 = 2.00$ eV. Treat the electron as having the free-electron mass $m_e = 9.109 \times 10^{-31}$ kg. For a bound state with $0 < E < V_0$, write $k = \sqrt{(2m_e E)/\hbar}$, $\kappa = \sqrt{(2m_e(V_0 - E))/\hbar}$, $z = ka$ and $z_0 = (a/\hbar)\sqrt{(2m_e V_0)}$. Take $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) JUSTIFY · U1

[2]

Justify looking for bound states that are either even or odd in x , rather than of general form.

- 1 $V(-x) = V(x)$, so if $\psi(x)$ solves the time-independent Schrödinger equation with energy E , then $\psi(-x)$ solves it with the same E
- 2 bound states in one dimension are non-degenerate, so $\psi(-x) = \lambda\psi(x)$; reflecting twice gives $\lambda^2 = 1$, so $\lambda = \pm 1$ and every bound state is even or odd as a matter of theorem, not merely of choice

Answers that say only that the well is symmetric, so the split is allowed, earn the first mark. The second needs non-degeneracy, which is what turns a convenient choice into a result.

(b) DERIVE · U2

[4]

Starting from the solutions of the Schrödinger equation in each region, derive the condition that the energies of the even bound states must satisfy, written as a single equation in z and z_0 alone.

- 1 inside the well $\psi = C \cos(kx)$; for $x > a$, $\psi = D e^{-\kappa x}$, mirrored for $x < -a$, the growing solution $e^{+\kappa x}$ being discarded because ψ must be normalisable
- 2 continuity of ψ at $x = a$: $C \cos(ka) = D e^{-\kappa a}$
- 3 continuity of ψ' at $x = a$, which holds because V is finite there: $-kC \sin(ka) = -\kappa D e^{-\kappa a}$; dividing the two conditions eliminates C and D and gives $k \tan(ka) = \kappa$
- 4 multiplying by a and using $k^2 + \kappa^2 = 2m_e V_0/\hbar^2$, so that $(ka)^2 = z_0^2 - z^2$, gives $z \tan z = \sqrt{(z_0^2 - z^2)}$

The order of the argument is what matters: normalisability removes $e^{+\kappa x}$ before any matching is done, and matching then fixes the amplitude ratio and the energy. Candidates who keep both exponentials are left with more unknowns than conditions, and candidates who stop at $k \tan(ka) = \kappa$ have not used $k^2 + \kappa^2 = 2m_e V_0/\hbar^2$ to remove κ .

(c) DETERMINE · U3

[2]

Calculate z_0 , and hence determine how many bound states the well has and the parity of each.

- 1 $z_0 = (0.40 \times 10^{-9} \text{ m} / 1.055 \times 10^{-34} \text{ J s}) \times \sqrt{(2 \times 9.109 \times 10^{-31} \text{ kg} \times 2.00 \times 1.602 \times 10^{-19} \text{ J})} = 2.90$
- 2 an even state exists for each $n\pi < z_0$ and an odd state for each $(n + \frac{1}{2})\pi < z_0$, with $n = 0, 1, 2, \dots$; since $\pi/2 < 2.90 < \pi$ there are two bound states, the even ground state and one odd state

Using the full width 0.80 nm in place of a doubles z_0 to 5.79 and predicts four states. Checking that $2z_0/\pi = 1.84$ rounds up to 2 is a quick guard against that slip.

(d) COMPARE · U3

[3]

The even condition has its lowest root at $z = 1.159$. Compare the ground-state energy with that of an infinite square well of the same width, 0.80 nm, and account for the difference.

- 1 $E = V_0(z/z_0)^2 = 2.00 \text{ eV} \times (1.159/2.897)^2 = 0.320 \text{ eV}$ (accept 0.319 eV from $z_0 = 2.90$)
- 2 infinite well of width $2a$: $E = \pi^2 \hbar^2 / (2m_e (2a)^2) = 0.588 \text{ eV}$, so the finite-well ground state is lower, about 0.54 of the infinite-well value
- 3 ψ leaks into the classically forbidden region over a decay length $1/\kappa = a/\sqrt{(z_0^2 - z^2)} = 0.15 \text{ nm}$, so it fits a longer wavelength inside the well: $ka = 1.159$ rather than $\pi/2 = 1.571$, giving a smaller k and a lower energy

Candidates often argue that the finite-well level must be higher because the electron needs extra energy to enter the barrier. The comparison has to rest on the curvature of ψ inside the well, which the leakage reduces.

In a scanning tunnelling microscope a sharp tungsten tip is held above a clean metal surface with a small bias voltage between them, and the tunnelling current I is recorded as a piezoelectric actuator withdraws the tip in equal steps. Δd is the increase in the tip-to-surface gap from its starting value d_0 , which is not known accurately but is estimated to be about 0.5 nm. The gap is modelled as a rectangular barrier of height ϕ above the energy of the tunnelling electrons. In the thick-barrier limit $I = I_0 \exp(-2\kappa d)$, with $\kappa = \sqrt{(2m_e\phi)/\hbar}$, $d = d_0 + \Delta d$, and I_0 independent of d . Take $m_e = 9.109 \times 10^{-31}$ kg, $\hbar = 1.055 \times 10^{-34}$ J s and $1 \text{ eV} = 1.602 \times 10^{-19}$ J.

(a) EXPLAIN · U4

[1]

(i) Explain why a plot of $\ln(I/nA)$ against Δd should give a straight line even though d_0 is not known.

- 1 $\ln I = (\ln I_0 - 2\kappa d_0) - 2\kappa \Delta d$, so $\ln I$ is linear in Δd , and the unknown d_0 only shifts the intercept

Candidates who think d_0 must be known before κ can be found have not separated the intercept from the gradient. The measurement is designed so that the gradient needs only changes in the gap.

(a) STATE · U4

[1]

(ii) State what the gradient of that line represents.

- 1 the gradient is -2κ

Giving the gradient as $-\kappa$, by dropping the 2 that comes from squaring the amplitude to get a probability, is the slip this catches; it carries through to a barrier height four times too large.

(b) DETERMINE · U4

[3]

Determine κ from the data.

- 1 computes $\ln(I/nA) = 1.163, 0.615, 0.122, -0.440, -0.947, -1.505$
2 gradient of the best-fit straight line of $\ln(I/nA)$ against $\Delta d = -0.0212 \text{ pm}^{-1}$ (accept -0.0205 to -0.0220 pm^{-1}), from a least-squares fit or from widely separated points on a drawn line rather than from adjacent readings
3 $\kappa = 0.0212 \text{ pm}^{-1} / 2 = 1.06 \times 10^{10} \text{ m}^{-1}$ (accept 1.03×10^{10} to $1.10 \times 10^{10} \text{ m}^{-1}$)

Adjacent pairs of readings give gradients anywhere from -0.0197 to -0.0225 pm^{-1} , because the readings scatter by 1 to 2 per cent; the least-squares gradient, -0.02124 pm^{-1} , uses all six. Forgetting the factor of 2 doubles κ and quadruples the barrier height in (c).

(c) CALCULATE · U3

[2]

Calculate the barrier height ϕ in electronvolts.

- 1 $\phi = \hbar^2 \kappa^2 / (2m_e) = (1.055 \times 10^{-34} \text{ J s} \times 1.06 \times 10^{10} \text{ m}^{-1})^2 / (2 \times 9.109 \times 10^{-31} \text{ kg}) = 6.9 \times 10^{-19} \text{ J}$
2 $\phi = 4.3 \text{ eV}$ (accept 4.0 to 4.6 eV when consistent with the value of κ from (b))

The squared κ makes this result sensitive: a 3% error in the gradient becomes a 6% error in ϕ . Candidates who leave ϕ in joules, or divide by e twice, are the ones this part catches.

(d) EVALUATE · U4

[2]

Evaluate whether the thick-barrier form is adequate for analysing these readings.

- 1 at the smallest gap $2\kappa d \approx 2 \times 1.06 \times 10^{10} \text{ m}^{-1} \times 0.5 \times 10^{-9} \text{ m} \approx 11 \gg 1$, so the exact rectangular-barrier transmission differs from the thick-barrier form only by corrections of relative order $e^{(-2\kappa d)}$, a few parts in 10^5 , and the prefactor that form discards does not depend on d , so it cannot change the gradient
2 the six values of $\ln I$ lie within about 0.02 of a straight line, which supports the model over this range; judgement: adequate for finding the gradient, but ϕ is an apparent barrier height, because the real barrier is not rectangular (image-charge forces round and lower it) and the tip geometry is not one-dimensional

A strong answer weighs both the size of $2\kappa d$ and the residuals of the fit, and then limits the claim: the analysis gives an effective ϕ for a model barrier, not the work function of the surface directly.

(e) ESTIMATE · U4

[2]

Estimate, stating your assumptions, the change in gap that would alter the current by 5%.

- 1 assumes I_0 stays fixed and the change is small enough to linearise the exponential, so that $|\Delta I|/I \approx 2\kappa|\Delta d|$
- 2 $|\Delta d| \approx 0.05 / (2.1 \times 10^{10} \text{ m}^{-1}) \approx 2 \times 10^{-12} \text{ m}$ (2 pm), a small fraction of an atomic diameter

The error to catch is reading the change off the 25 pm spacing of the table instead of using the local slope. The estimate is what makes sub-atomic height resolution plausible: a change of a couple of picometres already gives a measurable change in current.

14
CORE
COURSE

FREE RESPONSE · EXTENDED DERIVATION · DISCRIMINATING · UQMI-02.06 · UQMI-02.07 · UQMI-08.01 · UQMI-08.02

[15]

A beam of electrons of mass m_e and energy E travels in the $+x$ direction towards a potential step: $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, with V_0 real and positive. The electrons are described by stationary scattering states of the time-dependent Schrödinger equation $i\hbar \partial\Psi/\partial t = -(\hbar^2/2m_e)\partial^2\Psi/\partial x^2 + V(x)\Psi$. Take $m_e = 9.109 \times 10^{-31} \text{ kg}$, $\hbar = 1.055 \times 10^{-34} \text{ J s}$ and $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$.

(a) DERIVE · U2

[4]

Derive, from the time-dependent Schrödinger equation and its complex conjugate, the continuity equation $\partial\rho/\partial t + \partial j/\partial x = 0$ for $\rho = |\Psi|^2$, obtaining an expression for the probability current j and stating where the reality of V is used.

- 1 writes the conjugate equation $-i\hbar \partial\Psi^*/\partial t = -(\hbar^2/2m_e)\partial^2\Psi^*/\partial x^2 + V\Psi^*$, which uses $V^* = V$
- 2 $\partial\rho/\partial t = \Psi^*\partial\Psi/\partial t + \Psi\partial\Psi^*/\partial t$; on substituting both equations the potential terms $-(i/\hbar)V|\Psi|^2$ and $+(i/\hbar)V|\Psi|^2$ cancel, because V is real
- 3 $\partial\rho/\partial t = (i\hbar/2m_e)(\Psi^*\partial^2\Psi/\partial x^2 - \Psi\partial^2\Psi^*/\partial x^2) = (i\hbar/2m_e)\partial/\partial x(\Psi^*\partial\Psi/\partial x - \Psi\partial\Psi^*/\partial x)$, the terms in $(\partial\Psi^*/\partial x)(\partial\Psi/\partial x)$ cancelling
- 4 so $\partial\rho/\partial t + \partial j/\partial x = 0$ with $j = (\hbar/2im_e)(\Psi^*\partial\Psi/\partial x - \Psi\partial\Psi^*/\partial x) = (\hbar/m_e)\text{Im}(\Psi^*\partial\Psi/\partial x)$

The step candidates miss is recognising $\Psi^*\partial^2\Psi/\partial x^2 - \Psi\partial^2\Psi^*/\partial x^2$ as an exact derivative; without it no current can be identified. Candidates who never write $V^* = V$ have not shown where conservation of probability could fail, and a complex potential is exactly how absorption is modelled.

(b) SHOW THAT · U2

[4]

For $E > V_0$, take $\psi(x) = e^{ik_1x} + r e^{-ik_1x}$ for $x < 0$ and $\psi(x) = t e^{ik_2x}$ for $x > 0$, where $k_1 = \sqrt{2m_e E}/\hbar$ and $k_2 = \sqrt{2m_e(E - V_0)}/\hbar$. Explaining why there is no e^{-ik_2x} term, show that the reflection and transmission probabilities are $R = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = 4k_1k_2/(k_1 + k_2)^2$.

- 1 there is no e^{-ik_2x} term because no wave comes in from $x = +\infty$: electrons are sent in from the left only
- 2 continuity of ψ at $x = 0$ gives $1 + r = t$, and continuity of ψ' , which holds because V is finite, gives $ik_1(1 - r) = ik_2t$
- 3 solving: $r = (k_1 - k_2)/(k_1 + k_2)$ and $t = 2k_1/(k_1 + k_2)$
- 4 using the current from (a): $j = (\hbar k_1/m_e)(1 - |r|^2)$ for $x < 0$, with no interference term, and $j = (\hbar k_2/m_e)|t|^2$ for $x > 0$, so $R = |r|^2 = ((k_1 - k_2)/(k_1 + k_2))^2$ and $T = (k_2/k_1)|t|^2 = 4k_1k_2/(k_1 + k_2)^2$

Writing $T = |t|^2$ is the error this part is built to catch. The current, not the density, counts particles crossing a point per unit time, so the speed ratio k_2/k_1 has to multiply $|t|^2$; because T is printed in the prompt, a candidate who writes $T = |t|^2$ and then forces agreement earns nothing for the last line.

(c) CALCULATE · U3

[2]

For electrons with $E = 6.0$ eV incident on a step of $V_0 = 4.5$ eV, calculate R , T and $|t|^2$, and confirm that $R + T = 1$.

1 $k_2/k_1 = \sqrt{(E - V_0)/E} = \sqrt{1.5/6.0} = 0.50$, so $R = (0.5/1.5)^2 = 1/9 = 0.11$ and $T = 4(0.5)/(1.5)^2 = 8/9 = 0.89$, giving $R + T = 1$

2 $t = 2/(1 + 0.5) = 4/3$, so $|t|^2 = 16/9 = 1.78$, and $T = (k_2/k_1)|t|^2 = 0.89$ is less than 1

A value of $|t|^2$ greater than 1 is not an error: the transmitted electrons move at half the speed, so they bunch to a higher density while carrying less flux. Candidates who conclude that a probability has exceeded one have mistaken a density ratio for a flux ratio.

(d) SKETCH · U1

[3]

The electrons now have $E = 3.0$ eV, below the step. Sketch $|\psi(x)|^2$ against x from about $x = -1$ nm to $x = +0.5$ nm, for an incident wave of unit amplitude, marking the spacing of the maxima for $x < 0$, the value of $|\psi|^2$ at $x = 0$, and the length over which $|\psi|^2$ falls by a factor of e for $x > 0$.

1 for $x < 0$: a standing-wave pattern oscillating between 0 and 4, with true zeros because $|r| = 1$, and maxima spaced $\pi/k_1 = 0.35$ nm apart ($k_1 = 8.87 \times 10^9$ m $^{-1}$)

2 at $x = 0$: matching gives $t = 1 + r = 2k_1/(k_1 + i\kappa)$, so $|\psi(0)|^2 = 4k_1^2/(k_1^2 + \kappa^2) = 4E/V_0 = 2.7$, with the two sides joining with no jump in value or in slope

3 for $x > 0$: a smooth decay $|\psi(0)|^2 e^{-2\kappa x}$ with no oscillation, falling by a factor of e in $1/(2\kappa) = 0.080$ nm, where $\kappa = \sqrt{2m_e(V_0 - E)}/\hbar = 6.27 \times 10^9$ m $^{-1}$

Common sketches either carry the oscillation on into the step or drop $|\psi|^2$ to zero at $x = 0$ as though the step were an infinite wall. Both miss the evanescent tail, which is the point of the part.

(e) EXPLAIN · U1

[2]

Explain why every electron is reflected when $E < V_0$, even though $|\psi|^2$ is not zero inside the step.

1 for $x > 0$, $\psi = te^{-\kappa x}$ has the same complex phase at every x , so $\Psi^* \partial \Psi / \partial x = -\kappa |t|^2 e^{-2\kappa x}$ is real and $j = (\hbar/m_e) \text{Im}(\Psi^* \partial \Psi / \partial x) = 0$: no probability flows into the step, so $T = 0$

2 in a stationary state $\partial \rho / \partial t = 0$, so by the continuity equation j is the same at every x ; zero on the right means $(\hbar k_1/m_e)(1 - |r|^2) = 0$ on the left, so $R = |r|^2 = 1$, and the density in the step is a standing evanescent penetration rather than a flow of particles through it

Candidates who equate a nonzero density with transmission, or say that some electrons tunnel into the step and get through, are the target. The argument has to run through the current and the continuity equation derived in (a).

What the command words are asking for

Examiners mark to the command word as well as to the physics. An answer that explains where the question said state has spent time earning one mark; one that states a result where the question said derive has earned none, however correct the result.

State	Give the result, definition or law concisely. No justification is required.
Describe	Give the relevant features of a situation or process in words, precisely enough that someone else could reproduce it.
Explain	Give the physical reason a result or behaviour occurs, connecting it explicitly to the principle that governs it.
Calculate	Obtain a numerical value, showing the relation used, the substitution and the units of the result.
Determine	Obtain a result by any valid route the problem allows — analytic, graphical or numerical — with enough working to be followed.
Derive	Start from a named fundamental law or definition and reach the required result by explicit, justified steps.
Show that	Establish a result the question gives you. Every step must be shown, because the stated answer earns nothing on its own.
Estimate	Reach an order-of-magnitude value from stated, reasonable assumptions, and say what those assumptions are.
Sketch	Draw the shape of a graph or field with axes labelled and its key features marked — intercepts, asymptotes, turning points, limits. Exact scale is not required.
Justify	Support a claim or a choice with evidence, and with the reasoning that connects that evidence to the claim.
Evaluate	Weigh a model, argument or result against criteria, including its assumptions and domain of validity, and reach a stated judgement.
Compare	Identify the similarities and differences between two situations, models or results, relating each difference to its cause.
Predict	Say what will happen when a condition changes, reasoning from a principle rather than guessing, and give the reason.

WHAT IS BEING TESTED

University assessment objectives

Each course has the two papers a university course is examined by: a midterm on the first half of its units, and a cumulative final weighted towards the second half. Every paper combines short conceptual multiple-choice items with structured problems, a data or estimation question, and at least one extended derivation. Constants are given where a question needs them; a calculator and your own formula sheet are assumed.

U1 21 MARKS · 28%	Conceptual understanding Identify the governing principle and the conditions under which it holds, and reason qualitatively about what a result must look like before calculating it.
U2 24 MARKS · 32%	Mathematical modelling and derivation Translate a physical situation into mathematics: choose coordinates and approximations, set up the governing equations, solve them symbolically, and derive stated results from named laws.

U3

19 MARKS · 26%

Quantitative problem solving

Carry a multi-step calculation to a numerical result with correct units and sensible precision, and check it against dimensions, limiting cases and orders of magnitude.

U4

10 MARKS · 14%

Data, estimation and experimental reasoning

Analyse measurements and their uncertainties, linearise and fit data, estimate orders of magnitude from stated assumptions, and judge whether evidence supports a model.

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